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# SYNCHRONIZATION THROUGH COMPOUND CHAOTIC SIGNAL IN CHUA'S CIRCUIT AND MURALI–LAKSHMANAN–CHUA CIRCUIT

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In this letter the idea of synchronization of chaotic systems is further extended to the case where all the drive system variables are combined to obtain a compound chaotic drive signal. An appropriate feedback loop is constructed in the response system to achieve synchronization among the variables of drive and response systems. We apply this method of synchronization to the familiar Chua's circuit and Murali–Lakshmanan–Chua circuit equations.

## 1. Introduction

The concept of synchronized chaos [Pecora & Carroll, 1990, 1991] allows for the possibility of building a set of chaotic dynamical systems such that their common signals are synchronized. Generally there are two methods of chaos synchronization available in the literature [Lakshmanan & Murali, 1996]. In the first method due to Pecora and Carroll [1990], a stable subsystem of a chaotic system is synchronized with a separate chaotic system under suitable conditions. This method has been further improved to cascading chaos synchronization with multiple stable subsystems [Kocarev, *et al.*, 1992; Carroll & Pecora, 1993; Cuomo & Oppenheim, 1993; Wu & Chua, 1994; Kocarev & Parlitz, 1995]. The second method to achieve chaos synchronization is due to the effect of one-way coupling element between two identical nonlinear systems without requiring to construct any stable subsystem [Pyragas, 1993; Murali & Lakshmanan, 1994; Wu & Chua, 1994; Murali, Lakshmanan & Chua, 1995]. In both these approaches, only one chaotic signal from the drive system is utilized to drive the response systems. In the present Letter, the idea of synchronization of chaotic systems is further extended to the case where all the drive system variables are combined so that a compound chaotic drive signal is produced to drive the response system. A feedback loop in the response system is appropriately constructed to achieve synchronization among the variables of the drive and response systems. We consider only those cases in which the compound signal so produced does not resemble any of the drive system variables. In Sec. 2, we give a brief account of the theory of modified chaos synchronization approach. In Sec. 3, we focus on the details of applicability of this modified approach to the familiar Chua's circuit. In Sec. 4, we investigate the synchronization of chaos in the Murali–Lakshmanan–Chua circuit equations using the above method. Finally, in Sec. 5, a summary of the results and conclusion are given.

## 2. Theory of Synchronization using Compound Chaotic Signal

Let us consider a chaotic dynamical system described by a set of first order differential equations of the form

$$\dot{x} = f(x, y, z),\ \dot{y} = g(x, y, z),\ \dot{z} = h(x, y, z)\,. \quad (1)$$

Here $(\cdot)$ refers to the operation $d/dt$. Using the Pecora and Carroll approach [Pecora & Carroll, 1990, 1991; Carroll, 1995] a set of cascaded subsystems is created as

$$\dot{x}'' = f(x'', y', z'),\ \dot{y}' = g(x_d, y', z'),\ \dot{z}' = h(x_d, y', z')\,. \tag{2}$$

If all of the Lyapunov exponents of Eq. (2) are less than zero, then the response subsystem variables $y'$, $z'$ and $x''$ will converge to the drive system variables $y$, $z$ and $x$ respectively under the influence of the single drive variable $x_d = x$ [Pecora & Carroll, 1990, 1991].

However it is not necessary that one of the drive variables alone is used for synchronization. One can also combine and modify the drive signal appropriately, and then the transformation is undone using the response system. Along these lines, Carroll recently reported the synchronization of chaotic systems using filtered signals [Carroll, 1994, 1995]. Alternatively, instead of using one drive signal variable, one can transform the drive variables by appropriate combinations to produce a compound chaotic signal for use as the drive for synchronization. A suitable feedback loop can be deviced in the response system to achieve synchronization among the variables of the drive and response systems. The schematic diagram of this modified approach of cascading chaos synchronization using compound signal is illustrated in Fig. 1(a). The set of equations for this new synchronizing system is

Drive:

$$\dot{x} = f(x, y, z),\ \dot{y} = g(x, y, z),\ \dot{z} = h(x, y, z)\,, \tag{3a}$$

$$x_t = x + u(y, z)\,, \tag{3b}$$

Response:

$$x_d = x_t - u(y', z')\,, \tag{4a}$$

$$\dot{x}'' = f(x'', y', z'),\ \dot{y}' = g(x_d, y', z'),\ \dot{z}' = h(x_d, y', z')\,. \tag{4b}$$

Here $u(\cdot)$ can be either a linear or nonlinear function and $x_t$ is the compound chaotic signal used as the drive and $x_d$ is the signal generated in the response system feedback loop. If the Lyapunov exponents of the response system [Eq. (4b)] are negative under the influence of $x_d$ signal then the response system variables $y'$, $z'$ and $x''$ are synchronized with $y$, $z$ and $x$ variables respectively.

Similarly for the case of synchronization through one-way coupling approach [Pyragas, 1992; Murali & Lakshmanan, 1994], the compound drive signal can be used as shown schematically in Fig. 1(b). The associated set of equations are represented as

Drive:

$$\dot{x} = f(x, y, z),\ \dot{y} = g(x, y, z),\ \dot{z} = h(x, y, z)\,, \tag{5a}$$

$$x_t = x + u(y, z)\,, \tag{5b}$$

Response:

$$x_d = x_t - u(y', z')\,, \tag{6a}$$

$$\dot{x}' = f(x', y', z') + \varepsilon(x_d - x')\,, \tag{6b}$$

$$\dot{y}' = g(x', y', z')\,, \tag{6c}$$

$$\dot{z}' = h(x', y', z')\,. \tag{6d}$$

Here $\varepsilon$ is the one-way coupling parameter. If the Lyapunov exponents of the response system

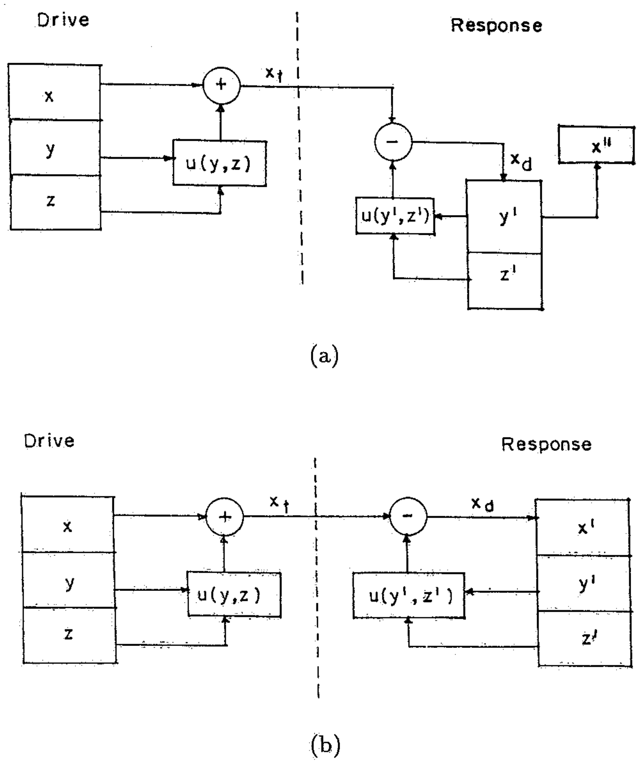



Fig. 1. (a) Block diagram of the cascading synchronization approach through compound chaotic signal. (b) Block diagram of the one-way coupling approach of synchronization through compound chaotic signal.

[Eqs. (6b)–(6d)] are negative under the influence of $x_d$ for appropriate $\varepsilon$ values, then the response system variables $x'$, $y'$ and $z'$ are synchronized with their drive counterpart.

## 3. Synchronization of Chaos in Chua's Circuit

The rescaled Chua's circuit equations [Chua *et al.*, 1993] are

$$\dot{x} = \alpha(y - x - f(x)),\ \dot{y} = x - y + z,\ \dot{z} = -\beta y\,, \quad (7)$$

where $f(x) = bx + 0.5(a - b)[|x + 1| - |x - 1|]$. The familiar double-scroll chaotic attractor as shown in Fig. 2 is observed for the fixed parameters $\alpha = 9.0$, $\beta = 14.87$, $a = -1.27$ and $b = -0.68$. The time series plot of the corresponding variables of Eq. (7) are shown in Fig. 3.

### 3.1. *Cascaded subsystems*

To achieve synchronization, a cascading system of equations along with a feedback loop having the simplest function $u(y, z) = y + z$ is represented as

Drive:

$$\dot{x} = \alpha(y - x - f(x))\,, \quad (8a)$$

$$\dot{y} = x - y + z\,, \quad (8b)$$

$$\dot{z} = -\beta y\,, \quad (8c)$$

$$x_t = x + u(y, z) \equiv x + y + z\,, \quad (8d)$$

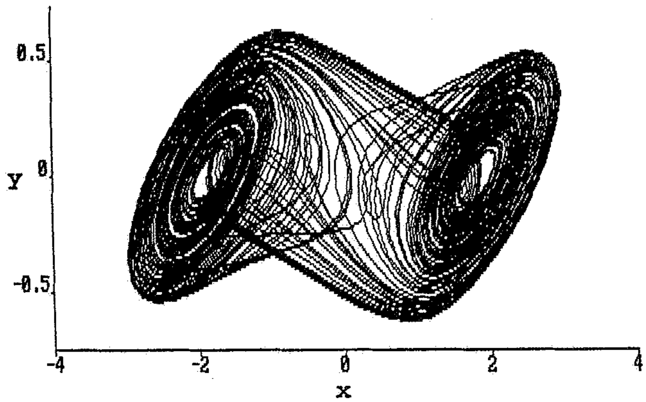



Fig. 2. The double-scroll chaotic attractor of Eq. (7) for $\alpha = 9.0$, $\beta = 14.87$, $a = -1.27$ and $b = -0.68$.

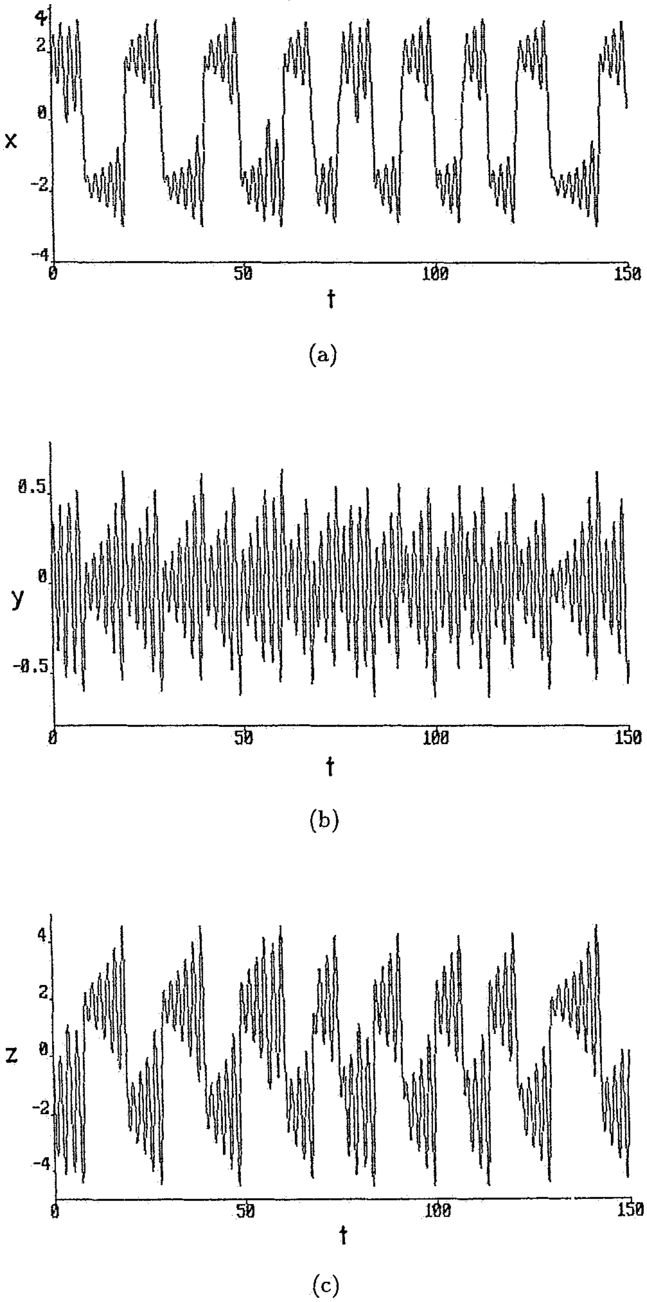



Fig. 3. Chaotic signals of Fig. 2. (a) variable $x$, (b) variable $y$ and (c) variable $z$.

Response:

$$x_d = x_t - u(y', z') \equiv x_t - (y' + z')\,, \quad (8e)$$

$$\dot{y}' = x_d - y' + z'\,, \quad (8f)$$

$$\dot{z}' = -\beta y'\,, \quad (8g)$$

$$\dot{x}'' = \alpha(y' - x'' - f(x''))\,. \quad (8h)$$

The compound chaotic signal $x_t$ is shown in Fig. 4(a). The numerically computed conditional

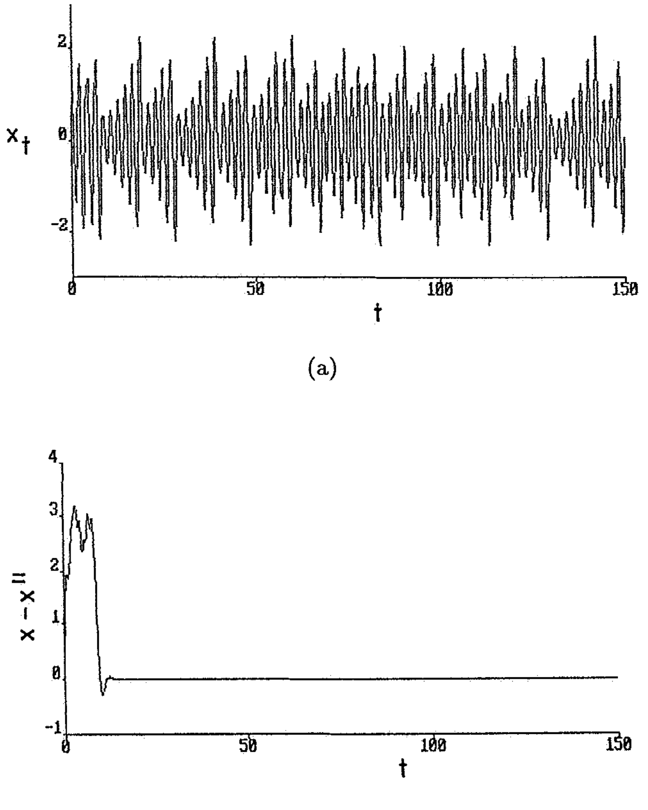



Fig. 4. (a) The compound chaotic signal $x_t$ of Eqs. (8a)–(8d) for $\alpha = 9.0$, $\beta = 14.87$. (b) The difference signal $(x - x'')$ versus $t$ of Eqs. (8a)–(8h).

Lyapunov exponents of the response system Eqs. (8f)–(8h) are (−0.05, −2.0019, −2.1942) and thus the response system variables synchronize with their drive counterpart. Figure 4(b) depicts the difference signal $(x - x'')$ of Eqs. (8a)–(8h). We may note that the compound chaotic signal $x_t$ does not resemble any of the drive signals $x$, $y$ or $z$.

### 3.2. *One-way coupled systems*

One can also study chaos synchronization among identical chaotic systems through one-way coupling scheme, without constructing any cascading stable subsystems [Pyragas, 1993; Murali and Lakshmanan, 1994; Murali, Lakshmanan & Chua, 1995]. We now apply this approach to Chua's circuit as follows. The normalized state equations are represented with $u(y, z) = y + z$ as

Drive:

$$\dot{x} = \alpha(y - x - f(x)), \quad (9a)$$

$$\dot{y} = x - y + z, \quad (9b)$$

$$\dot{z} = -\beta y, \quad (9c)$$

$$x_t = x + (y + z), \quad (9d)$$

Response:

$$x_d = x_t - (y' + z'), \quad (9e)$$

$$\dot{x}' = \alpha(y' - x' - f(x')) + \delta_x(x_d - x'), \quad (9f)$$

$$\dot{y}' = x' - y' + z', \quad (9g)$$

$$\dot{z}' = -\beta y'. \quad (9h)$$

Here $\delta_x$ is the one-way coupling parameter. In this set-up, we have effected a one-way coupling of two Chua's circuits through a linear resistor. The difference system of Eqs. (9a)–(9d) and Eqs. (9e)–(9h) is

$$\dot{x}^* = \alpha(y^* - x^* - (f(x) - f(x')) - \delta_x(x^* + y^* + z^*), \quad (10a)$$

$$\dot{y}^* = x^* - y^* + z^*, \quad (10b)$$

$$\dot{z}^* = -\beta y^*, \quad (10c)$$

where $x^* = (x - x')$, $y^* = (y - y')$ and $z^* = (z - z')$. Asymptotically $x^*$, $y^*$ and $z^*$ tend to zero, then we can easily see that $f(x) - f(x') = f'(\eta)(x - x') = f'(\eta)x^*$ and $f'(\eta)$ takes two values $a$ and $b$ [Chua *et al.*, 1993] asymptotically, depending upon the region of operation. Then

$$\dot{x}^* = \alpha(y^* - x^* - s_i x^*) - \delta_x(x^* + y^* + z^*), \quad (11a)$$

$$\dot{y}^* = x^* - y^* + z^*, \quad (11b)$$

$$\dot{z}^* = -\beta y^*, \quad (11c)$$

or

$$\begin{bmatrix} \dot{x}^* \\ \dot{y}^* \\ \dot{z}^* \end{bmatrix} = \begin{bmatrix} -\alpha - s_i\alpha - \delta_x & \alpha - \delta_x & -\delta_x \\ 1 & -1 & 1 \\ 0 & -\beta & 0 \end{bmatrix} \begin{bmatrix} x^* \\ y^* \\ z^* \end{bmatrix}, \quad (12)$$

where $s_i = a, b$; $i = 1, 2$. The characteristic equation is

$$\lambda^3 + \kappa\lambda^2 + \rho\lambda + \sigma = 0, \quad (13)$$

where $\kappa = 1 + \alpha + \alpha s_i + \delta_x$, $\rho = \alpha s_i + 2\delta_x + \beta$ and $\sigma = \beta\alpha + \beta\alpha s_i$.

If $\kappa > 0$, $\sigma > 0$ and $\kappa\rho - \sigma > 0$, then $x^* = y^* = z^* = 0$ is a stable point and the response

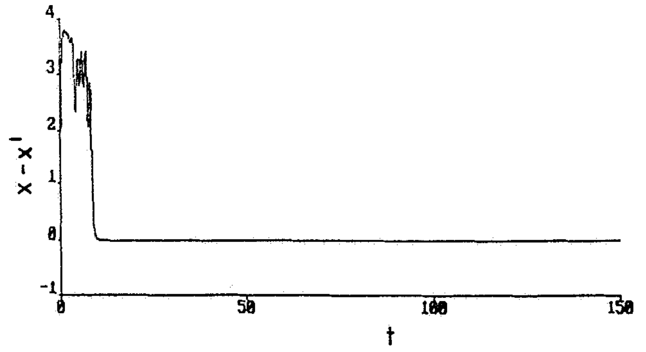



Fig. 5. The difference signal $(x - x')$ versus $t$ of Eqs. (9a)–(9h) for $\delta_x = 1.0$.

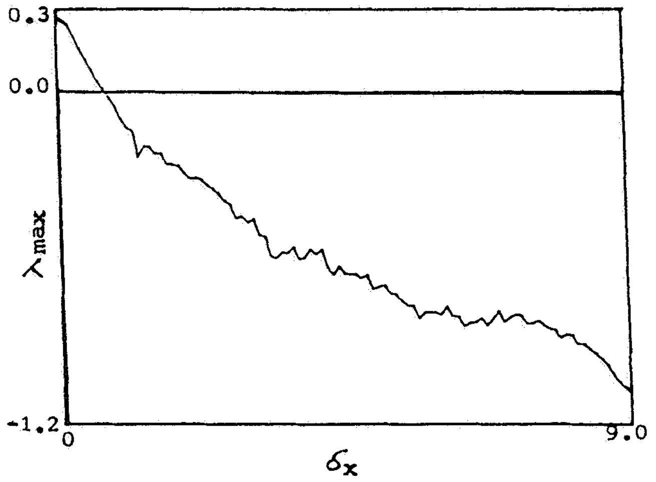



Fig. 6. The maximal conditional Lyapunov exponent $(\lambda_{\max})$ versus $\delta_x$ of the response system Eqs. (9e)–(9h).

system variables synchronize with the drive system. In the present case the critical value of $\delta_x(=\bar{\delta}_x)$ was found to be 0.5066, and thus for all $\delta_x > \bar{\delta}_x \approx 0.5066$ the response system will synchronize. Figure 5 depicts the synchronized chaos behavior of Eq. (9) for $\delta_x = 1.0$, calculated numerically. Also, the maximal conditional Lyapunov exponent of the response system Eqs. (9f)–(9h) as a function of the one-way coupling parameter $\delta_x$ is shown in Fig. 6. It is evident from this figure that the response system is synchronized with the drive counterpart, which is confirmed by a change in the sign of the maximal conditional Lyapunov exponent of the response system from positive to negative.

## 4. Synchronization of Chaos in Murali-Lakshmanan-Chua Circuit

In this section, we have applied the idea of synchronization through compound chaotic signal to the recently reported simplest non-autonomous chaotic circuit, namely, the MLC [Murali, Lakshmanan & Chua, 1994a, 1994b, 1995] circuit. The normalized state equations of two MLC circuits coupled through one-way coupling element are represented (with $u(y) = y$) as

Drive:

$$\begin{aligned} &\dot{x} = y - g(x),\ \dot{y} = -\sigma y - \beta x + F\sin(\omega t)\,, \\ &x_t = x + u(y) = x + y\,, \end{aligned} \tag{14a}$$

Response:

$$\begin{aligned} x_d &= x_t - u(y') = x_t - y'\,, \\ \dot{x}' &= y' - g(x') + \varepsilon(x_d - x')\,, \\ \dot{y}' &= -\sigma y' - \beta x' + F\sin(\omega t)\,, \end{aligned} \tag{14b}$$

where $\varepsilon$ is the one-way coupling parameter and $g(x) = bx + 0.5(a - b)[|x + 1| - |x - 1|]$. For the parameters $\sigma = 1.015$, $\beta = 1.0$, $F = 0.15$, $\omega = 0.75$, $a = -1.02$ and $b = -0.55$, a double-band chaotic attractor of Fig. 7 is observed [Murali, Lakshmanan & Chua, 1994a, 1994b, 1995].

The difference systems of Eqs. (14a) and (14b) are

$$\begin{aligned} \dot{x}^* &= y^* - (g(x) - g(x')) - \varepsilon(x^* + y^*)\,, \\ \dot{y}^* &= -\sigma y^* - \beta x^*\,, \end{aligned} \tag{15}$$

where $x^* = (x - x')$ and $y^* = (y - y')$. Since $g(x) - g(x') = g'(\eta)x^*$ and $g'(\eta)$ takes two slope values $a$ and $b$ [Chua *et al.*, 1993] depending upon the region of operation and basin of attraction asymptotically, we have

$$\dot{x}^* = y^* - s_i x^* - \varepsilon(x^* + y^*), \quad \dot{y}^* = -\sigma y^* - \beta x^*\,, \tag{16}$$

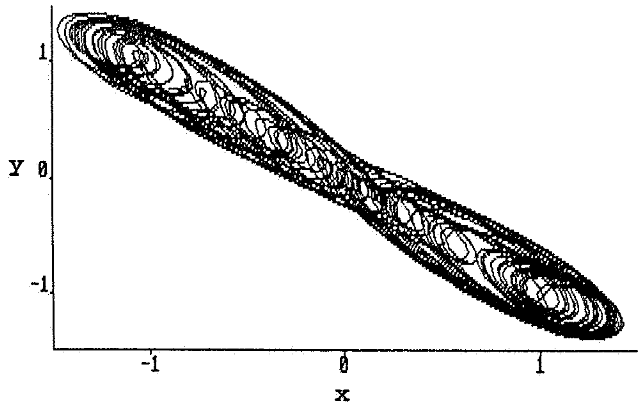
or

$$\begin{bmatrix} \dot{x}^* \\ \dot{y}^* \end{bmatrix} = \begin{bmatrix} -s_i - \varepsilon & 1 - \varepsilon \\ -\beta & -\sigma \end{bmatrix} \begin{bmatrix} x^* \\ y^* \end{bmatrix}, \tag{17}$$

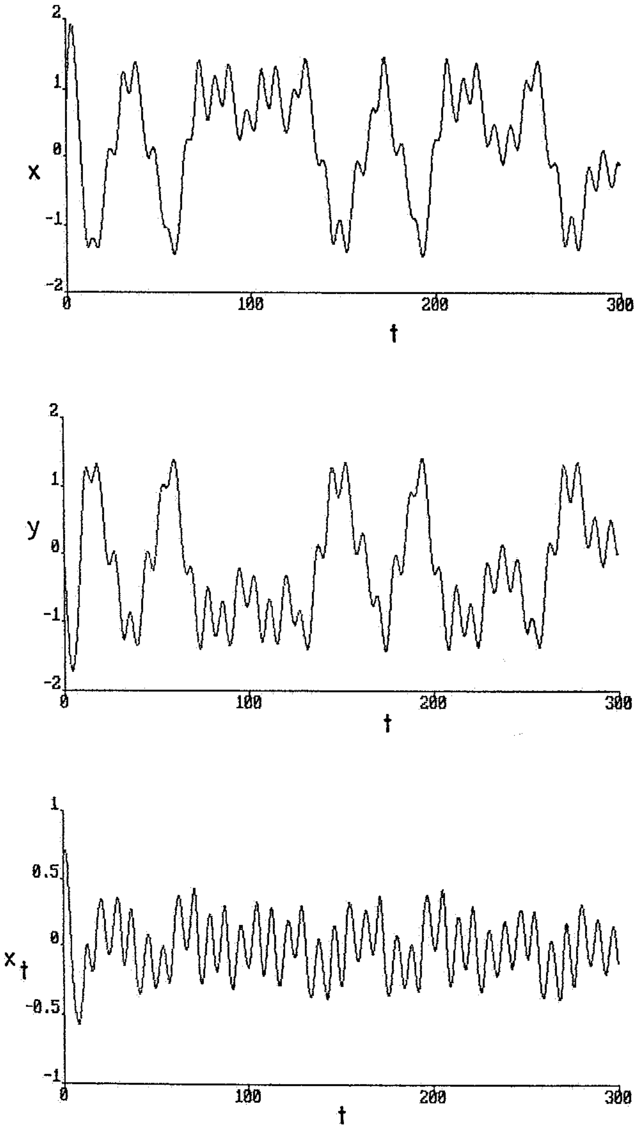
where $s_i = a, b$; $i = 1, 2$. The characteristic equation is

$$\lambda^2 + \mu\lambda + \xi = 0\,, \tag{18}$$

where $\mu = \sigma + s_i + \varepsilon$ and $\xi = s_i\sigma + \varepsilon(\sigma - \beta) + \beta$. If $\mu, \xi > 0$ then $x^* = y^* = 0$ is a stable point and the two systems (14a) and (14b) will synchronize asymptotically. In the present case, the critical value of $\varepsilon$ was calculated to be 2.354, and thus for all $\varepsilon > 2.354$ the two systems (14a) and (14b) will synchronize asymptotically.




(a)




(b)

Fig. 7. (a) A double-band chaotic attractor of Eq. (14a) for $\sigma = 1.015$, $\beta = 1.0$, $F = 0.15$, $\omega = 0.75$, $a = -1.02$ and $b = -0.55$. (b) The chaotic variables $x$, $y$ and $x_t$ of (a).

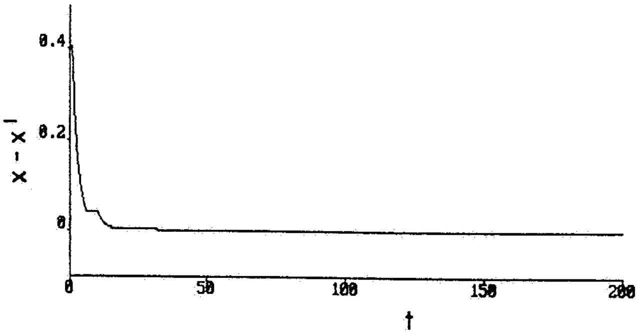



Fig. 8. The difference signal $(x - x')$ of Eqs. (14a) and (14b) for $\varepsilon = 2.4$ indicating synchronization.

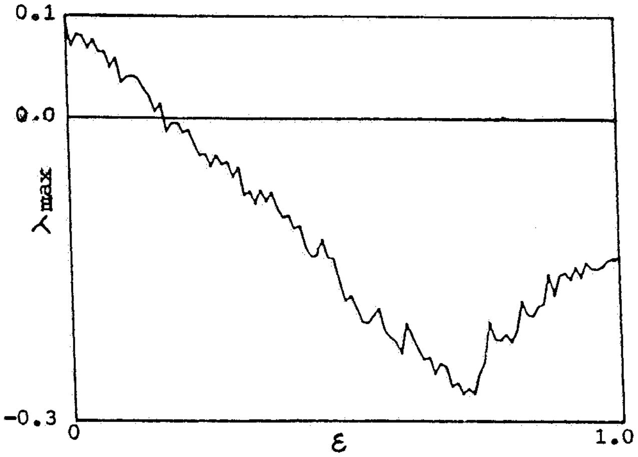



Fig. 9. The maximal conditional Lyapunov exponent $\lambda_{max}$ versus $\varepsilon$ of Eq. (14b).

In order to investigate this phenomenon, we have numerically integrated Eqs. (14a) and (14b) for $\varepsilon = 2.4$ and these two systems exhibit perfect synchronization among their variables, as indicated in Fig. 8, even if these two systems are integrated numerically with different initial conditions. Also, the conditional maximal Lyapunov exponent of Eqs. (14b) under the influence of the one-way coupling parameter is shown in Fig. 9.

## 5. Discussions

In the above examples we have considered the transformation function $u(\cdot)$ in Eq. (8) and Eq. (9) as a linear one but one can consider appropriate nonlinear functions also. As an example, in the following we use a typical nonlinear function $u(y, z) = \sin(y) + \sin(z)$ in Eq. (9d) and $u(y', z') = \sin(y') + \sin(z')$ in Eq. (9e). Then the Eq. (9) is numerically integrated. Figure 10(a) shows the compound chaotic signal $x_t$ of the present case and 10(b)

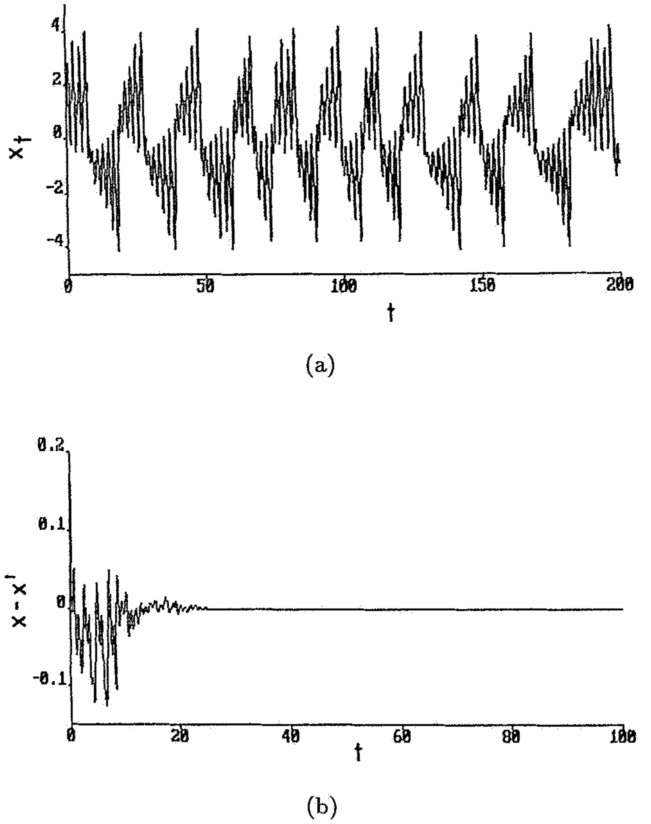


Fig. 10. (a) The compound chaotic signal $x_t$ of Eqs. (9a)–(9d) with $u(y, z) = \sin(y) + \sin(z)$. (b) The difference signal $(x - x')$ of Eqs. (9a)–(9e) for $\delta_x = 1.0$.

shows the difference signal $(x - x')$ versus $t$, indicating perfect synchronization among the variables. We have also tested other forms of function $u(y, z)$ in our numerical simulations, such as $e^y$, $yz$, $y^2z$, $yz^2$, etc. and in all these cases synchronized chaotic behavior has been achieved successfully.

To conclude, in this Letter, we have introduced a procedure of achieving an efficient synchronization using a compound chaotic signal. The procedure has been tested successfully for Chua's circuit and the MLC circuit equations. More importantly, the usage of a compound chaotic signal of more than one chaotic variables to synchronize two chaotic systems can improve the security of a chaos-based secure communication system and the wide choice of the transformation functions $u(\cdot)$ can also act as a key.

## Acknowledgment

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