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ON THE SYNCHRONIZATION OF CHAOTIC SYSTEMS BY USING STATE OBSERVERS

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We show that the synchronization of chaotic systems can be achieved by using the observer design techniques which are widely used in the control of dynamical systems. We prove that local synchronization is possible under relatively mild conditions and global synchronization is possible if the chaotic system has some special structures, or can be transformed into some special forms. We show that some existing synchronization schemes for chaotic systems are related to the proposed observer-based synchronization scheme. We prove that the proposed scheme is robust with respect to noise and parameter mismatch under some mild conditions. We also give some examples including the Lorenz and Rössler systems and Chua's oscillator which are known to exhibit chaotic behavior, and show that in these systems synchronization by using observers is possible.

1. Introduction

The concept of synchronization of chaotic systems may seem somewhat paradoxical since in such systems solutions starting from arbitrary close initial conditions quickly diverge and become uncorrelated. However, it has recently been shown that such synchronization is possible (see e.g. [Pecora & Carroll, 1990; Cuomo & Oppenheim, 1993]). This subject then received a great deal of attention among scientists in many fields (see e.g. [Pecora & Carroll, 1991; Chua *et al.*, 1993a; Ogorzalek, 1993; Kocarev *et al.*, 1992; Cuomo *et al.*, 1993], and the references therein). One of the motivations for synchronization is the possibility of sending messages through chaotic systems for secure communication, see e.g. [Cuomo *et al.*, 1993; Halle *et al.*, 1993; Kocarev *et al.*, 1992]. Such synchronized systems usually consist of two parts: A generator of chaotic signals (drive system) and a receiver (response system). The response system is usually a duplicate of a part (or the whole) of the drive system. A chaotic signal generated by the drive system may be used as an input in the response system to synchronize the common signals of both systems (see e.g. [Pecora & Carroll, 1990]). After the synchronization, one may add the message to the chaotic signal used for synchronization, and under certain conditions one may recover the message from the signals of the response system (see e.g. [Cuomo *et al.*, 1993]). We note that once the chaotic "drive" system is given, most of the synchronization schemes proposed in the literature do not give a systematic procedure to determine the "response" system and the drive signal. Hence most of these schemes depend on the choice of the drive system and could not be easily generalized to an arbitrary chaotic drive system.

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A related problem encountered in the systems and control theory is the estimation of the states of a dynamical system by using another system, called an "observer". The theory of the design of observers, although not fully exploited, is a relatively well-studied branch of system theory and is widely used in the state feedback control of dynamical systems (see e.g. [Kailath, 1980; Wonham, 1985; Callier & Desoer, 1991; Vidyasagar, 1993]). In this paper our aim is to show that this existing theory of observers may naturally be used in the relatively new field of synchronization of chaotic systems. In this approach, once the drive system is given, the response system could be chosen in the observer form, and the drive signal should be chosen accordingly so that the drive system satisfies certain conditions. Under some relatively mild conditions, local or global synchronization of drive and observer systems may be guaranteed. Moreover, the synchronization is achieved exponentially fast. Hence this synchronization scheme offers a systematic procedure, independent of the choice of the drive system. Moreover, the observer proposed in this paper is robust with respect to noise and paramater mismatch. The seemingly counter-intuitive idea of robust synchronization of chaotic systems also becomes quite expected with this approach, since regardless of whether the drive system is chaotic or not, what is important is the error dynamics. By an appropriate choice of the feedback gain, the error dynamics can be made locally or globally exponentially stable under some relatively mild conditions. Robustness is then a consequence of exponential stability.

This paper is organized as follows. In the next section we present some basic material for the design of observers and show that local synchronization is possible under certain conditions, which are not very restrictive. We consider the Lorenz and Rössler systems and show that for these systems, local synchronization may be possible by using the observers. We also show that some of the existing schemes for synchronization (e.g. [Pecora & Carroll, 1990; Cuomo & Oppenheim, 1993]) are related to the observer-based synchronization, and some techniques (e.g. [Murali & Lakshmanan, 1994; Murali *et al.*, 1995]) are exactly the same as the observer-based synchronization. In Sec. 3 we show that the proposed observer is robust with respect to noise and paramater mismatch. In Sec. 4 we consider some special classes of chaotic systems and show that in these cases the proposed observer, or a modified version, may be used to obtain global synchronization results. We also show that some of the chaotic systems (e.g. the Rössler system and Chua's oscillator) can be transformed into this form. In Sec. 5 we present some numerical simulation results and finally we give some concluding remarks.

2. Full Order Observer

We begin with the definition of observability for a linear system, which plays an important role in modern control theory. Consider the following linear system:

$$\dot{u} = Au\,, \qquad y = Cu\,, \tag{1}$$

where $A \in \mathbf{R}^{n\times n}$, $C \in \mathbf{R}^{m\times n}$ are constant matrices, y is called the "output" of the system. The problem of observability is related to the computation of initial condition $u(0) \in \mathbf{R}^n$ by only observing the output $y(\cdot)$ over an interval of time.

Definition. (*Observability*) Consider the system described by (1). Two states u_0 and u_1 are said to be *distinguishable* if $y(t, u_0) \neq y(t, u_1)$ for $t \geq 0$, where $y(t, u_i) = Ce^{At}u_i$, is the output $y(t)$ corresponding to the initial condition $u(0) = u_i$, $i = 0, 1$. The system given by (1) (or in short the pair (C, A)) is said to be observable if all distinct states are distinguishable (see e.g. [Callier & Desoer, 1991; Kailath, 1980; Wonham, 1985; Vidyasagar, 1993]).

We next state the following well-known fact.

Theorem 1. *Consider the system given by (1). Then the following are equivalent:*

(i) *The pair (C, A) is observable.*
(ii) *The following rank condition is satisfied:*

$$\mathrm{rank}\begin{pmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{pmatrix} = n\,. \tag{2}$$

(iii) *The following rank condition is satisfied:*

$$\mathrm{rank}\begin{pmatrix} \lambda I - A \\ C \end{pmatrix} = n\,, \qquad \forall \lambda \in \mathbf{C}\,. \tag{3}$$

(iv) *For any polynomial $p(\lambda) = \lambda^n + a_1\lambda^{n-1} + \cdots + a_{n-1}\lambda + a_n$, $a_i \in \mathbf{R}$, $i = 1, 2, \ldots, n$, there exists a constant matrix $K \in \mathbf{R}^{n\times m}$ such that $\det(\lambda I - A + KC) = p(\lambda)$.*

Proof. See e.g. [Kailath, 1980], p. 80, p. 136 and [Wonham, 1985], p. 61. ■

Consider the nonlinear system given below

$$\dot{u} = A(\mu)u + g(u, \mu) + h(t), \qquad y = Cu, \quad (4)$$

where $\mu \in \mathbf{R}^p$ is a parameter vector, for a fixed $\mu \in \mathbf{R}^p$, $A(\mu) \in \mathbf{R}^{n\times n}$ is a constant matrix, $C \in \mathbf{R}^{m\times n}$ is a constant matrix, $g : \mathbf{R}^n \times \mathbf{R}^p \to \mathbf{R}^n$ is a differentiable function and $h : \mathbf{R} \to \mathbf{R}^n$ is a known forcing function (or input). We assume that the following Lipschitz conditions are satisfied:

$$\|A(\mu_1) - A(\mu_2)\| \le k_1\|\mu_1 - \mu_2\|, \quad \mu_1, \mu_2 \in \mathbf{R}^p, \quad (5)$$

$$\|g(u_1, \mu) - g(u_2, \mu)\| \le k_2\|u_1 - u_2\|, \quad u_1, u_2 \in \mathbf{R}^n, \quad \mu \in \mathbf{R}^p, \quad (6)$$

$$\|g(u, \mu_1) - g(u, \mu_2)\| \le k_3\|\mu_1 - \mu_2\|, \quad \mu_1, \mu_2 \in \mathbf{R}^p, \quad u \in \mathbf{R}^n, \quad (7)$$

for some positive k_1, k_2, k_3 Lipschitz constants. Here, $\|v\|$ represents standard Euclidean norm in $\mathbf{R}^k$ for any positive integer k if $v \in \mathbf{R}^k$ and the induced matrix norm if $v \in \mathbf{R}^{k\times k}$. We note that, since all norms are equivalent in $\mathbf{R}^k$, the norm used in Eqs. (5)–(7) is arbitrary. Nevertheless, we will use the standard Euclidean norm throughout the paper, unless otherwise specified.

Remark 1. In most of the cases, $A(\mu) = \sum_{i=1}^{p} A_i\mu^i$ for some constant matrices $A_i \in \mathbf{R}^{n\times n}$ and $\mu = (\mu^1 \cdots \mu^p)^T$, $i = 1, \ldots, p$, where the superscript T denotes transpose. Hence, in this case Eq. (5) is satisfied for $k_1 \le \max_i\{\|A_i\|\}$. The condition in Eq. (6) might seem to be very restrictive. However, since we consider chaotic systems, the solutions which are of interest to us are bounded. Let the solutions be bounded in a convex and bounded region $B \subset \mathbf{R}^n$ and let the parameter vector μ be bounded in a compact region $M \subset \mathbf{R}^p$. Then we have

$$k_2 \le \sup_{\mu\in M}\sup_{u\in B}\left\|\frac{\partial g}{\partial u}\right\|, \quad k_3 \le \sup_{u\in B}\sup_{\mu\in M}\left\|\frac{\partial g}{\partial \mu}\right\|,$$

where $\frac{\partial g}{\partial u}$ and $\frac{\partial g}{\partial \mu}$ represent the Jacobian of g with respect to the corresponding variables (see e.g. [Marsden, 1974], p. 199). Hence, Eqs. (5)–(7) may be considered as a consequence of differentiability and boundedness of solutions. We note that Eq. (6) is required for the convergence analysis of the observer given below, whereas Eqs. (5) and (7) are required for the robustness analysis of the observer. We also note that the input (or forcing) term $h(t)$ is not important for the convergence analysis, since it cancels in the error equation. We simply included this term so that forced chaotic oscillators can also be put in our framework. We can also consider the feedback control of chaotic systems in this framework.

For the system given by (4), we choose the following "observer" for synchronization:

$$\dot{\hat{u}} = A(\mu)\hat{u} + g(\hat{u}, \mu) + K(y - \hat{y}) + h(t), \quad \hat{y} = C\hat{u}, \quad (8)$$

where $K \in \mathbf{R}^{n\times m}$ is a gain matrix to be determined. In this formulation, Eq. (4) represents the "drive" system, and Eq. (8) represents the "response" system. The output y of the drive system is used in the response system and our aim is to choose the gain K so that the solutions of the Eqs. (4) and (8) asymptotically synchronize, i.e. $\lim_{t\to\infty}\|u(t) - \hat{u}(t)\| = 0$, at least locally. The observer given by Eq. (8) is known as the full order observer (see e.g. [Kailath, 1980]). Let us define the synchronization error as $e = u - \hat{u}$. By using Eqs. (4) and (8) we obtain the following error equation

$$\dot{e} = (A(\mu) - KC)e + g(u, \mu) - g(\hat{u}, \mu). \quad (9)$$

We first state the following well-known result.

Lemma 1. *Consider the systems given by Eqs. (4) and (8). Let the parameter vector μ be fixed, Eq. (6) be satisfied. Let the pair $(C, A(\mu))$ be observable. There exists a K such that if k_2 given by Eq. (6) is sufficiently small, then the error given by Eq. (9) decays exponentially to zero, i.e. the following holds for some $M > 0$ and $\delta > 0$*

$$\|e(t)\| \le Me^{-\delta t}\|e(0)\|. \quad (10)$$

Proof. Since the pair $(C, A(\mu))$ is observable, there exists a K such that $A_c = A(\mu) - KC$ is stable. Hence, the following holds for some $M > 0$ and $\alpha > 0$:

$$\|e^{A_c t}\| \le Me^{-\alpha t}. \quad (11)$$

The solution of Eq. (9) can be given as:

$$e(t) = e^{A_c t}e(0) + \int_0^t e^{A_c(t-\tau)}[g(u(\tau), \mu) - g(\hat{u}(\tau), \mu)]d\tau. \quad (12)$$

By taking norms in Eq. (12), using Eqs. (6), (11) and the Bellman–Gronwal inequality (see e.g. [Callier & Desoer, 1991]), we obtain

$$\|e(t)\| \leq Me^{-(\alpha-Mk_2)t}\|e(0)\|\,. \tag{13}$$

Hence if $\alpha - Mk_2 > 0$, then Eq. (10) is satisfied with $\delta = \alpha - Mk_2$. Moreover, $\delta > 0$ if $0 < k_2 < \alpha/M$. ■

In the application of the observer theory given above, the main difficulty is in the Lipschitz property given by Eq. (6), which should be satisfied globally. But if Eq. (6) is satisfied, then the observer given by Eq. (8) works globally, i.e. for all $e(0) \in \mathbf{R}^n$, provided that $0 < k_2 < \alpha/M$. We may relax this condition as follows, but then Eq. (10) may hold locally, i.e. in a compact region for $e(0)$.

Lemma 2. *Consider the systems given by Eqs. (4) and (8). Let the parameter vector μ be fixed, Eq. (6) be satisfied. Let the pair $(C, A(\mu))$ be observable. Let the function g satisfy the following*

$$\lim_{u\to 0}\left\|\frac{\partial g}{\partial u}\right\| = 0\,, \qquad \mu \in \mathbf{R}^p\,. \tag{14}$$

Then there exists a matrix $K \in \mathbf{R}^{n\times m}$ and a real number $r > 0$ such that Eq. (10) holds if $\|e(0)\| \leq r$ and $\|u(t)\| \leq r$, $\forall t \geq 0$.

Proof. Choose a matrix $K \in \mathbf{R}^{n\times m}$ such that $A_c = A(\mu) - KC$ is stable. Hence A_c satisfies Eq. (11) for some $M > 0$ and $\alpha > 0$. Now choose $R > 0$ so that if $\|u\| < R$, then $k_2 < \alpha/M$. Note that because of Eq. (14), such a $R > 0$ always exists, see Remark 1. Let $\hat{u}(0) \leq r_1$ and $u(t) \leq r_1$, $\forall t \geq 0$ for some $r_1 > 0$. By using Eqs. (6), (8) and the Bellman–Gronwall inequality, (see e.g. [Callier & Desoer, 1991]), it can be proven that if $r_1 > 0$ is sufficiently small, then $\|\hat{u}(t)\| \leq r_2$ for some $r_2 > 0$, $\forall t \geq 0$. Moreover, as $r_1 \to 0$, we have $r_2 \to 0$ as well. Hence, by using standard continuity arguments it then follows that there exists a $r > 0$ satisfying $R > r$ such that if $\|u(t)\| \leq r$ and $\|e(0)\| \leq r$, then we have $\|\hat{u}(t)\| \leq R$, hence the Lipschitz constant k_2 given by Eq. (6) remains valid $\forall t \geq 0$. It then follows that (13) holds $\forall t \geq 0$. ■

Remark 2. Note that the condition given by Eq. (14) is less stringent than the Lipschitz condition Eq. (6). In applications, the differential equation given by Eq. (4) is obtained by linearization of a nonlinear system around an equilibrium point. In such cases, the function g necessarily contains at least second order terms, hence Eq. (14) is automatically satisfied. We also show later that this condition is satisfied for the Lorenz and the Rössler systems.

Remark 3. For a given pair $(C, A(\mu))$, for the observer given by Eq. (8), the results of Lemma 1 and 2 hold when the matrix $A_c = A(\mu) - KC$ is stable. For observable pairs, by Theorem 1 there always exists a matrix K such that A_c is stable. For some pairs $(C, A(\mu))$ there may exist a matrix K such that A_c is stable, even if the pair is not observable. Such pairs are called "detectable" (see e.g. [Wonham, 1985]), and for such pairs the observer given by Eq. (8) could still be used, and the results of Lemmas 1 and 2 still hold. We will show later that for the Lorenz system, this detectability condition is satisfied, hence the observer given by Eq. (8) could be used for synchronization.

At this point we compare the proposed observer given above with some proposed synchronization schemes in the literature.

Example 1. (Lorenz system) Consider the Lorenz system given below:

$$\begin{aligned} \dot{x}_1 &= \sigma(x_2 - x_1)\,,\\ \dot{x}_2 &= -x_1x_3 + rx_1 - x_2\,,\\ \dot{x}_3 &= x_1x_2 - bx_3\,. \end{aligned} \tag{15}$$

The parameters $\sigma > 0$, $r > 0$ and $b > 0$ are chosen so that the system exhibits chaotic behavior.

We may write Eq. (15) in the form given by Eq. (4) where $u = (x_1\ x_2\ x_3)^T$,

$$A = \begin{pmatrix} -\sigma & \sigma & 0\\ r & -1 & 0\\ 0 & 0 & -b \end{pmatrix}, \qquad g(u) = \begin{pmatrix} 0\\ -x_1x_3\\ x_1x_2 \end{pmatrix}. \tag{16}$$

It follows easily that the selection of $y = c_1x_1 + c_2x_3$, (i.e. $C = (c_1\ 0\ c_2)$), or $y = c_1x_2 + c_2x_3$, (i.e. $C = (0\ c_1\ c_2)$), yields the pair (C, A) observable for almost all values of c_1 and c_2, provided that $c_1 \neq 0$, $c_2 \neq 0$. For actual values, Eq. (2) should be checked. For $C = (c_1\ c_2\ 0)$ the pair (C, A) is not observable but detectable, i.e. one can easily find matrices of the form $K = (k_1\ k_2\ 0)^T$ such that $A - KC$ is stable. In particular, the selection of $y = x_1$, (i.e. $C = (1\ 0\ 0)$), or $y = x_2$, (i.e. $C = (0\ 1\ 0)$)

makes the pair $(C,\ A)$ detectable, hence by an appropriate choice of K, one can obtain a stable matrix $A - KC$ and use the observer given by Eq. (8) for synchronization of chaos.

At this point we compare the observer given by Eq. (8) with some synchronization schemes proposed in [Pecora & Carroll, 1991; Cuomo & Oppenheim, 1993]. Consider the following system:

$$\dot{\hat{x}}_1 = \sigma(\hat{x}_2 - \hat{x}_1)\,, \tag{17}$$

$$\dot{\hat{x}}_2 = -x_1\hat{x}_3 + rx_1 - \hat{x}_2\,, \tag{18}$$

$$\dot{\hat{x}}_3 = x_1\hat{x}_2 - b\hat{x}_3\,, \tag{19}$$

In [Pecora & Carroll, 1991], Eqs. (18) and (19) are called the response system and in [Cuomo & Oppenheim, 1993], Eqs. (17)–(19) are called the response system, for the drive system given by Eq. (15). Note that here x_1 is used as the drive signal, hence according to our observer design technique, the output of Eq. (15) is $y = x_1$. By using Lyapunov theory, it can be shown that $\lim_{t\to\infty} \|u(t) - \hat{u}(t)\| = 0$, where $u = (x_1\ x_2\ x_3)^T$ and $\hat{u} = (\hat{x}_1\ \hat{x}_2\ \hat{x}_3)^T$ (see [Pecora & Carroll, 1991; Cuomo & Oppenheim, 1993]. Note that Eqs. (17)–(19) could be written in the form

$$\dot{\hat{u}} = A\hat{u} + g(\hat{u}) + KC(u - \hat{u}) + F(\hat{u})C(u - \hat{u})\,, \tag{20}$$

where A and g are given in Eq. (16), $K = (0\ r\ 0)^T$, $C = (1\ 0\ 0)$ and $F(\hat{u}) = (0\ \ -\hat{x}_3\ \hat{x}_2)^T$. Note that $A - KC$ is a stable matrix with this choice. Hence, the response system given by Eqs. (17)–(19), and hence Eq. (20), are similar to the observer given by Eq. (8) except for the last term in Eq. (20). Without this term, Lemma 1 and 2 may be used for local synchronization. However, due to the special structure of this term, one can prove global (exponential) convergence of the error. We note that in this case, the synchronization error decays exponentially to zero, i.e. Eq. (10) is satisfied.

Example 2. (One way coupling) Another synchronization scheme proposed in the literature is the so-called one-way coupling considered in [Murali and Lakshmanan, 1994; Murali *et al.*, 1995]. Here we show that the synchronization scheme proposed in these references is exactly the same as the observer-based synchronization proposed in this paper. In both references, same technique is applied to different chaotic systems. We consider the system used in [Murali *et al.*, 1995] because of its simplicity. In [Murali *et al.*, 1995], a simple second order forced chaotic oscillator and the associated response systems given by the following equations are considered:

$$\dot{x}_1 = x_2 - g_n(x_1)\,, \tag{21}$$

$$\dot{x}_2 = -\beta(1+\nu)x_2 - \beta x_1 + F\ \sin \omega t\,, \tag{22}$$

$$\dot{\hat{x}}_1 = \hat{x}_2 - g_n(\hat{x}_2) + \varepsilon(x_1 - \hat{x}_1)\,, \tag{23}$$

$$\dot{\hat{x}}_2 = -\beta(1+\nu)\hat{x}_2 - \beta\hat{x}_1 + F\ \sin \omega t\,, \tag{24}$$

where instead of the notation of [Murali *et al.*, 1995], we used our notation in order not to cause confusion, (see Eqs. (3a) and (3b) in the cited reference). Here, Eqs. (21)–(22) represent the drive system and Eqs. (23)–(24) represent the response system. Equations (21) and (22) can be put into the form given by Eq. (4) with $u = (x_1\ x_2)^T$,

$$A = \begin{pmatrix} 0 & 1 \\ -\beta & -\beta(1+\nu) \end{pmatrix}\,, \quad g(u) = \begin{pmatrix} -g_n(x_1) \\ 0 \end{pmatrix}\,,$$

$$h(t) = \begin{pmatrix} 0 \\ F\ \sin \omega t \end{pmatrix}\,.$$

Since the signal x_1 used in the response system, according to our formulation we have $y = x_1$, i.e. $C = (1\ 0)$. Note that the pair $(C,\ A)$ is always observable in this case. By direct comparison, it follows easily that the response system given by Eqs. (23) and (24) has the same form given by Eq. (8) with $K = (\varepsilon\ 0)^T$. It can easily be shown that $A - KC$ is a stable matrix if all the coefficients ε, β, ν are positive, hence Lemmas 1 and 2, whichever appropriate, could be applied. This shows that the one-way coupling as used in this reference is just an electronic circuit implementation of the observer given by Eq. (8), and the role of the buffer op-amp and coupling resistor R_c in [Murali *et al.*, 1995] and similarly in [Murali & Lakshmanan, 1994] is just to implement the injection term $K(y - \hat{y})$ in Eq. (8).

3. Robustness with Respect to Noise and Parameter Mismatch

In this section we show that the observer given by Eq. (8) is robust with respect to noise and parameter mismatch. From Lemmas 1 and 2, it follows that for the error dynamics given by Eq. (9), the equilibrium point $e = 0$ is exponentially stable. Since exponentially stable systems are robust with respect to perturbations in the dynamics (see e.g. [Khalil, 1992]), we expect that the observer given by Eq. (8) is also robust.

We assume that the output y given in Eq. (4) is corrupted by noise, the parameter vector μ and the forcing term $h(t)$ in the observer given by Eq. (8) are slightly different than the ones used in the drive system given by Eq. (4). Hence, the observer equation in this nonideal case becomes:

$$\dot{\hat{u}} = A(\mu')\hat{u} + g(\hat{u}, \mu') + K(y + n(t) - \hat{y}) + h'(t), \quad \hat{y} = C\hat{u}. \tag{25}$$

Then by using Eqs. (4) and (25), after some algebraic but simple arrangements we obtain the following error dynamics:

$$\dot{e} = (A(\mu) - KC)e + [g(u, \mu) - g(\hat{u}, \mu)] + [g(\hat{u}, \mu) - g(\hat{u}, \mu')] + [A(\mu) - A(\mu')]\hat{u} + [h(t) - h'(t)] - Kn(t) \tag{26}$$

Note that, in Eq. (26), the dynamics with only the first two terms in the right hand side represents an exponentially stable system, (i.e. the case where $\mu = \mu'$, $n(t) = 0$ and $h(t) = h'(t)$). Hence, the last four terms may be considered as a perturbation to an exponentially stable dynamical system. We assume that the difference between the forcing terms is also bounded as follows:

$$\|h(t) - h'(t)\| \le h_M, \qquad \forall t \ge 0, \tag{27}$$

for some $h_M > 0$.

Theorem 2. *Consider the systems given by Eqs. (4) and (25). Let Eqs. (5)–(7) and (27) be satisfied. Let the noise and the solutions of (25) be bounded as $\|n(t)\| \le n_M$ and $\|\hat{u}(t)\| \le u_M$ for some $n_M > 0$ and $u_M > 0$, $\forall t \ge 0$. Let the pair $(C, A(\mu))$ be observable. If k_2 given by Eq. (6) is sufficiently small, then there exists a K such that the error given by Eq. (26) satisfies the following inequality:*

$$\|e(t)\| \le An_M + B\|\mu - \mu'\| + Ch_M + De^{-\delta t}, \quad \forall t \ge 0, \tag{28}$$

for some $A > 0$, $B > 0$, $C > 0$, $\delta > 0$ and $D \in \mathbf{R}$. Moreover, A, B, C and δ do not depend on $n(\cdot)$, $h(\cdot)$, μ and μ'.

Proof. Choose a matrix K so that $A_c = A(\mu) - KC$ is stable, hence Eq. (11) is satisfied. Assume that k_2 is sufficiently small so that $k_2 < \alpha/M$. The solution of Eq. (26) can be given as

$$e(t) = e^{A_c t}e(0) + \int_0^t e^{A_c(t-\tau)}[g(u(\tau), \mu) - g(\hat{u}(\tau), \mu)]d\tau + \int_0^t e^{A_c(t-\tau)}[g(\hat{u}(\tau), \mu) - g(\hat{u}(\tau), \mu')]d\tau + \int_0^t e^{A_c(t-\tau)}[A(\mu) - A(\mu')]\hat{u}(\tau)d\tau + \int_0^t e^{A_c(t-\tau)}[h(\tau) - h'(\tau)]d\tau - \int_0^t e^{A_c(t-\tau)}Kn(\tau)d\tau. \tag{29}$$

By taking norms in Eq. (29), using Eq. (11) and by simple integration we obtain:

$$\|e^{\alpha t}e(t)\| \le M\|e(0)\| + \frac{M}{\alpha}[k_1 u_M\|\Delta\mu\| + k_3\|\Delta\mu\| + \|K\|n_M + h_M](e^{\alpha t} - 1) + \int_0^t Mk_2\|e^{\alpha\tau}e(\tau)\|d\tau, \tag{30}$$

where $\Delta\mu = \mu - \mu'$. For simplicity, let us define

$$A_1 = M\|e(0)\|, \quad B_1 = \frac{M}{\alpha}[k_1 u_M\|\Delta\mu\| + k_3\|\Delta\mu\| + \|K\|n_M + h_M]. \tag{31}$$

Then, by using a generalized form of Bellman–Gronwal inequality [Callier & Desoer, 1991], we obtain

$$\|e^{\alpha t}e(t)\| \le A_1 + B_1(e^{\alpha t} - 1) + \int_0^t Mk_2(A_1 + B_1(e^{\alpha\tau} - 1))e^{Mk_2(t-\tau)}d\tau. \tag{32}$$

After simple integration and multiplication by $e^{-\alpha t}$ we obtain

$$\|e(t)\| \le \frac{\alpha B_1}{\alpha - Mk_2} + \left[A_1 - \frac{\alpha B_1}{\alpha - Mk_2}\right]e^{-(\alpha - Mk_2)t}, \quad \forall t \ge 0. \tag{33}$$

By comparing (28) and (33) we see that the former is satisfied with $\delta = \alpha - Mk_2 > 0$ and

$$A = \frac{M\|K\|}{\alpha - Mk_2}, \qquad B = \frac{M(k_1 u_M + k_3)}{\alpha - Mk_2},$$

$$C = \frac{M}{\alpha - Mk_2}, \qquad D = A_1 - \frac{\alpha B_1}{\alpha - Mk_2},$$

where A_1 and B_1 are given by Eq. (31). ■

Remark 4. From (28), (31) and (33) it follows that the effect of the initial error $e(0)$ appears in (28) only in the term $De^{-\delta t}$ and since $\delta > 0$, it follows that after sufficiently large time, this effect becomes negligible. Therefore, asymptotically, (i.e. for $t \geq T$, where $T > 0$ is sufficiently large), we may assume that only the first three terms are effective in (28), (i.e. the terms multiplying A, B and C). Note that the first three terms depend linearly on $\Delta\mu$, n_M and h_M. Hence, for small $\Delta\mu$, n_M and h_M, asymptotically the error bound will also be small. Theorem 2 requires that the Lipschitz constant k_2 be sufficiently small (i.e. $k_2 < \alpha/M$), but then the result holds globally, i.e. for all $e(0)$. On the other hand, if Eq. (14) holds, then a similar result may hold locally, see Lemma 2.

4. Global Results for Some Special Cases

In this section we will apply the observer theory given in the previous section to some class of systems and show that global synchronization results may hold for the considered classes. For simplicity, we suppress the dependence on the parameter vector μ throughout this section.

4.1. *Systems in Lur'e form*

We consider the class of systems having the structure shown in Fig. 1. Here $L(s)$ represents the transfer function of a linear time-invariant system and $n(\cdot) : \mathbf{R} \to \mathbf{R}$ represents a memoryless nonlinearity. This class of systems are called Lur'e type systems and have been investigated by many researchers (see e.g. [Vidyasagar, 1993]). It has also been shown that a lot of systems in this type exhibit chaotic behavior (see e.g. [Brockett, 1982; Cook, 1986; Amrani & Atherton, 1989; Genesio & Tesi, 1992]). Recently, in [Genesio & Tesi, 1992] a conjecture for the existence of chaotic behavior in this type of systems has been given.

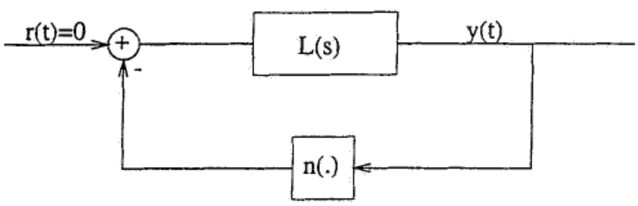

Fig. 1. A basic feedback system (Lur'e system).

Here we show that such a chaotic system can always be synchronized globally under some mild conditions by modifying the observer given by Eq. (8). To see that, assume that $L(s)$ is a strictly proper transfer function (i.e. $\lim_{|s|\to\infty} |L(s)| = 0$), and let (A, B, C) be an observable realization of $L(s)$, i.e. $L(s) = C(sI - A)^{-1}B$. Such a realization is always possible (see e.g. [Kailath, 1980]). Hence, the system in Fig. 1 can be given by the following equations:

$$\dot{u} = Au - Bn(y)\,, \qquad y = Cu\,. \tag{34}$$

For this system, we choose the following observer:

$$\dot{\hat{u}} = A\hat{u} - Bn(y) + K(y - \hat{y})\,, \qquad \hat{y} = C\hat{u}\,. \tag{35}$$

where $K \in \mathbf{R}^n$ is chosen such that $A_c = A - KC$ is stable. Since the pair (C, A) is observable, this is always possible. From the Eqs. (34) and (35), we obtain the following error equation:

$$\dot{e} = A_c e\,.$$

Hence Eq. (10) holds globally. Note that here y is the measurable output of the drive system given by Eq. (34). The nonlinearity in Eq. (4) is in the form $g(u) = -Bn(y)$ in Eq. (34), hence can be constructed from y. This is the rationale in using $-Bn(y)$ in Eq. (35), instead of using $-Bn(\hat{y})$ [cf. Eqs. (8) and (35)]. Note that the error equation given above holds independent of the nonlinearity $n(\cdot)$, hence the systems given by Eqs. (34) and (35) synchronize exponentially fast, even if $n(\cdot)$ is not Lipschitz (e.g. hysteresis-type nonlinearity). However, for the robustness of the observer, we still require Eqs. (5)–(7). The results of the Theorem 2 hold, provided that the Eqs. (5)–(7) are satisfied. This could easily be proven by using the steps in the proof of Theorem 2. ■

4.2. *Systems in Brunowsky canonical form*

In some cases, the local convergence result of the Lemma 1 could be extended to global convergence result, provided that the chaotic system given by Eq. (4) has a special form. Assume that the system is in the form Eq. (4) with

$$A=\begin{pmatrix}0 & 1 & 0 & \cdots & 0\\ 0 & 0 & 1 & \cdots & 0\\ & & \vdots & & \\ 0 & 0 & 0 & \cdots & 1\\ 0 & 0 & 0 & \cdots & 0\end{pmatrix},\quad g(u)=\begin{pmatrix}0\\ \vdots\\ 0\\ 1\end{pmatrix} f(u),$$

$$C=(1 \quad 0 \quad \cdots \quad 0), \tag{36}$$

where $f : \mathbf{R}^n \to \mathbf{R}$ is a differentiable function and that g satisfies the Lipschitz property given by Eq. (6). The form given by Eq. (36) is called the Brunowsky canonical form, and is frequently used in the control of nonlinear systems (see e.g. [Vidyasagar, 1993; Ciccarella *et al.*, 1993]). Since the pair (C, A) is observable and g is Lipschitz, the observer given by Eq. (8) could be used for global convergence of error, provided that $k_2 < \alpha/M$, see Lemma 1. However, it was shown in [Ciccarella *et al.*, 1993] that for any $k_2 > 0$, one can find a feedback matrix K, such that Eq. (10) is satisfied when the system is in Brunowsky canonical form. Obviously this result still holds if the system can be transformed into Brunowsky canonical form by means of a diffeomorphic coordinate transformation. The details can be found in [Ciccarella *et al.*, 1993]. Here we give a procedure to select the desired K, different than the one considered in [Ciccarella *et al.*, 1993].

For the design of the observer, choose $\lambda_1 < 0$ and $\lambda_2 = \gamma\lambda_1$, $\lambda_3 = \gamma^2\lambda_1, \ldots, \lambda_n = \gamma^{n-1}\lambda_1$, where $\gamma > 1$. Consider the following Vandermonde matrix:

$$V=\begin{pmatrix}\lambda_1^{n-1} & \lambda_1^{n-2} & \cdots & 1\\ \lambda_2^{n-1} & \lambda_2^{n-2} & \cdots & 1\\ \vdots & & & \\ \lambda_n^{n-1} & \lambda_n^{n-2} & \cdots & 1\end{pmatrix}. \tag{37}$$

It can easily be shown that the feedback matrix $K = (k_1\ k_2\ \cdots\ k_n)^T$ can be appropriately chosen so that

$$A_c = A - KC = V^{-1}\Lambda V\,, \tag{38}$$

is satisfied, where $\Lambda = \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$. Now consider the error equation given by Eq. (9), whose solution can be written as follows

$$e(t) = V^{-1}e^{\Lambda t}Ve(0) + V^{-1}\int_0^t e^{\Lambda(t-\tau)}VB[f(u(\tau)) - f(\hat{u}(\tau))]d\tau\,, \tag{39}$$

where $B = (0\ 0\ \cdots\ 1)^T$ and $e^{\Lambda t} = \text{diag}(e^{\lambda_1 t}, e^{\lambda_2 t}, \ldots, e^{\lambda_n t})$. By taking the max. norm $\|\cdot\|_\infty$ (see e.g. [Vidyasagar, 1993]), we obtain

$$\|e(t)\|_\infty \le \|V^{-1}\|_\infty\|V\|_\infty e^{\lambda_1 t}\|e(0)\|_\infty + \|V^{-1}\|_\infty\int_0^t e^{\lambda_1(t-\tau)}k_2\|e(\tau)\|_\infty d\tau\,, \tag{40}$$

where we now assumed that Eq. (6) is satisfied with the max. norm. Note that since in $\mathbf{R}^n$ all norms are equivalent, this only affects the Lipschitz constant $k_2 > 0$. Also, in Eq. (40), we used the matrix norm induced by the max. norm. By multiplying both sides of Eq. (40) by $e^{-\lambda_1 t}$, using the Bellman–Gronwall Lemma, we obtain

$$\|e(t)\|_\infty \le \|V^{-1}\|_\infty\|V\|_\infty e^{(\lambda_1+k_2\|V^{-1}\|_\infty)t}\|e(0)\|_\infty\,. \tag{41}$$

Now simple calculation shows that $\|V^{-1}\|_\infty = G(\gamma)$ for some rational function $G(\cdot)$, provided that γ and $|\lambda_1|$ are sufficiently large. Obviously once $\gamma > 1$ is chosen sufficiently big, then for any $\alpha > 0$ and $k_2 > 0$, one can choose λ_1 so that $\lambda_1 + k_2\|V^{-1}\|_\infty \le -\alpha$. Hence Eq. (10) is satisfied with $M = \|V^{-1}\|_\infty\|V\|_\infty$ and α given by the inequality stated above.

Note that some chaotic systems are already in the form given by Eqs. (4) and (36), see e.g. [Tesi *et al.*, 1994; Alexeyev & Shalfeev, 1995], hence the theory presented above can be directly applied for such systems. Some systems may be transformed into this form by a coordinate transformation $z = T(u)$ where $T : \mathbf{R}^n \to \mathbf{R}^n$ is a diffeomorphism. The details of finding such a transformation may be found in [Ciccarella, 1993]. Here we emphasize that for some systems this transformation may be linear, i.e. $T(u) = Tu$ for some invertible matrix $T \in \mathbf{R}^{n\times n}$, hence the required transformation is quite simple. Now assume that the matrix A given in Eq. (4) is in the following form:

$$A = \begin{pmatrix} * & \alpha_1 & 0 & 0 & \cdots & 0 \\ * & * & \alpha_2 & 0 & \cdots & 0 \\ & & & \vdots & & \\ * & * & * & * & \cdots & \alpha_{n-1} \\ * & * & * & * & \cdots & * \end{pmatrix}, \quad (42)$$

where the entries given by $*$ are arbitrary, and $\alpha_i \neq 0$ for $i = 1, 2, \ldots, n-1$. We also assume that g has the form given in Eq. (36). Under these conditions there exists a linear and invertible transformation $T \in \mathbf{R}^{n\times n}$ such that after the transformation $z = Tu$, in the transformed variables the system is given in the form Eqs. (4) and (36). We note that in this case the required transformation has the form:

$$T = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \\ * & \alpha_1 & 0 & 0 & \cdots & 0 \\ * & * & \alpha_1\alpha_2 & 0 & \cdots & 0 \\ & & & \vdots & & \\ * & * & * & * & \cdots & \alpha_1\alpha_2\cdots\alpha_{n-1} \end{pmatrix}, \quad (43)$$

hence is always invertible. Hence, after this state transformation, by using the procedure given above global synchronization can be achieved. We summarize these results in the following Corollary.

Corollary 1. *Consider the system given by Eqs. (4) and (36), where the matrix A is either in the form given by Eq. (36) or by Eq. (42). Let the function f in Eq. (36) satisfy the following Lipschitz condition for some* $k > 0$:

$$\|f(u_1) - f(u_2)\| \leq k\|u_1 - u_2\|, \quad u_1, u_2 \in \mathbf{R}^n.$$

Then there exists a K such that the systems given by Eqs. (4) and (8) synchronize globally and exponentially fast, i.e. Eq. (10) holds.

Example 3. (Rössler system) Consider the Rössler system given below:

$$\begin{aligned} \dot{x}_1 &= x_2 + ax_1, \\ \dot{x}_2 &= -x_1 - x_3, \\ \dot{x}_3 &= b - cx_3 + x_2x_3, \end{aligned} \quad (44)$$

where the parameters $a > 0$, $b > 0$ and $c > 0$ are chosen so that the system exhibits chaotic motion. This system may be written in the form given by Eq. (4) where $u = (x_1\ x_2\ x_3)^T$,

$$A = \begin{pmatrix} a & 1 & 0 \\ -1 & 0 & -1 \\ 0 & 0 & -c \end{pmatrix}, \quad g(u) = \begin{pmatrix} 0 \\ 0 \\ b + x_2x_3 \end{pmatrix}.$$

It can easily be shown that the selection of $y = c_1x_1 + c_2x_2 + c_3x_3$, (i.e. $C = (c_1\ c_2\ c_3)$) yields the pair (C, A) observable for almost all c_1, c_2 and c_3, provided that $|c_1| + |c_2| \neq 0$. For actual values, Eq. (2) should be checked. In particular, with the selection of $y = x_1$ or $y = x_2$, the corresponding pairs (C, A) are observable, hence by choosing the feedback matrix K appropriately, the observer given by Eq. (8) may achieve local synchronization. Note that with the selection of $y = x_3$, the corresponding pair (C, A) is not even detectable, hence the observer given by Eq. (8) could not be used for synchronization with this output.

Consider the Rössler system given above. Note that A given above is in the form given by Eq. (42). By choosing the transformation:

$$\begin{aligned} z_1 &= x_1, \\ z_2 &= ax_1 + x_2, \\ z_3 &= (a^2 - 1)x_1 + ax_2 - x_3, \end{aligned}$$

the Rössler system can be transformed into the form given by Eq. (4) and Eq. (36), where

$$\begin{aligned} f(z) = &-cz_1 + (ca - 1)z_2 + (a - c)z_3 - az_1^2 - az_2^2 \\ &+ (a^2 - 1)z_1z_2 - az_1z_3 + z_2z_3 - b. \end{aligned}$$

Since the function f given above is differentiable, it follows that the Lipschitz condition Eq. (6) is satisfied in any compact region. Since the Rössler system exhibits chaotic behavior for certain values of the parameters a, b and c, these chaotic solutions are bounded by a compact region, and in this region Eq. (6) is satisfied for some $k_2 > 0$. An estimate of k_2 can be found by using the Jacobian matrix, see Remark 1. Hence by using the technique presented above, an observer for which the synchronization error satisfies Eq. (10) globally, can be designed.

Example 4. (Chua's oscillator) We consider the well-known Chua's oscillator. This circuit is well studied and is known to exhibit many forms of chaotic behavior (see [Kennedy, 1993; Chua *et al.*, 1993b]) and the references therein. The state equations for this circuit are given as

$$\begin{aligned}\dot{x}_1 &= -\frac{R_0}{L}x_1 - \frac{1}{L}x_2\,,\\ \dot{x}_2 &= \frac{1}{C_2}x_1 - \frac{G}{C_2}x_2 + \frac{G}{C_2}x_3\,,\\ \dot{x}_3 &= \frac{G}{C_1}x_2 - \frac{G}{C_1}x_3 - \frac{1}{C_1}f(x_3)\,,\end{aligned} \tag{45}$$

where $x_1 = i_3$, $x_2 = v_2$, $x_3 = v_1$, $G = \frac{1}{R}$, see [Chua *et al.*, 1993b]. The nonlinear resistor (i.e. $f(x_3)$) is given by the characteristics $i_R = f(v_R)$ where the nonlinear function $f : \mathbf{R} \to \mathbf{R}$ is a three segment piecewise linear function given as, (note that $v_R = x_3$), $f(x_3) = G_2x_3 + 0.5(G_1 - G_2)(|x_3 + E| - |x_3 - E|)$ and $G_1 < 0$, $G_2 < 0$, $E > 0$ are some constants, for details see e.g. [Chua *et al.*, 1993b]. For this system, we consider three possible outputs, $y = x_1$, $y = x_2$ and $y = x_3$, separately.

1. Assume that $y = x_1$ is used for synchronization signal, (i.e. $C = (1\ 0\ 0)$). Note that the equations given above are in the form given by Eq. (4), where $g(u) = -(0\ 0\ 1)^T \frac{1}{C_1} f(x_3)$. Note that g is also in the form given by Eq. (36) and satisfies Eq. (6) globally; in fact $k_2 = \frac{1}{C_1}\max\{|G_1|, |G_2|\}$. Since the matrix A for this system has the form given in Eq. (42), by a linear transformation $z = Tu$, these equations can be transformed into the form given by Eq. (4) and Eq. (36). Since g satisfies Eq. (6) globally, by using the technique given above, the synchronization can be achieved globally, i.e. Eq. (10) holds for any initial error $e(0)$.

2. If $y = x_2$, (i.e. $C = (0\ 1\ 0)$), then the condition in Eq. (2) implies that for this case the pair (C, A) is observable if $GL \neq R_0C_1$, and under this condition, the observer given by Eq. (8) may be used for the local synchronization. Since the nonlinearity is piecewise linear, Eq. (14) is not satisfied, hence Lemma 2 cannot be used directly. But by exploiting the structure of the equations and the nonlinearity, the observer given by Eq. (8) could be used. To see this, note that by using the linear part of the nonlinearity $f(x_3)$, the last equation of the Chua's circuit given above can be rewritten as:

$$\dot{x}_3 = \frac{G}{C_1}x_2 - \frac{G + G_1}{C_1}x_3 - \frac{1}{C_1}f_n(x_3)\,,$$

where $f_n(x_3) = f(x_3) - G_1x_3$. With this new formulation, the new pair (C, A) is observable if $(G + G_1)L \neq R_0C_1$. Moreover, since $G + G_1 < 0$, this condition is always satisfied with $L > 0$, $C_1 > 0$ and $R_0 \geq 0$. The matrix A becomes unstable in this case and we have $f_n(x_3) = 0$ when $|x_3| < E$. Hence Eq. (14) is satisfied locally with $k_2 = 0$ in (6). Hence by choosing a feedback gain matrix K such that $A - KC$ is stable, which is always possible in this case, the observer given by Eq. (8) may be used for local synchronization, and Lemma 2 may be used.

3. If $y = x_3$ is chosen as the output, (i.e. $C = (0\ 0\ 1)$), then the Chua's oscillator can be transformed into Lur'e form (see [Genesio & Tesi, 1992]). Hence, after reorganization of the equations, the system can be put into the form given by Eq. (34). It follows from Eq. (2) that the pair (C, A) is observable for $G \neq 0$, hence the observer given by Eq. (35) can be used for global synchronization. Note that in this case A is a stable matrix, hence we could even choose $K = 0$. We note that this is the synchronization scheme used in [Wu & Chua, 1993].

4.3. *Forced oscillators*

Consider the systems given by the following equation:

$$x^{(n)} + F(x, \dot{x}, \ldots, x^{(n-1)}) = h(t)\,, \tag{46}$$

where $x^{(i)}$ represents the ith time derivative of x, $h(t)$ is a scalar forcing function. We assume that F is a differentiable function of its arguments. This class of systems covers a wide range of chaotic forced oscillators such as Duffing equation, Van der Pol oscillator, etc. By using the standard variables $x_1 = x$, $x_2 = \dot{x}, \ldots, x_i = x^{(i-1)}$, $i = 1, \ldots, n$, the system given by Eq. (46) can be transformed into the Brunowsky canonical form given by Eqs. (4) and (40). Hence by the techniques presented above, global synchronization may be possible when we choose $y = x_1$ as output.

5. Simulation Results

In this section we present some numerical simulations of various chaotic systems.

In the first set of simulations, we considered the chaotic system introduced in [Brockett, 1982]. The system is in the Lur'e form in Fig. 1 with

$$L(s) = \frac{1}{s^3 + s^2 + 1.25s},$$

$$n(y) = \begin{cases} -ky & |y| \le 1 \\ 2ky - 3k\ sgn\ y & 1 \le |y| \le 3 \\ 3k\ sgn\ y & |y| \ge 3 \end{cases},$$

with $k = 1.8$. Here, *sgn* denotes the signum function. This system can be put into the form given by Eq. (34) with

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -1.25 & -1 \end{pmatrix},\ B = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},\ C = (1\ 0\ 0).$$

Here we have $y = x_1$ and $f(y) = n(y)$. Note that the matrix A is not stable since $\lambda = 0$ is an eigenvalue, hence the feedback term K is necessary for synchronization. Note that the pair (C, A) is observable. For the observer, we first considered the observer given by Eq. (35) with the feedback gain $K = (2.9\ 0.55\ \ -2.375)^T$. The simulation results are shown in Fig. 2. We also considered the observer given by Eq. (8) with the same feedback gain K, and the results are given in Fig. 3. In both simulations, we used the initial conditions in the drive system as $(1\ -1\ -0.1)^T$ and in the observer as $(-2\ -2\ 1)^T$. Note that the initial error is $\|e(0)\| = 4.383$ in this case, which is not particularly small. This

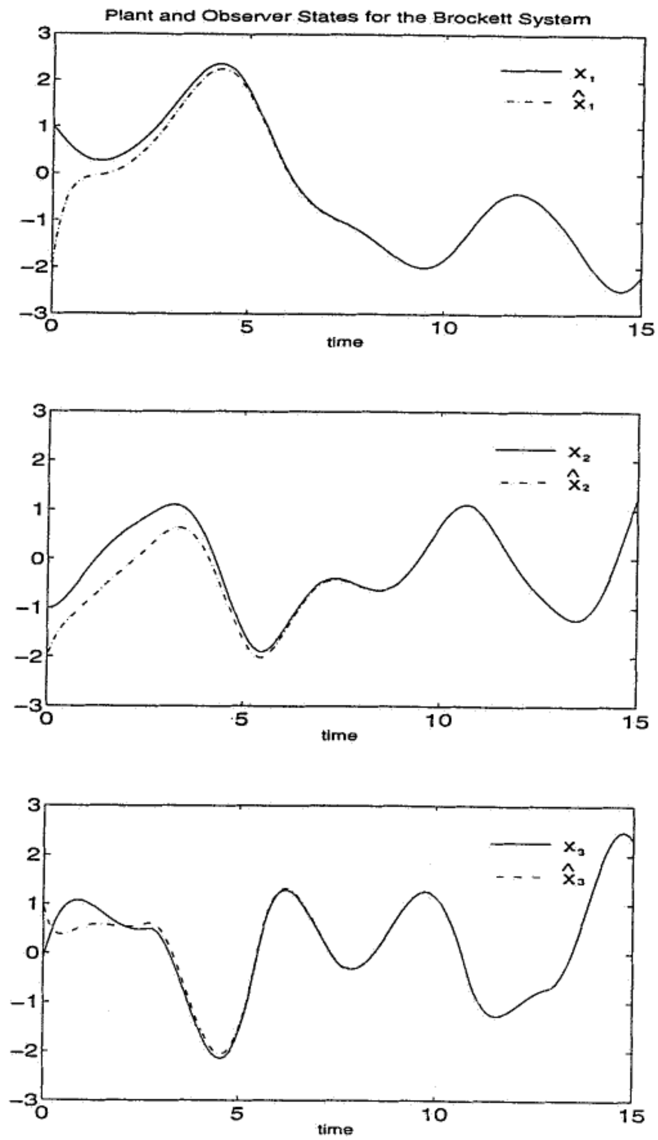

Fig. 2. Synchronization in the Brockett system. The output of the nonlinearity is calculated using the system output. Gains are $k_1 = \frac{29}{10}$, $k_2 = \frac{11}{20}$, $k_3 = -\frac{19}{8}$. Time in seconds.

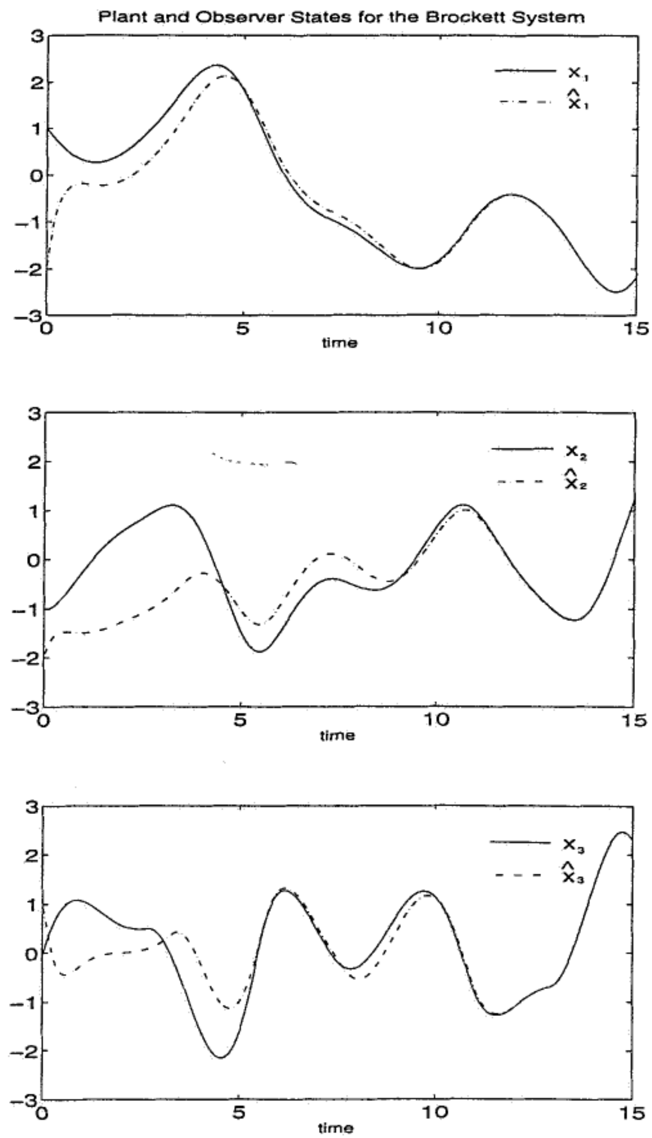

Fig. 3. Synchronization in the Brockett system. The output of the nonlinearity is calculated using the observer output. Gains are $k_1 = \frac{29}{10}$, $k_2 = \frac{11}{20}$, $k_3 = -\frac{19}{8}$. Time in seconds.

example shows the importance of the feedback term K.

In the second set of simulations we considered the Rössler system given by Eq. (44) as the drive system with parameters $a = 0.2$, $b = 0.2$ and $c = 5$. For the output, we choose $y = x_1$, i.e. $C = (1\ 0\ 0)$. For the observer, we considered the observer given by Eq. (8) with the feedback gain $K = (10.2\ 17\ \ -6)^T$. In the first simulation we assumed the ideal conditions, (i.e. without measurement noise and parameter mismatch) and the results are shown in Fig. 4. In the second simulation, we assumed the measurement noise is not present but the parameters in the observer are changed to $\hat{a} = 0.21$, $\hat{b} = 0.19$ and $\hat{c} = 5.25$. We used the same feedback matrix K and the results are shown in Fig. 5. Note that here we have $\|\Delta\mu\| = 0.2503$ and the largest change in the parameters is approximately 5%, which is not particularly small. In the third simulation, we assumed that the parameters of the drive system and the observer are the same, but the measurement is corrupted by a bounded and uniformly distributed noise generated by the computer. The noise magnitude is bounded by 10^{-3}. We used the same feedback matrix K and the results are shown in Fig. 6. As can be seen

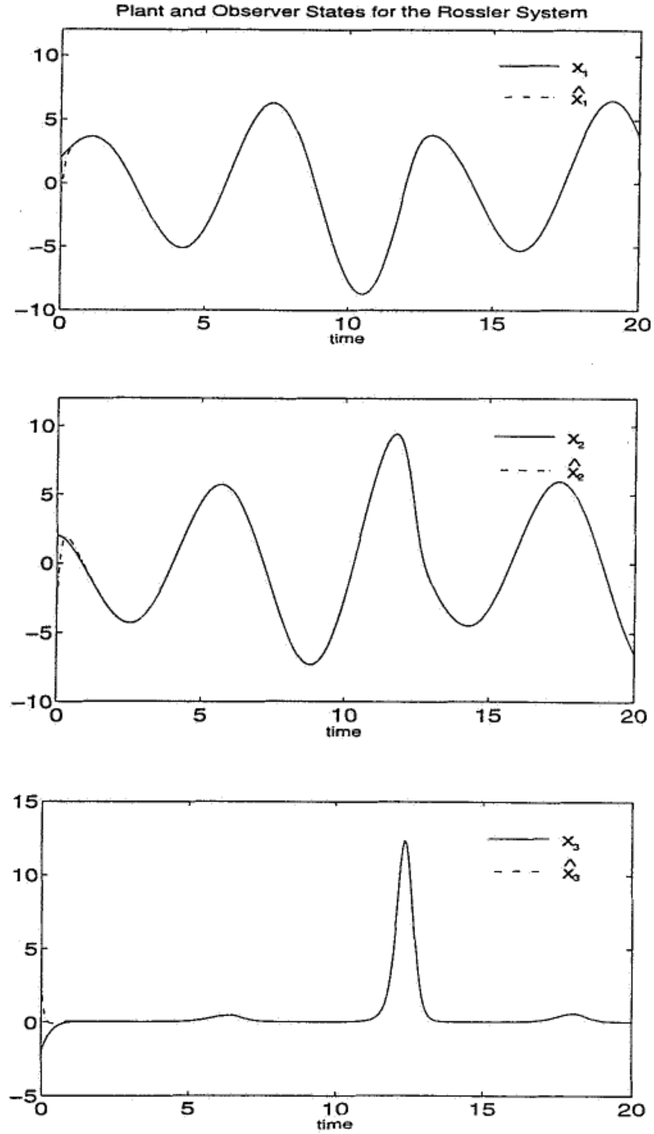

Fig. 4. Synchronization in the Rössler system. Observer has the same parameters as those of the system, $a = \hat{a} = 0.2$, $b = \hat{b} = 0.2$, $c = \hat{c} = 5.0$. Gains are $k_1 = \frac{51}{5}$, $k_2 = 17$, $k_3 = -6$. Time in seconds.

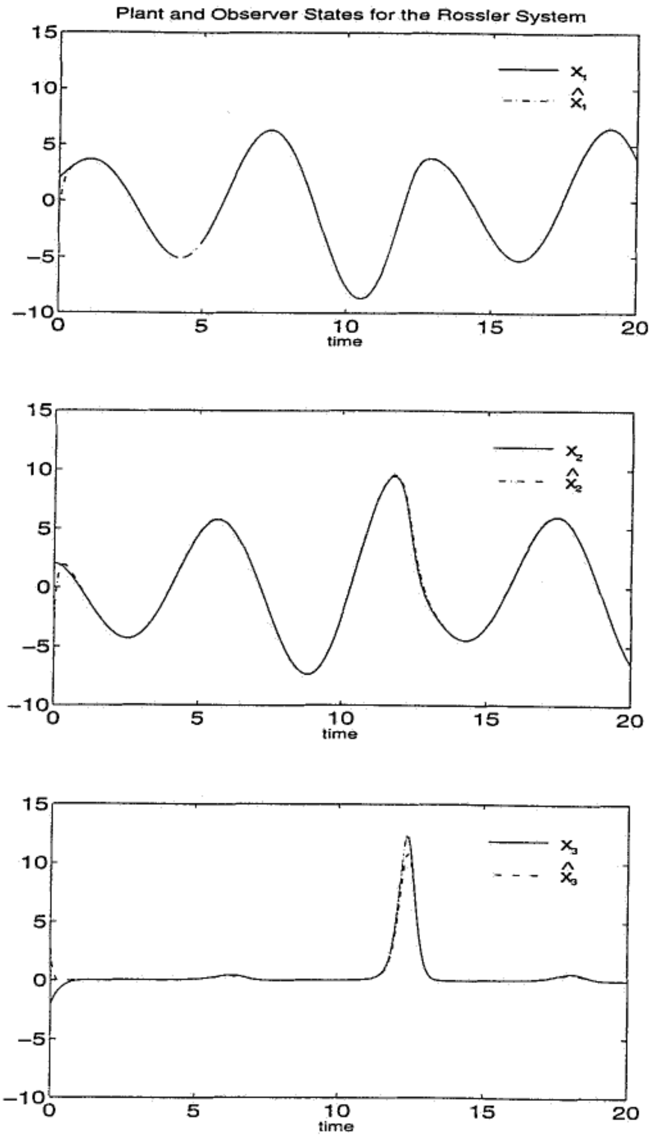

Fig. 5. Synchronization in the Rössler system. Observer parameters differ from those of the system, $a = 0.2$, $\hat{a} = 0.21$, $b = 0.2$, $\hat{b} = 0.19$, $c = 5.0$, $\hat{c} = 5.25$. Gains are $k_1 = \frac{51}{5}$, $k_2 = 17$, $k_3 = -6$. Time in seconds.

from the simulations, the proposed observer is quite robust with respect to parameter mismatch and the measurement noise. It is not possible to show the synchronization error in Fig. 6 because of the signal levels. The synchronization error has approximately the same bound as the noise, which is expected, and is shown in Fig. 7. In these simulations, we used the initial conditions in the drive system as $(2\ 2\ -2)^T$ and in the observer as $(-1\ -3\ 3)^T$. Note that the initial error is $\|e(0)\| = 7.68$ in this case, which is not particularly small. This example also shows the importance of the feedback term K.

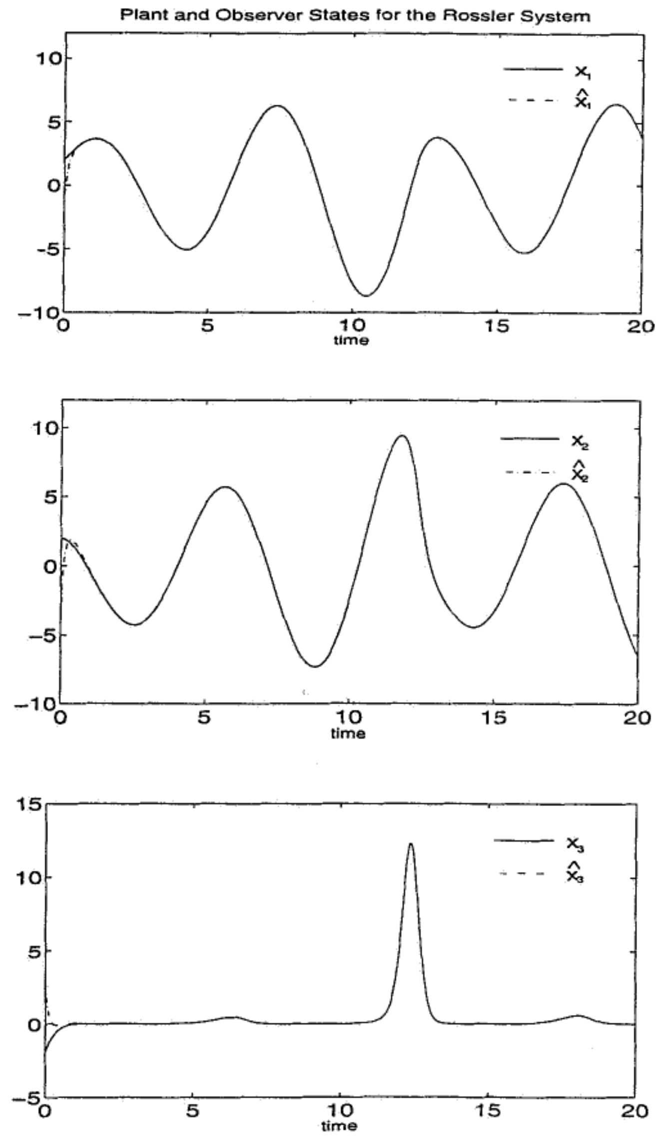

Fig. 6. Synchronization in the Rössler system. System output is corrupted by a bounded noise of uniform distribution, $|n(t)| \leq 10^{-3}$, $a = 0.2 = \hat{a} = 0.2$, $b = 0.2 = \hat{b} = 0.2$, $c = 5.0 = \hat{c} = 5.0$. Gains are $k_1 = \frac{51}{5}$, $k_2 = 17$, $k_3 = -6$. Time in seconds.

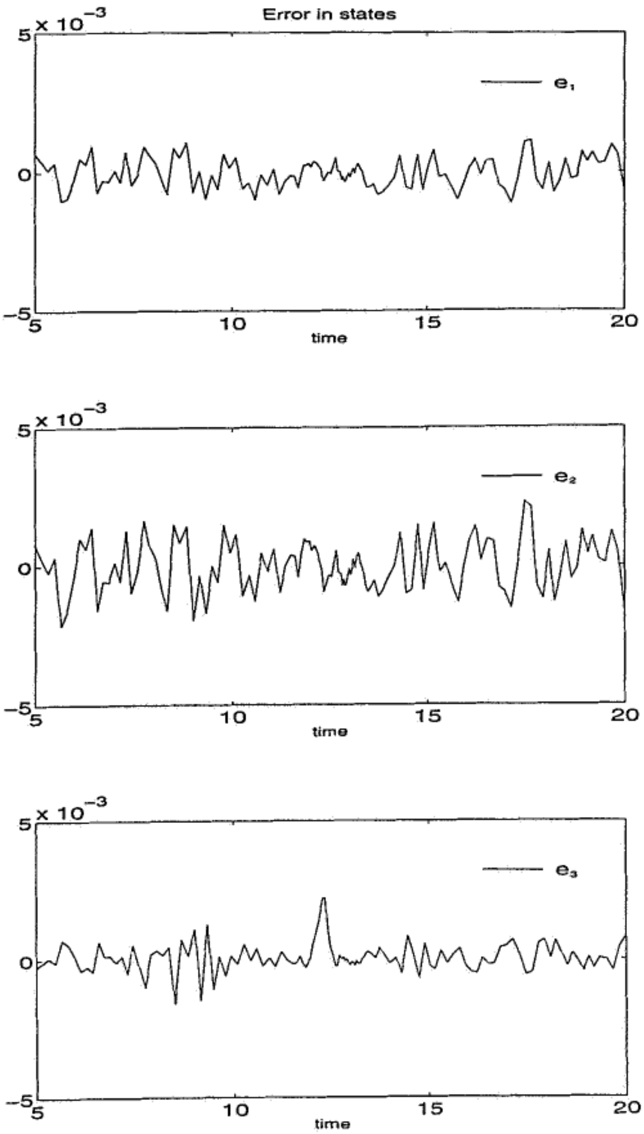

Fig. 7. Synchronization error in the Rössler system. System output is corrupted by a bounded noise of uniform distribution, $|n(t)| \leq 10^{-3}$, $a = 0.2 = \hat{a} = 0.2$, $b = 0.2 = \hat{b} = 0.2$, $c = 5.0 = \hat{c} = 5.0$. Gains are $k_1 = \frac{51}{5}$, $k_2 = 17$, $k_3 = -6$. Time in seconds.

In the last simulation, we considered Chua's oscillator given by Eq. (45). In the simulations we chose $R_0 = 0$, which does not affect the chaotic behavior, but simplifies Eq. (45), [Chua *et al.*, 1993b]. For actual values of the parameters to observe chaotic behavior, see [Kennedy, 1993; Chua *et al.*, 1993b]. For these actual values, the parameters in Eq. (45) may be too large, especially the Lipschitz constant in Eq. (6) may be in the range of 10^6, which causes problems in determining the observer. To overcome this difficulty, we first scaled the time and used $\tau = \frac{G}{C_2}t$ as the new independent variable and also scaled the variable x_1 by $1/G$. After these changes, Eq. (45) now becomes:

$$\dot{x}_1 = -\beta x_2\,,$$
$$\dot{x}_2 = x_1 - x_2 + x_3\,,$$
$$\dot{x}_3 = \alpha x_2 - \alpha x_3 - \tfrac{\alpha}{G} f(x_3)\,,$$

where $\alpha = \frac{C_2}{C_1}$ and $\beta = \frac{C_2}{LG^2}$. Following [Genesio & Tesi, 1992], we choose the parameters as $G_1 = -0.8$, $G_2 = -0.5$, $\alpha = 8$, $\beta = 11$, $E = 1$ and $G = 0.7$. As is shown in [Genesio & Tesi, 1992], with these parameters, the equations given above exhibit a double scroll type chaotic behavior. For the feedback matrix, we choose $K = (14.875\ 16.375\ 13)^T$ and the output is chosen as $y = x_3$, i.e. $C = (0\ 0\ 1)$. Note that with this choice, $A - KC$ is a stable matrix. In this case, the Chua oscillator is in the Lur'e form given in Fig. 1, see [Genesio & Tesi, 1992], and both the observer given by Eq. (35) or the standard observer given by Eq. (8) could be used. We simulated both cases and obtained good results. We report only the results related to the observer given by Eq. (8). We also assumed that the parameter β of

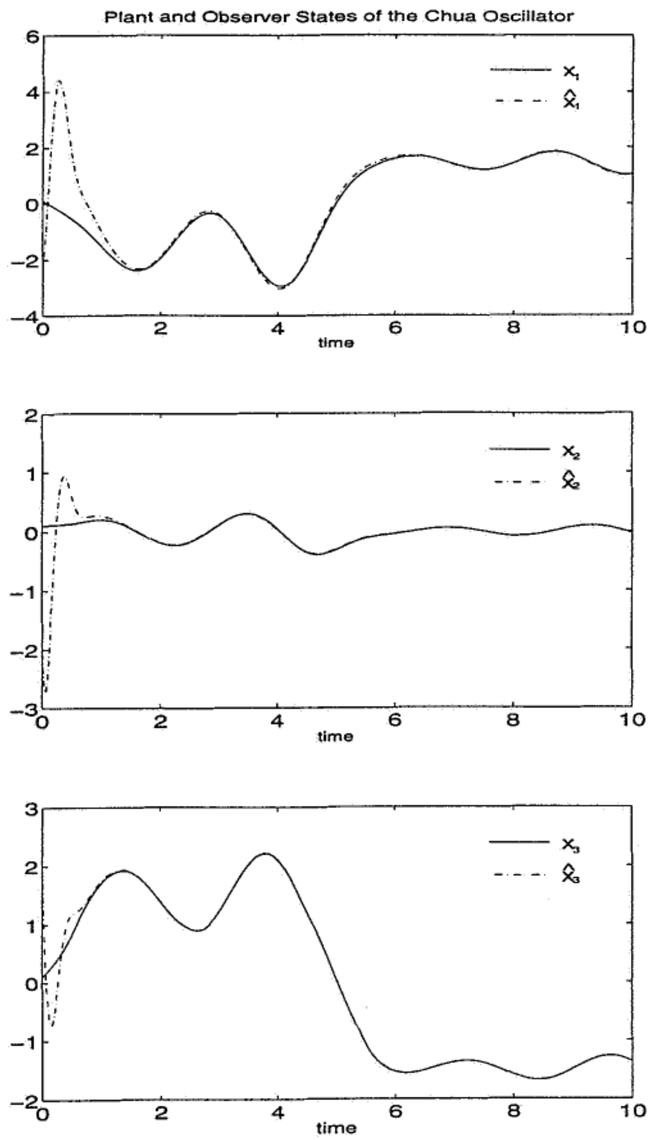

Fig. 8. Synchronization in the Chua's oscillator. The output of the nonlinearity is calculated using the observer output which is corrupted by a bounded noise of uniform distribution, $|n(t)| \leq 10^{-3}$, and the observer parameters differ from those of the system, $\alpha = \hat{\alpha} = 8$, $\beta = 11$, $\hat{\beta} = 12.1$. Gains are $k_1 = \frac{119}{8}$, $k_2 = \frac{131}{8}$, $k_3 = 13$. Time in seconds.

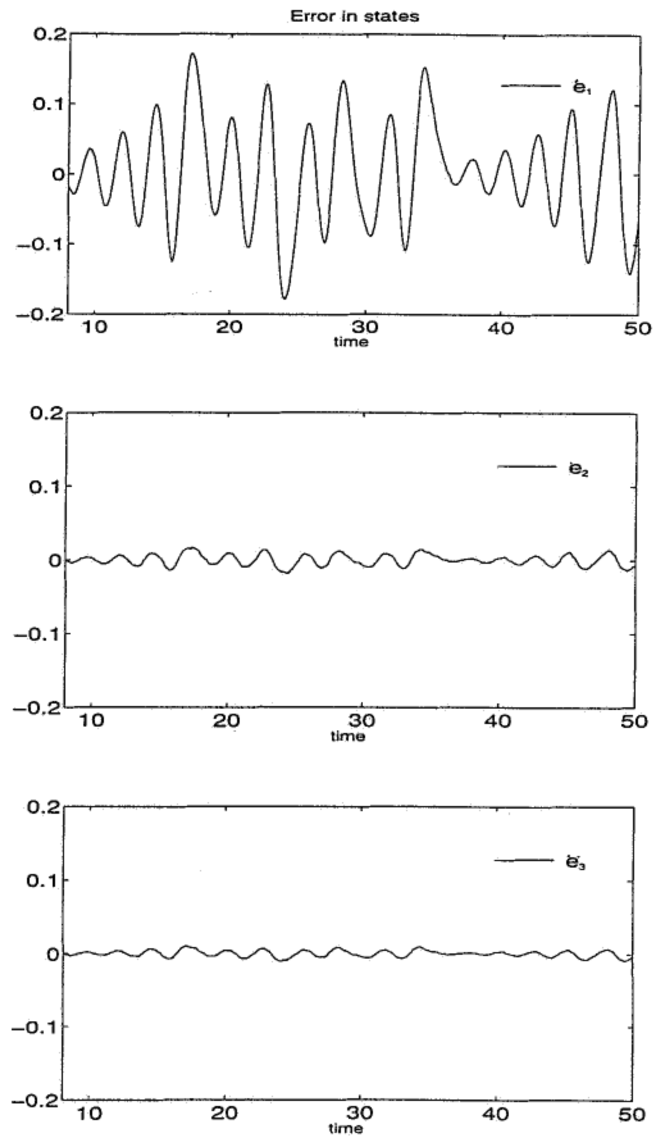

Fig. 9. Synchronization error in the Chua's oscillator. The output of the nonlinearity is calculated using the observer output which is corrupted by a bounded noise of uniform distribution, $|n(t)| \leq 10^{-3}$, and the observer parameters differ from those of the system, $\alpha = \hat{\alpha} = 8$, $\beta = 11$, $\hat{\beta} = 12.1$. Gains are $k_1 = \frac{119}{8}$, $k_2 = \frac{131}{8}$, $k_3 = 13$. Time in seconds.

the observer is changed to $\hat{\beta} = 12.1$ and the parameter α is kept the same as in the drive system, i.e. as $\hat{\alpha} = 8$. We also assumed that the measurement is corrupted by a bounded and uniformly distributed noise, and the bound on noise is 10^{-3}. The results are shown in Figs. 8 and 9. We see from Fig. 9 that the errors are all bounded, only e_1 seems to be a little bigger than the other error components. This is basically due to the large change in the parameter β which is 10%. For smaller parameter changes, we obtained smaller error. The reason to present these particular simulation results is to show that the observer presented here can tolerate relatively large changes in the parameters, hence is quite robust. Initial conditions are chosen as $(0.1\ 0.1\ 0.1)^T$ in the drive system and $(-2\ \ -2\ \ -2)^T$ in the response system which implies that $\|e(0)\| = 3.53$, which is not particularly small.

6. Conclusions

Most of the synchronized chaotic systems proposed in the literature consist of two parts: A drive system which generates the chaotic signals, and a response system. Some signals called drive signals are generated by the drive system and are used in the response system to synchronize the common signals of both systems. In most of the cases, once the drive system is given, the determination of the response system and the drive signals are not systematic and one scheme proposed for a particular drive system could not be easily generalized to an arbitrary chaotic drive system.

In this paper we considered the observer-based synchronization of chaotic systems. Observers are widely used in systems and control theory to estimate the states of a given system, hence may naturally be used in the synchronization of chaotic systems. In this approach, once the chaotic drive system is given in a form [see (4)], then the response system could be chosen as an observer [see (8)], provided that the output corresponding to the selected drive signal satisfies some conditions (i.e. observability or detectability, see Theorem 1, Remark 3). These conditions are not very restrictive and are satisfied by most of the chaotic systems (see Lemma 2, Remark 2). Then we stated a general result on the local synchronization of the drive system and the observer (see Lemma 1). We also stated a global convergence result, provided that the system could be transformed into a special form. We also showed that some of the existing schemes for the synchronization of chaos are related to the observer based synchronization proposed in this paper. We also presented some numerical simulation results for the Lorenz, Rössler systems and Chua's oscillator, which are known to exhibit many forms of chaotic behavior.

We note that the form of the observer given in this paper is not the only possible form. There are many observer design techniques and some of them may give better results in the synchronization of chaotic systems. This point requires further research and the results will be presented elsewhere.

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