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HIGHER-ORDER SPECTRAL ANALYSIS TO DETECT POWER-FREQUENCY MECHANISMS IN A DRIVEN CHUA'S CIRCUIT

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Higher-order spectra have been used to investigate nonlinear interactions between frequency modes in a driven Chua's circuit. The spectra show that an energy transfer takes place to the dominant frequencies in the circuit, i.e. the input frequency, the primary peak frequency and the harmonics of both frequencies. Other frequencies couplings become less important. Obviously, powers are (nonlinearly) related at different frequencies. When the circuit undergoes a period doubling sequence to chaos, the gain is increasing.

1. Introduction

Recently, higher order spectral techniques were used to study nonlinear dynamics equations, circuits and systems like the well-known Chua's circuit [Elgar & Chandran, 1994]. Second order or bispectral analysis isolates the nonlinearly induced phase coupling between triads of Fourier modes in quadratically nonlinear systems. Therefore, bispectral analysis allows us to study quadratic nonlinear interactions in the nonlinear system. In the same manner, the third order spectral or trispectral analysis isolates the nonlinearly induced phase coupling between quartets of Fourier modes in cubical nonlinear systems.

Elgar and Kennedy [1993] applied the bispectral analysis technique to study the nonlinear interactions in Chua's circuit. They showed that the quadratic nonlinear modal interactions in the Rössler-type attractor are important, but that also cubic nonlinear interactions are important to the dynamics. For circuit parameters that lead to the double scroll chaotic attractor Elgar and Kennedy showed that the system then is certainly not dominated by quadratic nonlinearities.

An interesting phenomenon in many forced nonlinear dynamics equations is the fact that during chaotic behavior, signal amplification of the input signal or driven term is maximal [Halle *et al.*, 1992; Anishchenko *et al.*, 1994]. This behavior can be used to advantage in developing high sensitivity detectors or low-noise amplification [Leenaerts, 1996]. However, so far this property of nonlinear dynamic systems is not yet understood very well.

The primary purpose of the presented study is to apply the higher-order spectral analysis technique to study the nonlinear modal interactions in a driven Chua's circuit and to obtain further understanding of the underlying physics of Chua's circuit [Madan, 1993].

Definitions and properties of the spectral analysis technique, in particular bispectrum and trispectrum analysis, are reviewed in Sec. 2. Chua's circuit and some computational aspects related to this research are discussed in Sec. 3. Section 4 deals with the bicoherence spectra of the period-1 and period-4 limit cycle behavior of Chua's circuit when the circuit is driven with a sinusoidal signal. In Sec. 5 the Rössler-type attractor and double scroll chaotic attractor will be studied by means of the bi- and tricoherence spectra. This is followed by conclusions in Sec. 6.

2. Definitions and Properties of Bispectrum and Trispectrum

Let a stationary random process be represented as

$$x(t) = \sum_{n=1}^{N} A_n e^{i\omega_n t} + A_n^* e^{-i\omega_n t} \tag{1}$$

where ω is the radian frequency, t is time, the subscript n is a frequency modal index, asterisk indicates complex conjugation, and the A_n are complex Fourier coefficients. Define $E[\cdot]$ as the expected value or average operator and suppose that the data $x(t)$ is discretely sampled. Then the power spectrum and discrete bispectrum can be defined as

$$P(\omega_k) = 0.5E[A_{\omega k}A_{\omega k}^*] \tag{2}$$

$$B(\omega_k, \omega_j) = E[A_{\omega k}A_{\omega j}A_{\omega k+\omega j}^*] \tag{3}$$

respectively [Haubrich, 1965; Nikias & Raghuveer, 1987].

If for instance the modes are independent, the average triple product of Fourier coefficients is zero, resulting in a zero bispectrum. For a discrete time series with Nyquist frequency ω_N, the bispectrum is uniquely defined within a triangle in the (ω_1, ω_2)-space with vertices at $(\omega_1 = 0, \omega_2 = 0)$, $(\omega_1 = \omega_{N/2}, \omega_2 = \omega_{N/2})$ and $(\omega_1 = \omega_N, \omega_2 = 0)$. In this paper, we will use the normalized magnitude of the bispectrum, called the squared bicoherence, defined as

$$b^2(\omega_k, \omega_j) = \frac{|B(\omega_k, \omega_j)|^2}{E[|A_{\omega k}A_{\omega j}|^2]E[|A_{\omega k+\omega j}|^2]} \tag{4}$$

The bicoherence does give an indication of the relative degree of phase coupling between triads of Fourier components with $b = 0$ for random phase relationships and $b = 1$ for maximum coupling.

In a similar manner the discrete trispectrum and tricoherence can be defined as

$$T(\omega_k, \omega_j, \omega_i) = E[A_{\omega k}A_{\omega j}A_{\omega i}A_{\omega k+\omega j+\omega i}^*] \tag{5}$$

$$t^2(\omega_k, \omega_j, \omega_i) = \frac{|T(\omega_k, \omega_j, \omega_i)|^2}{E[|A_{\omega k}A_{\omega j}A_{\omega i}|^2]E[|A_{\omega k+\omega j+\omega i}|^2]} \tag{6}$$

respectively. The tricoherence is a measure of the fraction of the power of the quartet of Fourier components $(\omega_k, \omega_j, \omega_i, \omega_{k+j+i})$ that is owing to cubic nonlinear interactions. Again owing to symmetry relations, the trispectrum needs only to be defined in a subset of the complete $(\omega_1, \omega_2, \omega_3)$-space. For sum interactions, this reduced region of computation is a tetrahedron with base equal to the triangle given above for the region of computation of the bispectrum [Haubrich, 1965].

A 95% significance level on zero bicoherence is given by Haubrich [1965] as

$$b_{95\%}^2 = 6/(\text{degree of freedom in the estimate of } b^2) \tag{7}$$

This means that there is only a 5% chance that a bicoherence estimate would exceed this value if the process were truly Gaussian. This relationship holds for any higher-order coherence [Chandran *et al.*, 1993].

3. Chua's Circuit

Chua's circuit is given in Fig. 1 where the input sinusoidal signal is injected as voltage in series with the inductor. Tests with the input signal injected at other places in the circuit indicate no significant difference in the obtained results.

For the circuit the following parameters are assumed: $C_1 = 10$ nF, $C_2 = 100$ nF, $L = 18$ mH, $R_0 = 12.5\ \Omega$ (inductor's resistance), $G_a = -757.576\ \mu$S, $G_b = -409.091\ \mu$S and $E = 1$ V. We will vary R as indicated in Table 1.

The amplitude of the input signal is chosen at -60 dBV to assure that the circuit will remain in its chosen mode. For instance, applying an input signal with larger amplitude to the circuit operating in period-4 limit cycle without the signal, the circuit is forced into the Rössler-type attractor. The software program INSITE [Parker & Chua, 1989] was used to verify that this kind of behavior would not occur during our experiments.

A fourth order explicit Runge–Kutta method was used to compute the trajectories of the circuit. The trajectory $v_r(t)$ (see Fig. 1) was sampled (sampling frequency was 49.9 kHz) to obtain 192 short records, each 512 points long. A Hanning window with 75% overlap was applied to each short time series to reduce spectral leakage. Power and higher-order spectra were computed with a final frequency resolution of 97 Hz. The spectra are statistically

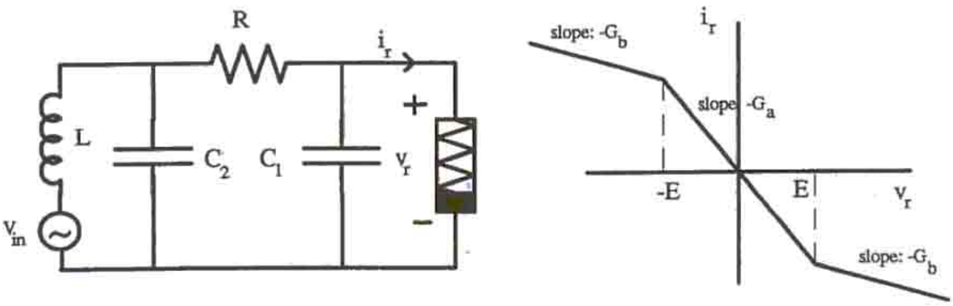


Fig. 1. Chua's circuit and the characteristic of the nonlinear resistor. Dimensional values for the circuit are given in the text.

Table 1. Values of R for the cases studied.

Case	R(Ohm)
period-1	1887
period-4	1855
Rössler-type attractor	1848
double scroll attractor	1770

Table 2. Color scale.

minimum contour plottted		
	b=0.7	b=0.4
	0.7	0.4
	0.75	0.5
	0.8	0.6
	0.85	0.7
	0.9	0.8
	0.95	0.9
	1	1

significant for $b_{95\%} = 0.125$. The correctness of the used methods was verified by obtaining similar results as Elgar & Kennedy [1993].

The used scale for the contours in the spectra plots is given in Table 2.

4. Bicoherence of Period-1 and Period-4

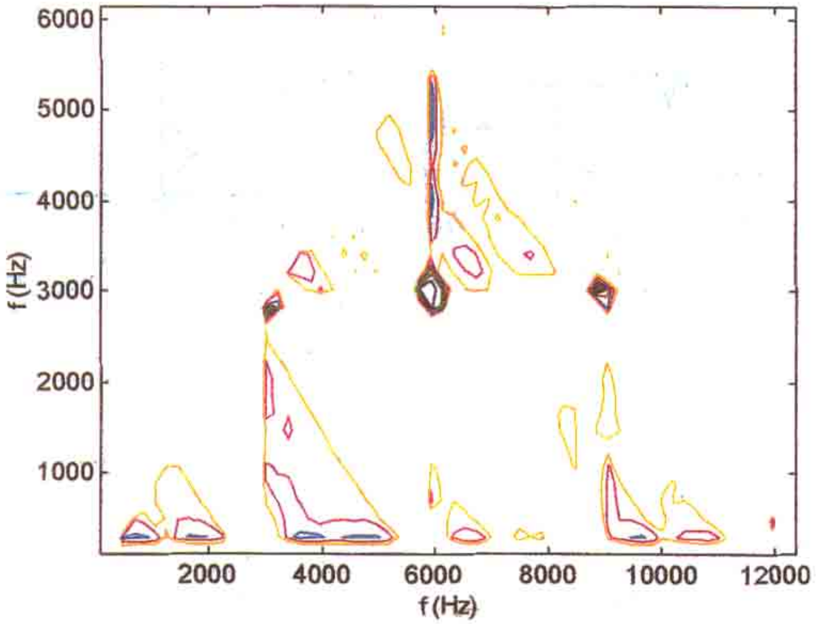
In this section the bicoherence and power spectra for data obtained from a forced Chua's circuit operating in the limit cycle region are presented.

Let us first examine the period-1 limit cycle. Figure 2(a) shows the bicoherence spectrum for the situation without input signal. Clearly visible is the strong quadratic coupling between motions at the primary central peak frequency and its harmonics [e.g. $f_1 = 2.9$ kHz, $f_2 = 2.9$ kHz, $f_1 = 2.9$ kHz, $f_2 = 5.8$ kHz, ...]. Figure 2(b) shows the bicoherence spectrum when an input signal at 1200 Hz is applied. Besides the fact that there is a coupling between this input frequency and the central peak frequency and its harmonics, energy is transferred to the primary peak motion and its harmonics. Quadratic interactions between other frequencies are reduced as can seen by comparing Fig. 2(a) with 2(b).

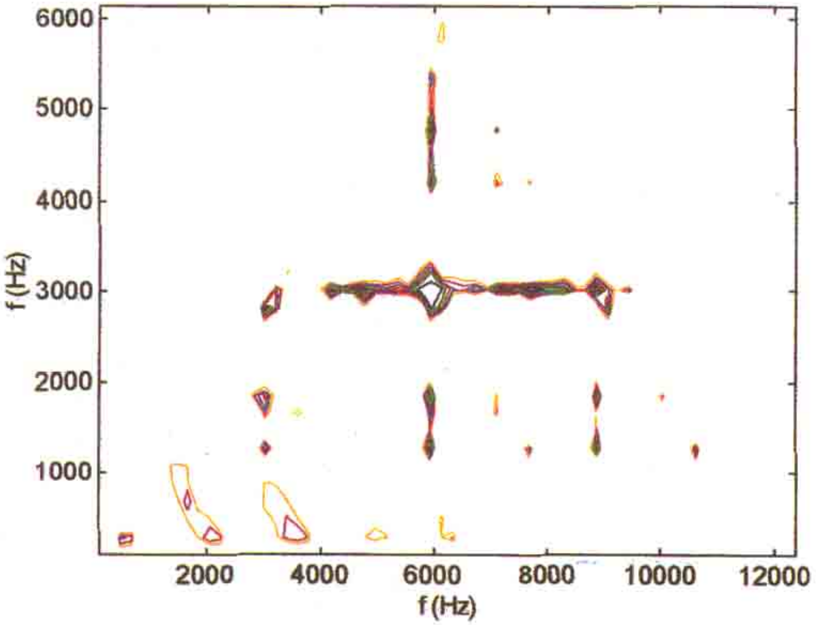
This effect of transferring energy to only the primary peak and its harmonics and the input frequency can also be observed for a period-2 and period-4 limit cycle. The latter situation is shown in Fig. 3. The bicoherence for the situation with applied input signal (frequency 1200 Hz) shows quadratic coupling only between the sums and differences of the input frequency and the central peak frequency and between the central peak frequency and its higher harmonics. Other frequency couplings are not significance anymore.

5. Rössler-Type Attractor and Double Scroll Attractor

The Rössler-type attractor is chaotic and has a fairly broadband power spectrum. Motions corresponding to the remnant of the primary peak ($f = 2.9$ kHz) are quadratically coupled to both higher frequencies (horizontal band of contours) and to lower frequencies as can be seen from the vertical bands of contours in Fig. 4(a). Applying an input signal with frequency of 1 kHz the bicoherence does not differ [Fig. 4(b)]. This means that the coupling of the



(a)

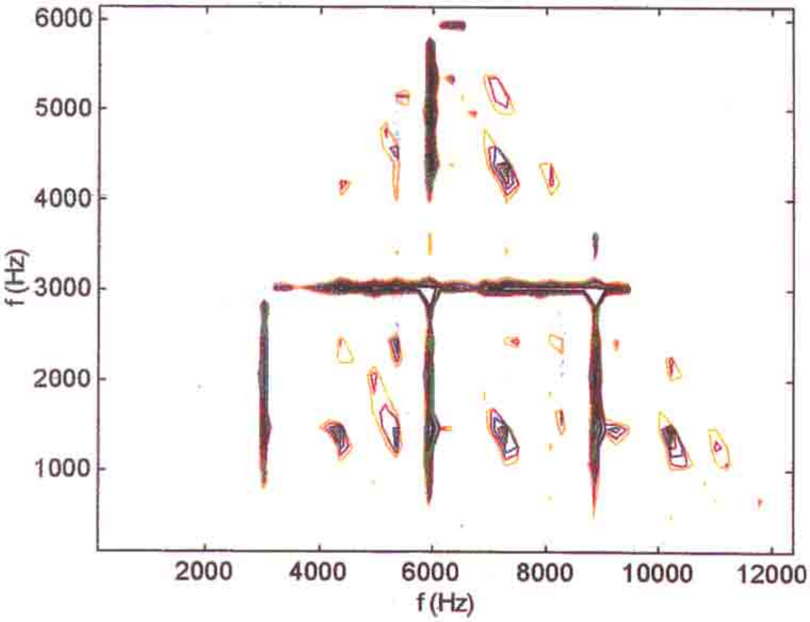


(b)

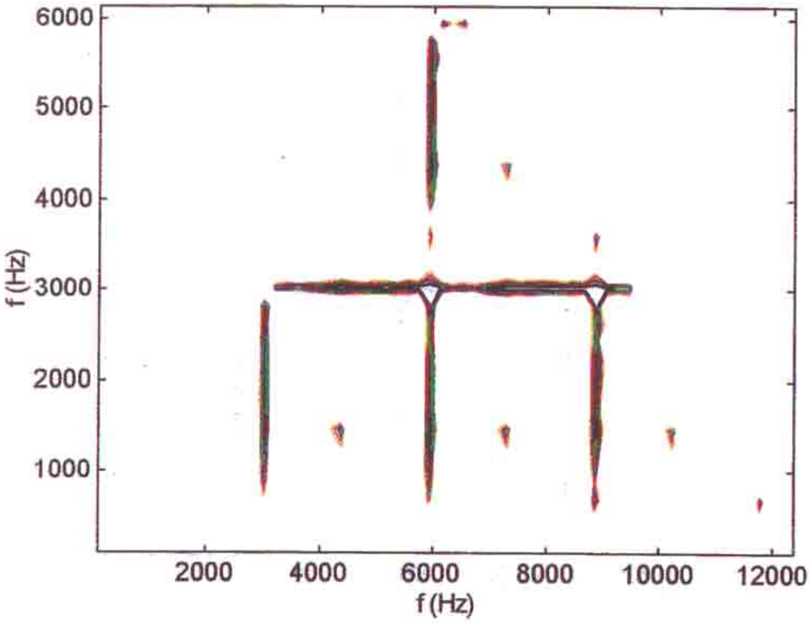
Fig. 2. Bicoherence spectra of a period-1 limit cycle. The minimum contour plotted is $b = 0.7$. (a) without applied input signal, (b) with input signal at $f = 1200$ Hz.

input frequency with the primary peak and its harmonics is not quadratic.

To verify whether or not the coupling is cubic we computed the tricoherence. The plots in Fig. 5 are generated for a sum frequency $f_4 = f_1 + f_2 + f_3 = 6800$ Hz, i.e. twice the primary peak frequency plus the input frequency. Figure 5(a) represents the situation without an input signal and shows that there exist cubic couplings between nearly all the components of the system, conform the work of Elgar & Chandran [1991]. However, these couplings are weaker than the quadratic couplings, the minimum contour plotted is $b = 0.4$. When the input signal is applied ($f = 1000$ Hz), an energy



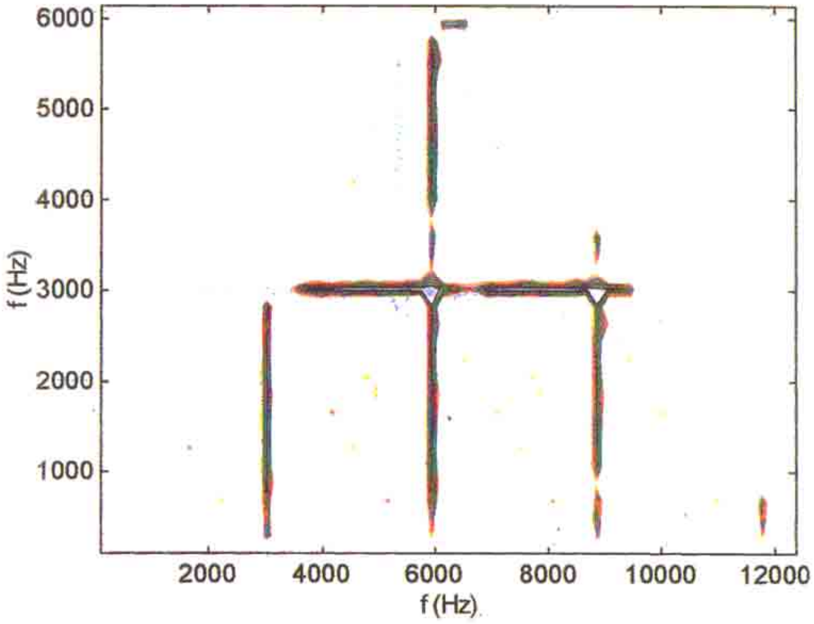
(a)



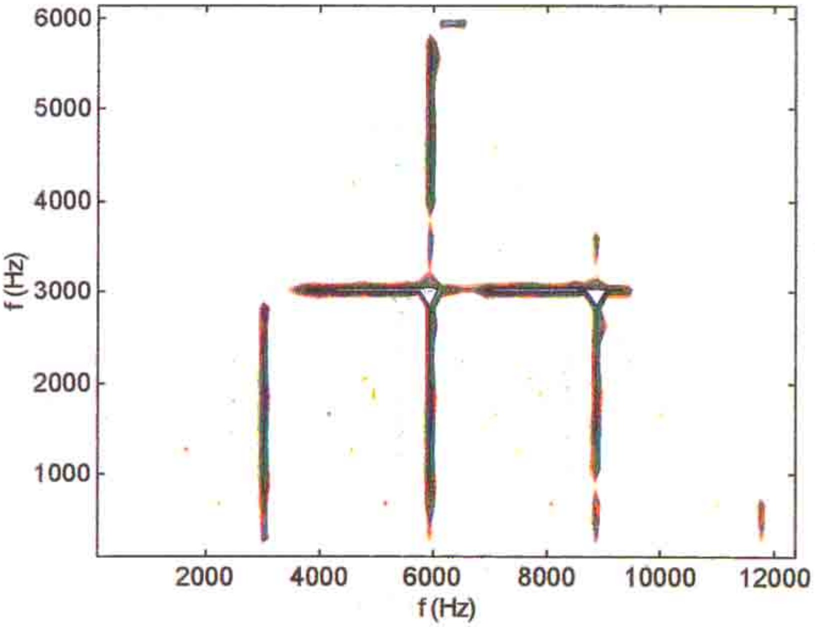
(b)

Fig. 3. Bicoherence spectra of a period-4 limit cycle. The minimum contour plotted is $b = 0.7$. (a) without applied input signal, (b) with input signal at $f = 1200$ Hz.

transfer can be observed in the tricoherence [Fig. 5(b)]. The cubic coupling between the input frequency and the peak frequency ($f_1 = f_2 = 2900$ Hz, $f_3 = 1000$ Hz) becomes visible as well as between the input frequency and the higher harmonics. These couplings are strong, other frequencies cubic couplings play a less important role. The observation that the input frequency is at least cubically and not quadratically related to the primary peak frequency of the Chua's circuit means also that the amplification of this input signal is higher than in the limit cycle regions. The limit cycle regions were dominated by quadratic couplings between the frequency modes.



(a)

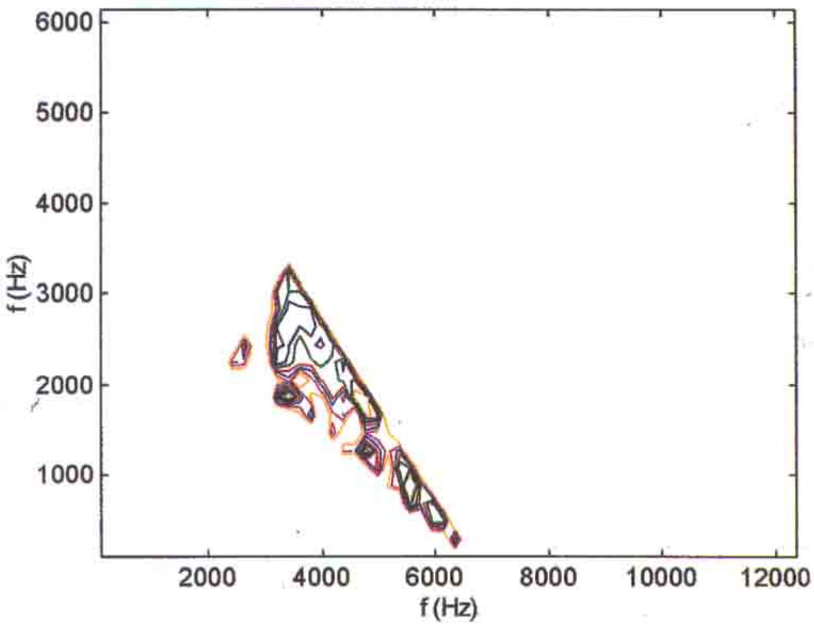


(b)

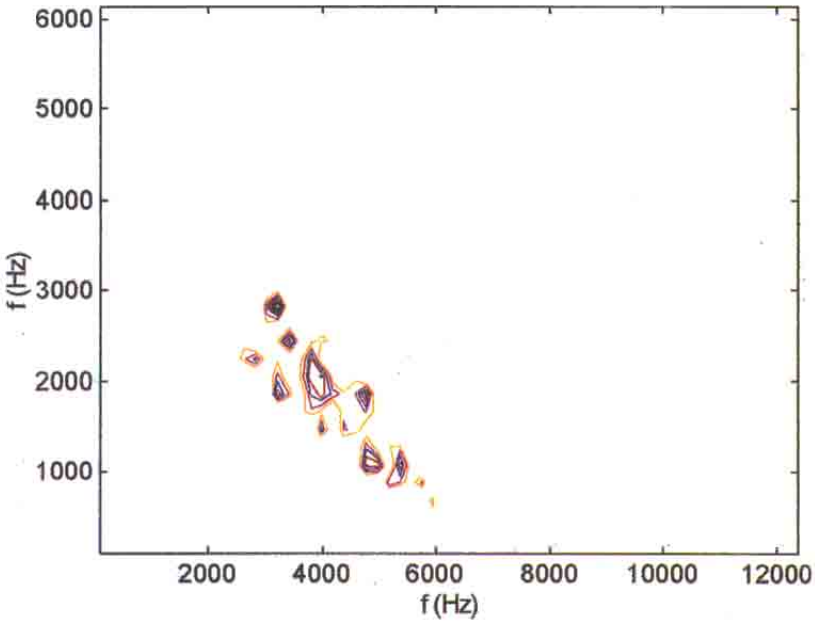
Fig. 4. Bicoherence spectra of a Rössler-type attractor. The minimum contour plotted is $b = 0.7$. (a) without applied input signal, (b) with input signal at $f = 1000$ Hz.

Finally the double scroll chaotic attractor is investigated. As already shown by Elgar & Kennedy [1993], the double scroll is not dominated by quadratic interactions, the bicoherence spectrum is empty. Higher-order interactions play here an important role and therefore we investigated the tricoherence (Figs. 6–7). The sum frequencies were chosen to be 3900 Hz and 6800 Hz, while the minimum contour plotted are $b = 0.4$ and $b = 0.7$ respectively.

First, we can observe a cubic coupling between the first subharmonic of input frequency and the primary peak frequency and vice versa [Fig. 6(b)], a cubic coupling between the second harmonic of the



(a)



(b)

Fig. 5. Tricoherence spectra of a Rössler-type attractor. The minimum contour plotted is $b = 0.4$. The constant sum frequency is $f_4 = 6800$ Hz. (a) without applied input signal, (b) with input signal at $f = 1000$ Hz.

primary frequency and harmonics of the input frequency etc. Notice that there exists only a weak cubic relation between the main frequencies 2900 Hz, 2900 Hz and 1000 Hz [Fig. 7(b)]. Higher order spectral analysis is needed to investigate whether higher order couplings between the primary frequencies are involved or not.

Second, we can again observe an energy transfer. With applied input signal the couplings between the primary frequency and its harmonics and the input frequency and its harmonics become more dominate than without input signal. In general the couplings in the double scroll attractor are stronger than in the Rössler-type attractor. This will

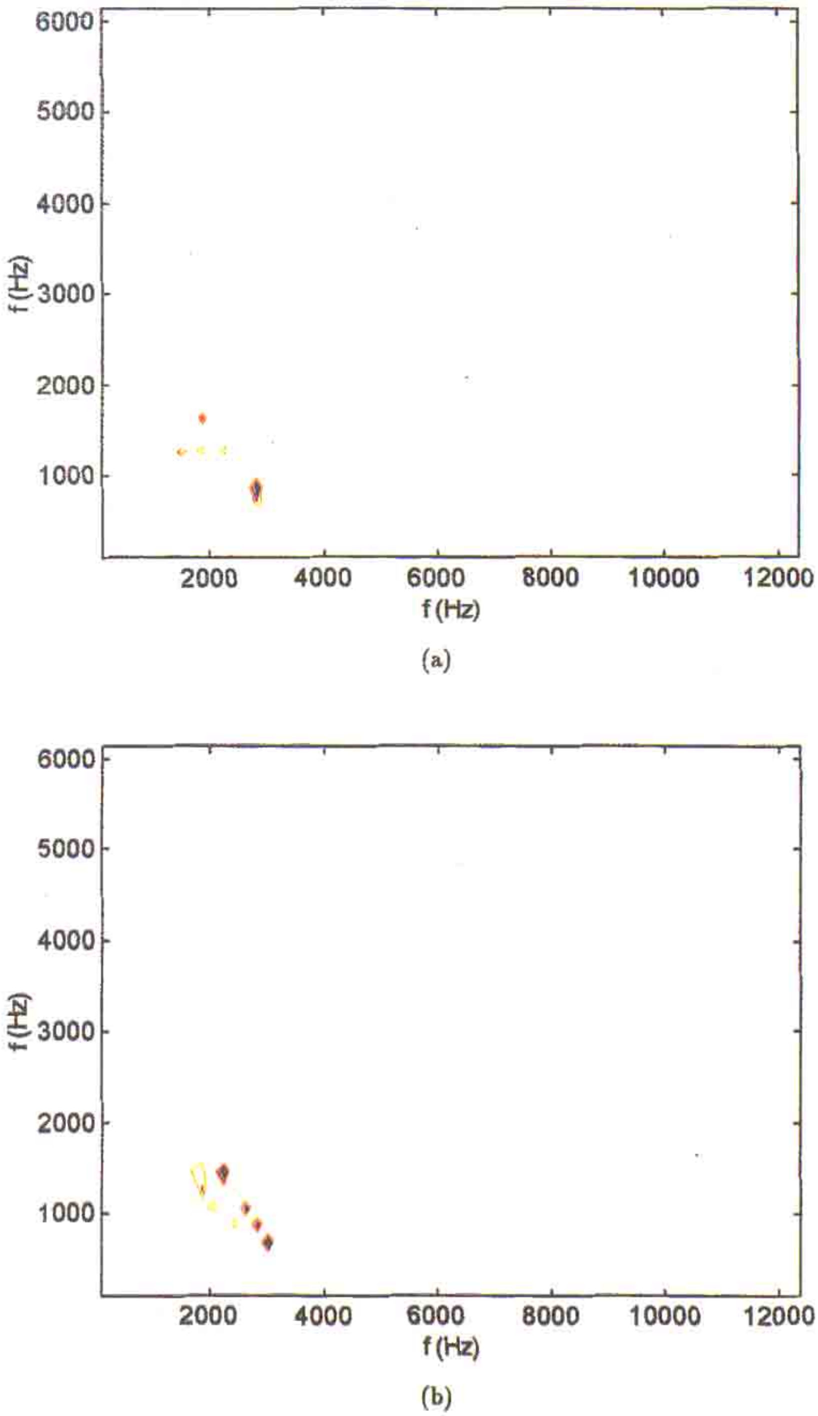


Fig. 6. Tricoherence spectra of a double scroll attractor. The minimum contour plotted is $b = 0.4$. The constant sum frequency is $f_4 = 3900$ Hz. (a) without applied input signal, (b) with input signal at $f = 1000$ Hz.

influence the gain, the amplification of the input signal will be higher in this case than in the Rössler-type attractor.

6. Discussion and Conclusion

From the experiments we may conclude that when an input signal is applied, the nonlinear coupling between the input frequency and the central peak frequency and higher order harmonics are dominating the bi- and tricoherence spectra. An energy transfer is taken place to the dominating frequencies in the circuit, nonlinear couplings between all kinds of frequencies become less important.

We conjecture the existence of a power-frequency mechanism, similar to the well known

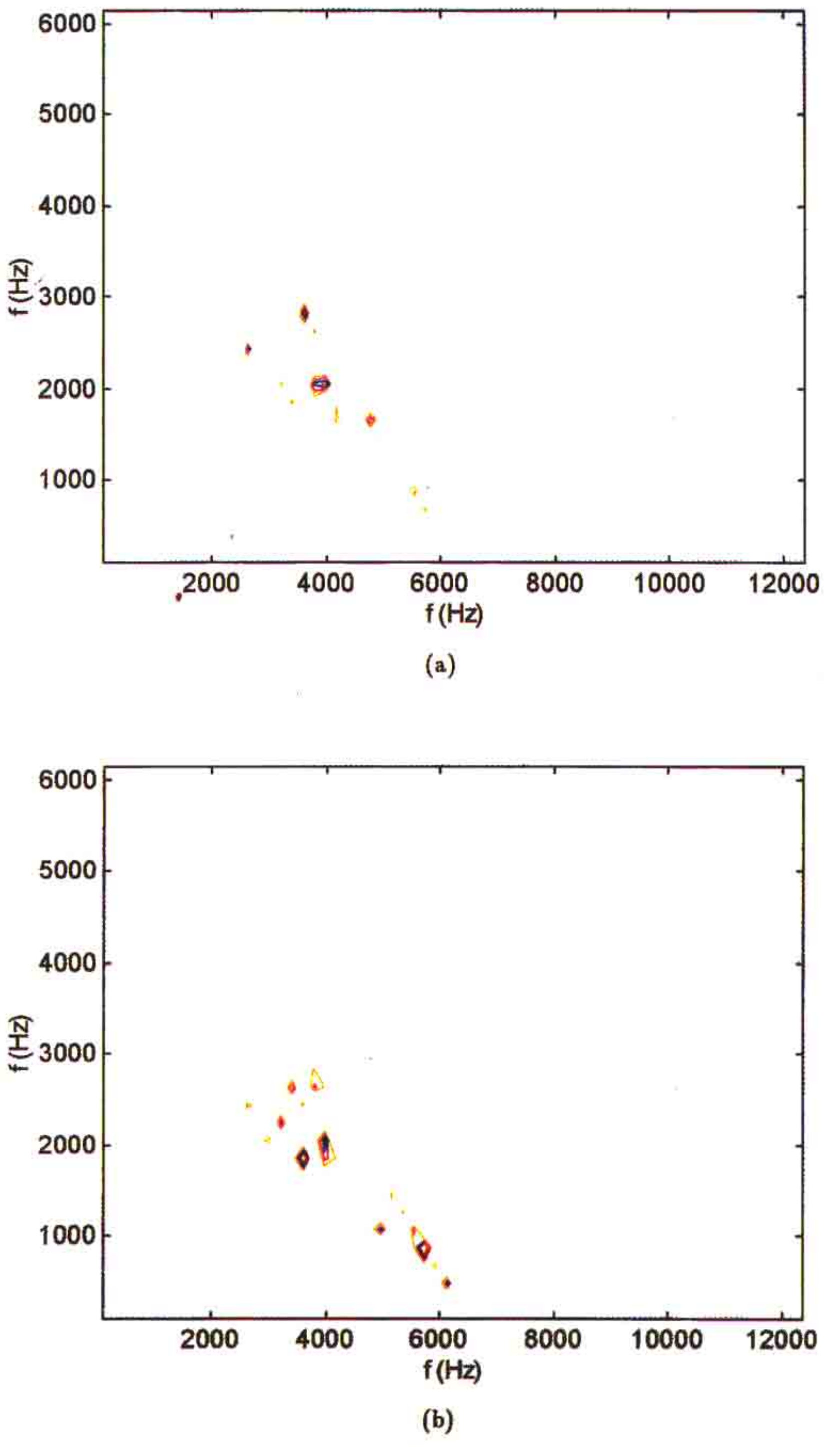


Fig. 7. Tricoherence spectra of a double scroll attractor. The minimum contour plotted is $b = 0.7$. The constant sum frequency is $f_4 = 6800$ Hz. (a) without applied input signal, (b) with input signal at $f = 1000$ Hz.

Manley–Rowe equations [Manley & Rowe, 1956] or its extensions [Tanaka, 1986]. These energy equations relate the average powers at different frequencies in nonlinear time dependent elements. The results of this research show that also in Chua's circuit powers are (nonlinearly) related at different frequencies, certainly when the circuit is driven by a sinusoidal signal.

The obtained results are in agreement with the experiments done by Halle *et al.* [1992]. There it was shown that the amplification rises when the circuit undergoes a period-doubling sequence to chaos.

We observe a similar effect. In the Rössler-like attractor the coupling is weakly cubic where in the limit cycle regions it is mainly quadratic. Therefore, the amplification of the amplitude of the input signal will also be related to a more cubic nonlinearity rather than a quadratic nonlinearity, resulting in a higher gain when the circuit exhibits a Rössler-like attractor. A similar reasoning can be held for the double scroll region where the coupling is certainly not quadratic and at least cubic.

Acknowledgments

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