Analysis of Regularization of Dynamics in a Circular Chain of Bistable Chaotic Elements with Variable Number of Couplings

A.S. Kuznetsov and V.D. Shalfeev
Institute of Applied Physics, Russian Academy of Sciences
46 Uljanov St., Nizhny Novgorod 603600, Russia
e-mail: alexey@ Hale.appl.sci-nnov.ru

Abstract

Dynamics of a circular chain of coupled bistable active elements is investigated for different number of couplings in the chain. It is found that dependences of the boundaries of existence domains for all the considered modes are characterized by a sharp change in the region with the smallest number of couplings.

1 Introduction

Dependence of collective dynamics of an ensemble of coupled bistable active elements on the number of couplings between them is investigated in this paper. Consider a model of bistable active elements in which by varying the parameter it is possible to pass from a chain of locally coupled elements in an ensemble to an ensemble with global couplings. The chain of identical Chua oscillators [1] is taken as such a model. Each element in this chain is coupled with $S$ right-side and $S$ left-side neighboring elements:

$$\frac{dx_i}{dt} = \alpha(y_i - \varphi(x_i)) +$$
$$\frac{d}{2S} \sum_{k=1}^{S} (F(x_{i-k}) + F(x_{i+k})),$$

$$\frac{dy_i}{dt} = x_i - y_i + z_i,$$

$$\frac{dz_i}{dt} = -\beta y_i.$$  

Here, $i$ is the number of the element, $i = 1, N$, $N$ is the number of elements in the system, and $d$ is the parameter of coupling between the elements. Let us choose $N = 55$ and take the coupling function of the form $F(x) = \frac{2x}{1+4x^2}$, where $\delta = 3$. Nonlinearity of a partial element is approximated by a smooth function $\varphi(x) = x + \alpha x^3 - \frac{2x^5}{1+4x^2}$.

Parameters of an isolated element in the existence domain of a strange attractor, spiral attractor to be more precise, are chosen to be the following: $\alpha = 6.4, \beta = 10, \delta = 0, \omega_0 = 0.7, c_3 = 0.05$. For an ensemble with global couplings to be the limiting case of the ensemble (1) for $S = \frac{N}{2}$ the boundary conditions need to be periodic: $x_{i-k} = x_{i-k+N} \forall i-k < 1$; $x_{i+k} = x_{i+k+N} \forall i+k > N$. The other limiting case of the ensemble (1) – a chain of locally coupled elements – is realized at $S = 1$.

2 Homogeneous modes

Investigations of the system (1) with global couplings [2] have shown that dynamics of an ensemble of bistable elements possessing chaotic dynamics in an uncoupled state is regularized when global couplings are introduced. Then, two homogeneous modes: active (oscillatory) and passive (equilibrium state) are realized in such an ensemble. The existence domain of the first of them is $0.849 \leq d \leq 0.864, and of the second $d \geq 0.613$. The corresponding coordinates of partial elements of the system are identical in these modes:

$$x_j = x(t), y_j = y(t), z_j = x(t); j = 1/N.$$  

For the homogeneous mode (2), the system (1) gives the following equations:

$$\dot{x} = \alpha(y - \varphi(x)) + dF(x),$$

$$\dot{y} = x - y + z,$$

$$\dot{z} = -\beta y.$$  

2.1 Homogeneous passive mode

Consider in more detail a homogeneous passive mode. Equilibrium state corresponds to it in phase space of the system. Coordinates of the equilibrium state are found from the following set of equations:

$$y = 0,$$

$$z = -x,$$

$$-\alpha(\varphi(x)) + dF(x) = 0.$$
A characteristic equation describing stability of this equilibrium state can be obtained analytically for an arbitrary number of couplings. Such an equation splits into $N$ equations of the form

$$ (\sigma - \lambda) [\lambda^2 + \lambda + \beta] + \alpha \lambda =$$

$$-\frac{d}{S} F'_{z}(a) [\lambda^2 + \lambda + \beta] \sum_{k=1}^{S} \cos \left(\frac{2\pi kn}{N}\right), \quad (5)$$

where $n = \overline{1,N}$, $\sigma = -a\varphi'_{z}(a)$, and $a$ is the $z$-coordinate of the equilibrium state. It is a third-order equation with respect to $\lambda$. For a cubic characteristic equation in a general form

$$\lambda^3 + a\lambda^2 + b\lambda + c = 0 \quad (6)$$

the condition on coefficients is known under which a bifurcation of the birth of limit cycle from equilibrium state or contraction of limit cycle to equilibrium state occurs:

$$ab - c = 0. \quad (7)$$

This condition gives $N$ curves in parameter space on each of which there occurs a bifurcation of the birth of saddle limit cycle from equilibrium state:

$$d^2 \Omega(n)^2 + d\Omega(n) [2\sigma - 1 + a] +$$

$$[\beta - \sigma - a + \sigma^2 + \sigma a] = 0, \quad (8)$$

where $n = \overline{1,N}$ and the following notation is used:

$$\Omega(n) = \frac{1}{S} F'_{z}(a) \sum_{k=1}^{S} \cos \left(\frac{2\pi kn}{N}\right). \quad (9)$$

The boundary of stability domain of the equilibrium state is shown in fig. 1(a) as a function of the number of couplings. As follows from the plot, the strongest effect is a sharp shift of the boundary of stability domain in the interval where the number of couplings is small.

2.2 Homogeneous active mode

Consider now a homogeneous active mode. Computer experiment for a fixed coupling parameter $d = 0.7$ and the number of couplings varying from $S = \sum_{j=1}^{S-1} = 27$ (the case of global couplings) to $S = 1$ verified that, when $S$ decreases down to $S = 2$, the mode does not collapse. Moreover, its characteristics (time mean, etc.) do not change, which is in a complete conformity with the low-dimensional model (3).

Consider the existence domain of this mode with respect to coupling parameter for different numbers of couplings. For this we conduct the experiment in a different fashion: we change the coupling parameter at different fixed numbers of couplings. For a homogeneous active mode in an ensemble with global couplings, both the boundaries of its existence domain are described by the low-dimensional model (3). When the number of couplings is decreased, the upper boundary of this domain with respect to coupling parameter remains unchanged. Consequently, this boundary is described by a low-dimensional model for an arbitrary number of couplings. Let us introduce the notion of synchronization threshold for a definite mode as the lower boundary of its existence domain. Investigations show that, close to the synchronization threshold for an active homogeneous mode, the system is still close to the bifurcation boundary throughout the interval of coupling parameter. Therefore, we will restrict ourselves to construction of a qualitative form of synchronization threshold for a homogeneous active mode. The existence domain of a homogeneous active mode is plotted in fig.1(b) where the dotted curve corresponds to the qualitative plot. Note that for $S = 1$ the considered mode does not exist for any values of coupling parameter. The domain of its existence at $S = 2$ is $0.659 \leq d \leq 0.864$. The homogeneous mode collapses in the transition from $S = 2$ to $S = 1$ for arbitrary values of coupling parameter in the above interval. We can state that the
absence of the existence domain of an active homogeneous mode at \( S = 1 \) is caused by a sharp increase of the value of synchronization threshold. An analogous feature was described for the stability boundary of a homogeneous passive mode.

3 A pair of clusters

It follows from investigation of an ensemble with global couplings [2] that a pair of clusters is formed when the coupling parameter is increased in such an ensemble. Part of the elements in such a mode belong to one cluster, all the remaining ones to the other. All elements inside each cluster are synchronized so that the corresponding coordinates are equal to

\[
\begin{align*}
    x_i &= a_1(t), \quad y_i = b_1(t), \quad z_i = c_1(t); \quad i = 1, M, \\
    x_j &= a_2(t), \quad y_j = b_2(t), \quad z_j = c_2(t); \quad j = M + 1, N.
\end{align*}
\]

These modes differ from each other by the number of elements \( M \) belonging to one of clusters \( M \in (1, N - 1) \). Because of the difference in the amplitudes of oscillations of the elements from different clusters one of them is called a passive cluster, the other an active one. A distinguishing feature of global couplings is that it is meaningless to speak about a spatial structure of the ensemble. Indeed, no changes occur when two partial elements of the ensemble exchange places. Therefore, it is correct to classify all cluster modes only by the number of elements belonging to one of the clusters.

Let us analyze the dependence of these modes and their existence domains on the number of couplings in the ensemble. When the number of couplings is decreased starting from the global couplings, the ensemble acquires a spatial structure of a circular chain of coupled elements. For cluster modes, this is expressed as appearance of dependence of the dynamics of the element on its position relative to the cluster boundary. This also affects mean characteristics and phase relationships between oscillations of the elements. Thus, distributions of instantaneous and mean values become nonidentical when \( S \) is decreased. In spite of this, it is possible to unambiguously classify the elements as active and passive ones by the values of mean characteristics for arbitrary number of couplings.

3.1 Domains of existence of cluster modes

Consider the dependence of existence domains of cluster modes on the number of couplings in an ensemble. Towards this end, let us analyze the cases when an active cluster consists of neighboring elements, the number of which is multiple to 10, i.e., \( S = 10, 20, 30, 40, 50 \).

Figure 2: (a) Existence domain of cluster mode in which the active cluster consists of 10 neighboring elements (curve 1) and stability boundary of a homogeneous passive mode (curve 2) in the coupling parameter-number of couplings parameter plane; (b) existence domains of cluster modes in which the active cluster consists of 20 (curve 1) and 30 (curve 2) neighboring elements, respectively.

The existence domain of a mode for ten active elements \( M = 10 \) is depicted in fig. 2(a). Comparison of the curve for the synchronization threshold of this mode with the boundary of stability region of a homogeneous passive mode shows that, if the number of couplings is sufficiently large, then these curves are identical and differ by a slight shift only. The region of sharp changes of synchronization threshold is shifted upwards by the number of couplings as compared to the corresponding region of stability boundary of equilibrium state. Analogously to the case of the homogeneous active mode, the cluster mode of interest does not exist for any coupling parameters if \( S = 1 \), whereas for \( S = 2 \) its existence domain is broad enough. Therefore, analogously to the two cases of homogeneous modes considered above, in this case too the increase of the synchronization threshold is the sharpest in the transition \( S = 2 \rightarrow S = 1 \). Dynamics of the ensemble in this mode is chaotic everywhere near the curve of
synchronization threshold. The transition to chaos with the approach to the synchronization threshold with decreasing coupling parameter occurs through a cascade of period doubling bifurcations of a stable limit cycle corresponding to this mode in the region of large coupling parameters (within its existence domain). When the coupling parameter is increased, the mode persists to be periodic up to the upper boundary of its existence domain. The value of the coupling parameter corresponding to this boundary is a smooth monotonic function of the number of couplings $S$.

Dependence of synchronization threshold in the region of the smallest number of couplings remains unchanged in all the considered cases (see fig. 2(b) as an example). Here, similar collapse of all cluster modes considered above is observed in the transition $S = 2 \rightarrow S = 1$. Thus, this region is the region of the sharpest dependence of synchronization threshold on the number of couplings for a mode with arbitrary number of active elements, limiting cases inclusive.

Analysis of the plots leads us to a conclusion that a cluster mode with the minimal number of active elements $M = 1$ is the most probable one for $S = 1$. The existence domain of this mode is plotted in fig. 3. Dependence of synchronization threshold is analogous qualitatively to the ones presented above and is characterized by a sharp increase of its value in the region of a small number of couplings. In addition, this curve repeats completely the shape of the curve for the stability boundary of a homogeneous passive mode and differs from it by a slight shift. In the region with a large number of couplings, the upper boundary of the existence domain of the considered mode is actually independent of the number of couplings. The dependence of the upper boundary is nonmonotonic in the region of small values of the number of couplings. The change of the number of couplings $S = 2 \rightarrow S = 1$ leads to a sharp decrease of the value at the upper boundary of existence domain of this mode. Its existence domain is much smaller at $S = 1$ than in all the other cases. Thus, for the upper boundary of existence domain of a cluster mode with one active element, a sharp dependence of this boundary is observed in the region with the smallest number of couplings.

4 Conclusion

In this paper a circular chain of coupled bistable active elements was investigated as a function of the number of couplings in the ensemble. In the case of global couplings, the ensemble possesses a high-order multistability but it is not a spatial one, i.e., it has no spatial structure due to degeneration (symmetry). A decrease of the number of couplings removes degeneration and the ensemble acquires a spatial structure. In the case of global couplings, synchronous modes differ only by the number of elements in one cluster and, consequently, the number of combinations is $N$; whereas in a general case, they are characterized by spatial distribution too, which increases the number of possible combinations multiply. However, a decrease of the number of couplings reduces the region in which multistability occurs. Dependence of the boundaries of existence domains for all the considered modes is characterized by a sharp change in the region with the smallest number of couplings. Such an identity, evidently, characterizes the change of the common collective features of the ensemble when the number of couplings is changed.

Acknowledgements

This research was supported by the Russian Foundation for Basic Research (project 99–02–17742) and the Program of Support of the Leading Scientific Schools of the Russian Federation (grant 00–15–96582).

References
