Synchronization of a One-Dimensional Array of Chua's Circuits by Feedback Control and Noise

Gang Hu, Ladislav Pivka, and A. L. Zheleznyak

Abstract—**A one-dimensional array of coupled Chua's circuits is investigated. Without control, the synchronization between sites is poor, and the total output is very weak due to the spatial disorder. By feedback injections at certain fraction of sites (i.e., pinnings), sychronization can be established between the sites. If the pinning density is low, some sites may be left unsynchronized. In this case, by properly applying noise for certain time interval, spatial disorder can be perfectly excluded. The optimal noise intensity for synchronization (or say, stochastic resonance for synchronization) is briefly discussed.**

I. Introduction

RECENTLY, synchronization of motions (both regular and chaotic) of two identical systems has attracted much attention [7], [10]–[13]. The key point for this interest is its great potential of practical applications. The application of synchronization of chaos for secure communication has been emphasized [4], [6]. However, the following application, very important as well, has not been given enough consideration. In practice, one often needs a collective operation of a set of large number of subsystems. For instance, each semiconductor laser device can emit a weak laser beam. A great number of semiconductor units may be used to produce a strong beam with the required frequency. In this case, synchronization of these units is absolutely necessary to yield in-phase output and produce a strong total collective light beam. The similar situations, when synchronization of a great number of subunits is required for the coherent output of large global systems, can be found in many practical cases in physics, chemistry, biology, and so on. In this paper, we focus on the problem of synchronization of coupled ordinary differential equations.

In the next section, we specify our model, a one-dimensional array of Chua's circuits. In Sections III and IV, we present numerical results to show synchronizations of cells by using deterministic feedback and stochastic force, respectively. Some conclusions and a discussion are given in Section V.

Manuscript received November 1, 1994; revised May 16, 1995. This paper was recommended by Guest Editor L. O. Chua.

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IEEE Log Number 9414473.

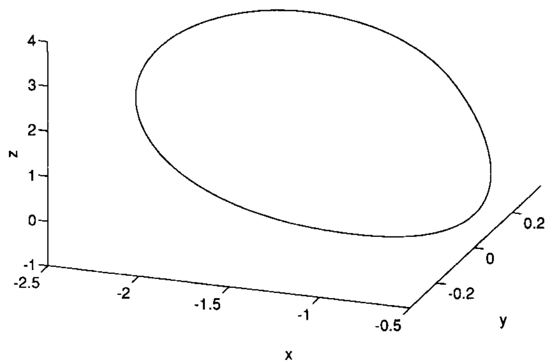


Fig. 1. A stable limit cycle of a single Chua's circuit (i.e., the system (1) with $D_x = D_y = D_z = 0$). The other stable limit cycle can be obtained by the inversions $x, y, z \to -x, -y, -z$.

II. Array of Chua's Circuits

We use the following one-dimensional array of coupled Chua's circuits as our model [3]:

$$\begin{aligned} \dot{x}(t) &= \alpha[y_i - x_i - g(x_i)] + D_x(x_{i+1} + x_{i-1} - 2x_i) \\ \dot{y}_i(t) &= x_i - y_i + z_i + D_y(y_{i+1} + y_{i-1} - 2y_i) \\ \dot{z}_i(t) &= -\beta y_i + D_z(z_{i+1} + z_{i-1} - 2z_i) \\ & \qquad i = 1, 2, \cdots, L; \quad D_x, D_y, D_z \geq 0 \end{aligned} \tag{1}$$

$$g(x_i) = S_1 x_i + \tfrac{1}{2}(S_0 - S_1)(|x_i + 1| - |x_i - 1|) \tag{2}$$

where $g(x)$ is a 3-segment piecewise-linear function and S_0 and S_1 are the slopes of the middle and the two side segments, respectively. A periodic boundary condition $x_{i+L}(t) = x_i(t), y_{i+L}(t) = y_i(t), z_{i+L}(t) = z_i(t), i = 0, 1$, is used. Throughout the paper, we use $L = 200$ for our lattice length. Without diffusive coupling ($D_x = D_y = D_z = 0$), (1) are reduced to L independent Chua's circuits. Each single Chua's circuit can exhibit extremely rich behavior, via various bifurcations, as the parameters α and β vary. The simple nonlinear term $g(x)$ plays a key role in all these complicated dynamic variations.

In this paper, we focus on the combination of parameters $\alpha = 9$, $\beta = 19$, and $S_0 = -1.143$, $S_1 = 0.714$. For these parameters, the single Chua's circuit has three asymptotic solutions, one is the unstable fixed point at the origin $x_i = y_i = z_i = 0$, and the other two are the stable limit cycle solutions given in Fig. 1 (another limit cycle can be obtained from Fig. 1 by the symmetrical inversions $x_i, y_i, z_i \to$

$-x_i, -y_i, -z_i$). It should be noticed that though, without coupling, all sites approach one of the two limit cycles (Fig. 1), the phases of different Chua's circuits may be completely different if the initial states are given randomly. Then the total output, $\langle x\rangle(t) = \Sigma_{i=1}^{L} x_i(t)/L$, may be very weak for very large L. (Throughout the paper, we will take $x_i(t)$ as our measurable variable.) Moreover, with coupling (i.e., with nonzero $D_x + D_y + D_z$), the attractors in Fig. 1 may not be asymptotically approached, and various complicated spatially disordered configurations, depending on initial conditions of the system, can be observed, and the total output is weak and unpredictable due to the phase mismatch and the spatial disorder. In the following two sections, we will apply feedback control and noise to drive the individual units to the same attractor with an identical phase.

III. Synchronization of Chua's Circuit Ring to a Limit Cycle with an Identical Phase by Feedback Control

Introducing diffusive couplings in (1) essentially changes the dynamics and the asymptotic state of the system. In Fig. 2, we fix $D_x = 0.5$, $D_y = 6.5$, and $D_z = 0$, and let each site take random initial value uniformly distributed in the range [−0.5, 0.5]. The dynamics and the results are rather interesting. In Fig. 2(a), we plot a snapshot of the system state at $t = 40$, from which two features can be clearly seen: First, some sites take, approximately, one limit cycle motion, some others the other limit cycle, between these two types of motions there remain a small number of sites showing sharp transitions from one type to the other. Second, the cluster lengths of different types of sites vary randomly in space, and spatial disorder is apparent in the configuration. Different random initial conditions may produce alternative realizations of configurations. However, the above general features are kept, independent of the initial condition if the randomness of initial site values is guaranteed. In Fig. 2(b), we plot the total x output

$$\langle x\rangle(t) = \frac{1}{L}\sum_{i=1}^{L} x_i(t) \tag{3}$$

against t; the amplitude of the output oscillation is small due to the spatial disorder. Now we try to synchronize the system sites and make the motions as spatially homogeneous as possible. In [5], Hu *et al.* suggested a pinning scheme to control spatiotemporal chaos by applying feedback to certain local variables. In [7], Kocarev *et al.* suggested a unified approach to control and synchronize chaos. These methods are used here to synchronize our spatiotemporal system. To do this, we modify (1) to

$$\begin{aligned}
\dot{x}_i(t) &= f_{xi} + \lambda_x \left(\sum_{n=1}^{L/I} \delta_{i,nI}\right)[\hat{x}(t) - x_i(t)] \\
\dot{y}_i(t) &= f_{yi} + \lambda_y \left(\sum_{n=1}^{L/I} \delta_{i,nI}\right)[\hat{y}(t) - y_i(t)] \\
\dot{z}_i(t) &= f_{zi} + \lambda_z \left(\sum_{n=1}^{L/I} \delta_{i,nI}\right)[\hat{z}(t) - z_i(t)]
\end{aligned} \tag{4}$$

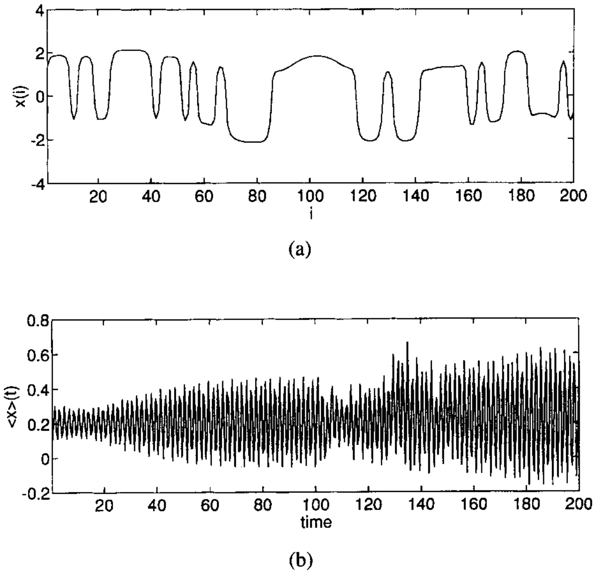


Fig. 2. (a) A snapshot of the state of (1) at $t = 30$, $D_x = 0.5$, $D_y = 6.5$, and $D_z = 0$. The cells take random values from −0.5 to 0.5 initially. Spatial disorder is apparent. (The same D_x, D_y, D_z are taken in the following figures.) (b) The time evolution of $\langle x\rangle(t)$. The total output is rather weak due to the spatial disorder.

where f_{xi}, f_{yi}, and f_{zi} are given in the right sides of (1). $\lambda_{x,y,z}$ are the control coefficients with respect to x, y, z, respectively, $\lambda_{i,nI}$ is the Kronecker delta notation, I is an integer, and $1/I$ is the pinning density. We hope to use as few injections as possible and to synchronize as many cells as possible. The reference trajectory $[\hat{x}(t), \hat{y}(t), \hat{z}(t)]$ is the asymptotic limit cycle trajectory of the single Chua's circuit with a fixed phase shown in Fig. 1. Our task is to force all the sites to make homogeneous collective motions by feeding back the nIth sites, $n = 1, 2, \cdots, L/I$, with respect to the aim orbit $[\hat{x}(t), \hat{y}(t), \hat{z}(t)]$.

Apparently, the number of synchronized cells is considerably influenced by the feedback strength λ and the pinning density $1/I$. A site is regarded to be synchronized to the aim trajectory if

$$\Delta_i = |x_i(t) - \hat{x}(t)| + |y_i(t) - \hat{y}(t)| + |z_i(t) - \hat{z}(t)| < 0.1 \tag{5}$$

for every $t \in [T_S, T_F]$ where T_F is the total time length of simulation, $T_S < T_F$. In our case, we chose $T_S = T_F - 20$. In order to measure the portion of synchronized cells, we define

$$Q = M/L \tag{6}$$

where M is the number of synchronized cells according to (5), and L is the chain length. In Fig. 3, we plot a diagram of the level of synchronization Q as a function of the pinning density and the feedback intensity ($\lambda_x = \lambda_y = \lambda_z = \lambda$). The functions $Q(\lambda, I)$ with I fixed are monotonously increasing with respect to λ, while a saturation is reached for a large λ for each I. Defining $S(I)$ as the saturated $Q(\lambda, I)$, we can see in general $S(I)$ increases as the pinning density $1/I$ increases, however, a certain fluctuation can be observed. The reason for the fluctuation is not quite clear. For very high pinning density ($I \leq 5$ in our case) we get complete synchronization as λ is sufficiently large.

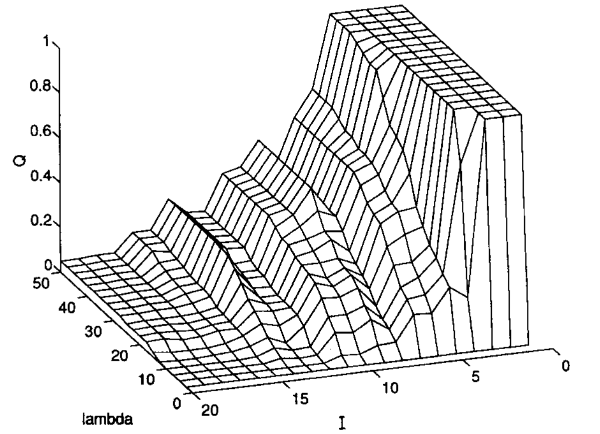


Fig. 3. $Q(\lambda, I)$ versus λ and I. The simulation runs were started from the same random initial conditions for each pair $[\lambda, I]$.

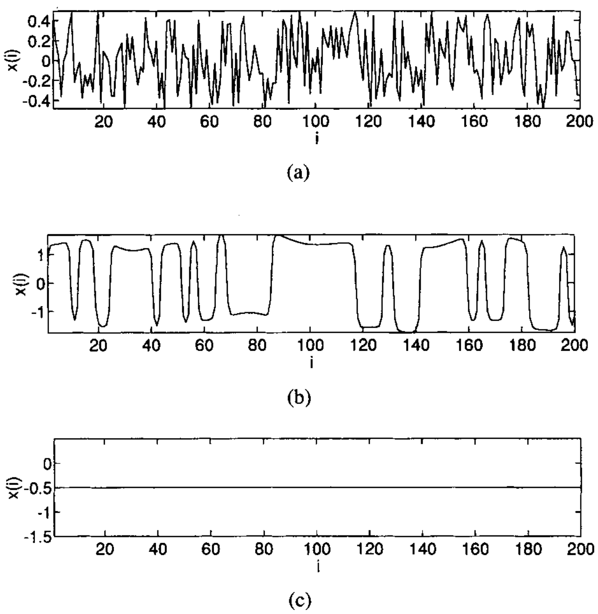


Fig. 4. The snapshots of the state variable x_i of (4) at different times during the system evolution. $\lambda_x = \lambda_y = \lambda_z = \lambda = 20$. Control is applied for $t > 30, I = 4$. (a) $t = 0$, random initial conditions. (b) $t = 20$, state without control. (c) $t = 60$, complete synchronization is established by feedback control.

In Fig. 4, we take $I = 4$ with random initial conditions. After certain transient time, all cells are driven to the same limit cycle, and their phases are perfectly locked to the phase of the aim state. While the pinning density is lower, synchronization may not be complete. Nevertheless, the improvement in synchronization due to the controlling can be still clearly seen. In Fig. 5, we take the same λ as in Fig. 4 and increase I to 6. We find that some cells are left unsynchronized after a very long time (much longer than the relaxation time in Fig. 4). However, after initiating the control, some changes in comparison with Fig. 2(a) appear. First, in Fig. 5, the balance of the numbers of the sites making the two limit cycle motions breaks (this balance holds in Fig. 2(a)), and more sites take the limit cycle motion of Fig. 1 (i.e, the reference trajectory of the feedback control). Second, sites in the same limit cycle attractor fall into a better in-phase status, and spatial homogeneity is established between these sites. The quantity Q measured in Fig. 5 can be up to 50%, while without control the same quantity is down to zero.

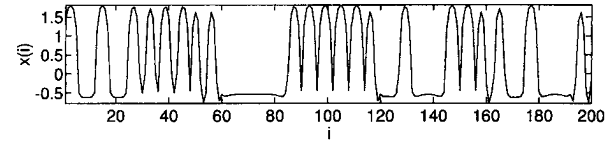


Fig. 5. The same as Fig. 4 with $I = 4$ replaced by $I = 6, t = 200$. Some defects remain after control has been applied for a long time.

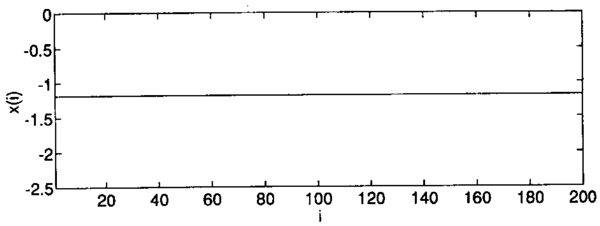
IV. Synchronization of Cells by Control and Noise; Stochastic Resonance (SR)

In Fig. 5, a number of sites are still not synchronized. On one hand, these desynchronized sites may considerably reduce the intensity of the total output, on the other hand, these defects may be fatally harmful for certain practical purposes (e.g., for the quality of products, or for the security of biological bodies, and so on). It has been found that these defects cannot be eliminated by increasing λ. Reducing I (i.e., increasing the pinning density) may effectively wipe out the defects. Nevertheless, in experiments, feedbacks require instant measurements of the system variables and quick responses to the system evolution, and then it is not convenient to make too many feedback injections. It would be of great convenience to make better synchronization without increasing pinning density. Fortunately, these remained defects can be much more conveniently eliminated by properly applying noise. Let us further modify (4) to

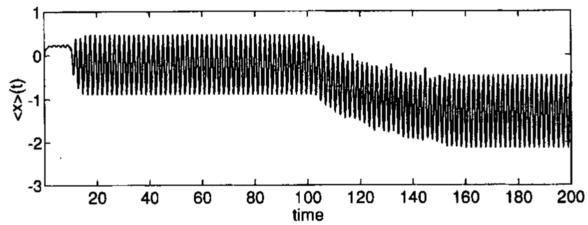
$$\begin{aligned} \dot{x}_i(t) &= \varphi_{xi} + \theta(t)\mu_x N_{xi}(t) \\ \dot{y}_i(t) &= \varphi_{yi} + \theta(t)\mu_y N_{yi}(t) \\ \dot{z}_i(t) &= \varphi_{zi} + \theta(t)\mu_z N_{zi}(t) \\ \theta(t) &= \begin{cases} 1 & t \in [T_1, T_2] \\ 0 & t \in [T_1, T_2] \end{cases} \end{aligned} \tag{7}$$

where $\varphi_{x,y,z;i}$ are the functions given in the right hand sides of (4). $N_{x,y,z;i}(t)$ are random numbers uniformly distributed in the interval $[-1, 1]$, which are kept constant in each simulation time step while totally uncorrelated for different time steps, different sites, and different variables. $\mu_{x,y,z} = \mu$ are the noise intensities, and $[T_1, T_2]$ is the time interval when noise functions. Here, we take noise correlation time equal to the time step of numerical simulations only for computational simplicity. Larger or smaller correlation times of noise do not change the main features of all results of the following parts if the noise correlation time is much smaller than the system relaxation time. The step function $\theta(t)$ indicates that noise will be used only as a tool to eliminate the spatial disorder. After the synchronization is realized, the noise is excluded. Therefore, the array system will not be subject to these random attacks in the final synchronized coherent state.

In Fig. 6(a), the snapshot is taken at $t = 180$, i.e., well after the noise is cleared away. Most of the defects in Fig. 5 are wiped out by the noise, and a perfect synchronization is realized. In Fig. 6(b), we plot the total output $\langle x\rangle(t)$ against time; a periodic oscillation is observed for large t,



(a)



(b)

Fig. 6. Synchronization by using both control and noise. Noise coefficients are $\mu_x = \mu_y = \mu_z = \mu = 5$. The other parameters are the same as in Fig 5. Noise is applied in the time interval $t \in [100, 150]$. (a) Snapshot at $t = 180$. Perfect synchronization is reached after applying noise. (b) The evolution of $\langle x \rangle (t)$. The amplitude of the total output is much larger than that of Fig. 2(b) due to the collective effect.

the amplitude of which is considerably larger than that in Fig. 2(b). In Fig. 6(b), one can interestingly find four time stages. From $t = 0$ to $t = 10$, without control and noise, the total output is rather weak. From $t = 10$ to $t = 100$, control is applied, a rather regular oscillation is induced by forcing though spatial homogeneity is not established (see Fig. 5). However, the oscillation of $\langle x(t) \rangle$ is smaller than that of the aim limit cycle (and located higher than the aim orbit) due to the fact that some cells are still trapped in the basin of the other limit cycle. From $t = 100$ to $t = 150$, one can clearly see that noise migrates the system state towards the aim limit cycle. After $t > 150$, noise is cleared and complete synchronization to the aim trajectory is established.

Noise is usually regarded as a negative factor in practice. It destroys order and coherence, makes events unpredictable and uncontrollable. Recently, in the field of nonlinear science, scientists are more and more aware of the active role played by noise and become more and more interested in this active role. Now, we again find that noise plays a rather desirable role in improving synchronization and coherence. Actually, what we do in Fig. 6 is to exclude unpredictable defects and randomness by applying random force!

Whenever noise plays an active role, there must exist an optimal noise intensity at which the best active role can be achieved, and too small or too large noises are not good for the active effect. This phenomenon is analyzed in great detail in the study of SR [1], [8], [9]. For our synchronization, we find similar results. In Fig. 7, we plot Q against $\mu = \mu_z = \mu_y = \mu_z$ at $I = 6$. The Q–L curve has a nice resonance shape peaked at the optimal noise intensity $\mu \approx 7.5$. Fig. 7 reminds us of the SR phenomenon. We term the behavior of Fig. 7 stochastic resonance in the synchronization of our array. In the early stage, the SR study focused on a model of periodically forced

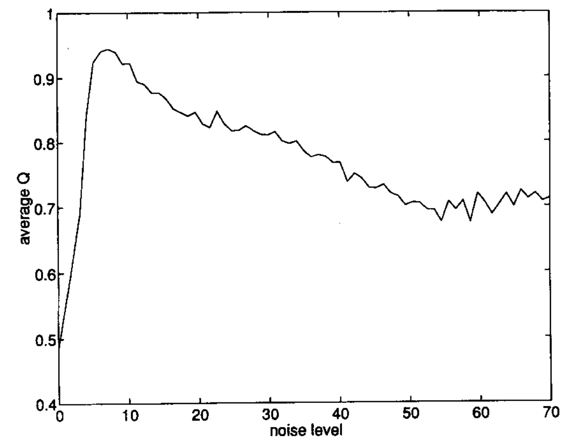


Fig. 7. Q plotted versus μ. $I = 6$. An SR-like response is obvious. Each plot at given μ is obtained by averaging 100 Q-μ curves with different noise signals of the given intensity. The feedback control ($\lambda = 20$) is applied after $t > 10$, while noise is used in the time interval $t \in [30, 60]$.

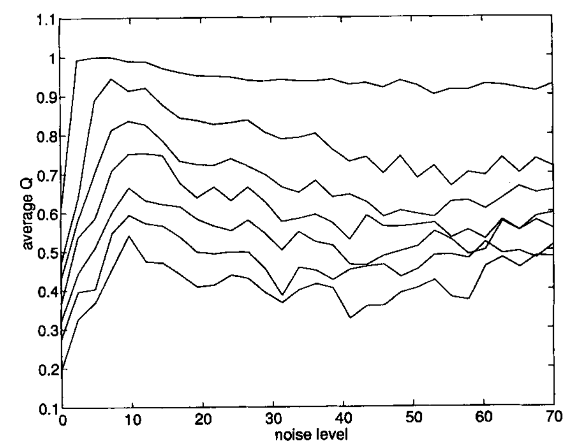


Fig. 8. Q versus μ for various I. All other parameters are the same as in Fig. 7. From the top curve to the bottom one, we take I =5, 6, 7, 8, 9, 10, and 11, respectively. For every pair $[I, \mu]$, the data is obtained by averaging results of 20 runs.

bistable system subject to white noise. Recently, the scope of this study has been very much enlarged [2], [14]. Here, we find a new application of SR.

For a global view on how the pinning density and noise play their roles in synchronizing coupled extended systems, we present Fig. 8 where Q is plotted against μ for various I. Apparent stochastic resonance phenomenon for synchronization is generally observed.

V. Conclusion

In conclusion, we would like to make the following remarks.

The synchronization approach can be used to produce a desirable collective motion of systems of wide usefulness, especially for systems with a great number of degrees of freedom.

In our case, we consider a spatiotemporal system consisting of a large number of identical sites, i.e., Chua's circuit with diffusion-like couplings. This system enjoys some symmetry properties and seems to be simple. However, the dynamics of the system is actually very complicated. In this paper, we focused on the realization of spatially homogeneous state. As each single site has a certain stable state, the corresponding spatially homogeneous state is not necessarily stable for the array system in general because the couplings may induce new instability. Even if its homogeneous state is stable, it is still often very difficult to realize this state from practical random initial conditions since a great number of stable states may exist for the coupled system, and the probability for the system to fall into the attracting basin of the synchronized state may be negligibly small. Our investigation shows that feedback injections at certain local points in space may very effectively enlarge the basin attraction of the homogeneous state and drive sites to the phase-locked status.

The feedback injections with low density may not be enough to induce full synchronization. Some defects may inevitably exist. However, the feedbacks still play a role in making the basin of the synchronized state much larger than the basins of nonsynchronized states. By applying noise, we can force the system to jump between various basins. In this case, noise has more probability to drive the system from small basins to large basins than that from large ones to small ones. This is why noise, together with feedback control, can induce perfect synchronization while many off-synchronization basins exist.

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Ladislav Pivka, for a photograph and biography, see this issue, p. 637.

A. L. Zheleznyak, photograph and biography not available at the time of publication.