ORDERING CHAOS IN CHUA'S CIRCUIT: A SAMPLED-DATA FEEDBACK AND DIGITAL REDESIGN APPROACH*

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In this paper, we develop and apply some digital design and redesign techniques for ordering the chaotic Chua's circuit. The idea of using sampled-data feedback for controlling the circuit was previously suggested [Yang & Chua, 1998], which relies on small sampling periods. We show how this sampled-data feedback control method can be significantly improved, so that large sampling times are allowed, for the same purpose of ordering the nonlinear circuit, from anywhere within the chaotic attractor towards a predesired periodic cycle of the circuit.

1. Introduction

Within the areas of dynamics and control for nonlinear systems, the research on controlling chaos has seen a dramatic increase in the last decade (see e.g. [Chen & Dong, 1998; Chen, 1999; Chen & Ogorzalek, 2000]).

There have been many successful methods for controlling or ordering chaos, many of which make use of some essential properties of chaotic systems such as their sensitivity to initial conditions, the feature of possessing a dense set of unstable periodic orbits of different periods, and the dependence on various bifurcations. In particular, generating chaos via control techniques, when chaos is useful and beneficial, utilizes the very features of chaos and the essence of feedback control principles [Chen & Dong, 1998].

It is now known that controlling chaos in continuous-time systems can be implemented by analog circuits. However, in order to take advantage of the modern high-speed computers and microelectronics, it is more preferable to use

digital controllers instead of analog circuits, particularly in aerospace systems and industries [Dedieu & Ogorzalek, 1994; Yang & Chua, 1998].

The process of converting an existing continuous-time controller to an equivalent discrete-time controller is called *digital redesign* [Shieh *et al.*, 1989, 1991, 1992a, 1992b; Tsai *et al.*, 1989, 1991]. Digital redesign is generally technical, since there are some critical issues such as the sensitive instability problem existing in the redesign when the desired short sampling periods are used in digitalization (i.e. if the sampling time is too short then it can be very CPU time-consuming and may cause instability in its digital version of the control system). Moreover, fast-rate sampling devices can be very expensive or even physically impossible. For these reasons, digital redesign techniques have never been applied to nonlinear systems, with perhaps the only exception [Xu *et al.*, 1996], particularly the numerically extremely sensitive chaotic systems like Chua's circuit.

Digital redesign was first studied in detail by Kuo [1980]. He proposed a discrete-state

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matching method and applied it to a simplified one-axis sky-lab satellite system. Recently, Shieh and his colleagues [Shieh *et al.*, 1989, 1991, 1992a, 1992b; Tsai *et al.*, 1989, 1991] have thoroughly investigated this type of digital redesign and various types of (sub)optimal digital redesign methods, so that the digital redesign technology is greatly advanced.

In this paper, we develop and apply the successful digital redesign techniques to the ordering of the chaotic Chua's circuit. We will show how the sampled-data feedback control method using a small sampling time [Yang & Chua, 1989] can be significantly improved to be an even better digital controller, with large sampling times. Our approach is again to first develop a successful analog controller, for controlling the chaotic trajectories of Chua's circuit, and then to redesign it via advanced digital control technique, so as to control the circuit dynamics from anywhere within its chaotic attractor to a predesired periodic trajectory.

The material presented in this paper is organized as follows. First, a very brief introduction to Chua's circuit is given, under the format to be used in the digital redesign framework of this paper. Then, an existing continuous-time feedback control approach for controlling the chaotic Chua's circuit is described. To that end, an appropriate digitalization of the continuous-time controller is suggested. The digitally controlled Chua's circuit is then studied. The behaviors of the digitally controlled Chua's circuit with different sampling

periods are compared by computer simulations. Finally, a good digital redesign method for improving the critical control performance of a long sampling-period digital controller for the circuit is introduced and simulated.

2. The Chaotic Chua's Circuit

Chua's circuit is a simple electronic circuit that exhibits a wide variety of nonlinear dynamical phenomena such as bifurcations and chaos. This circuit, as shown in Fig. 1(a), can be described by the following set of dynamical equations:

$$\begin{aligned}\frac{dv_1}{dt} &= \frac{1}{C_1} [G(v_2 - v_1) - f(v_1)], \\ \frac{dv_2}{dt} &= \frac{1}{C_2} [G(v_1 - v_2) + i_L], \\ \frac{di_L}{dt} &= -\frac{1}{L} v_2,\end{aligned}\quad (1)$$

where $G = 1/R$, R is a resistor, i_R is the current through the inductor L , v_1 and v_2 are the voltages across the capacitors C_1 and C_2 , respectively, and

$$\begin{aligned}f(v_1) &= G_b v_1 \\ &+ \frac{1}{2} (G_a - G_b) (|v_1 + E| - |v_1 - E|)\end{aligned}\quad (2)$$

with E being the breakpoint voltage of the diode, and $G_b < 0$ and $G_a < 0$ being some appropriate

Fig. 1. Chua's circuit: (a) the circuitry; (b) the piecewise-linear resistance.

constants, representing slopes of the piecewise-linear resistance.

The dimensionless version of the circuit is given by

$$\begin{cases} \dot{x}_1 = \alpha(-x_1 + x_2 - f(x_1)), \\ \dot{x}_2 = x_1 - x_2 + x_3, \\ \dot{x}_3 = -\beta x_2, \end{cases} \quad (3)$$

where $x_1 = v_1/E$, $x_2 = v_2/E$, $x_3 = i_R/(EG)$, $\alpha = C_2/C_1$, and $\beta = 1/(LG) = C_2R^2/L$ by letting $G = C_2$, and

$$\begin{aligned} f(x_1) &= m_0x_1 + 0.5(m_1 - m_0)(|x_1 + 1| - |x_1 - 1|) \\ &= \begin{cases} m_0x_1 + m_1 - m_0 & x_1 \geq 1.0 \\ m_1x_1 & |x_1| \leq 1.0 \\ m_0x_1 - m_1 + m_0 & x_1 \leq -1.0, \end{cases} \end{aligned} \quad (4)$$

in which $m_0 = RG_b$ and $m_1 = RG_a$. This piecewise-linear function is shown in Fig. 1(b), with $E = 1$ therein.

Now, putting the circuit equation into the state-space format, we have the following systems, which are *continuously connected together*:

$$\dot{x}(t) = A_i x(t) + B_i u_i(t), \quad i = 1, 2, 3, \quad (5)$$

with the linear segments contained in (A_i, B_i) for $i = 1, 2, 3$ in (6)–(8), respectively:

$$\begin{aligned} \dot{x}(t) &= \begin{bmatrix} -\alpha(1+m_0) & \alpha & 0 \\ 1 & -1 & 1 \\ 0 & -\beta & 0 \end{bmatrix} x(t) \\ &+ \begin{bmatrix} -\alpha(m_1-m_0) & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} u_1(t), \\ &x_1(t) > 1, \end{aligned} \quad (6)$$

$$\begin{aligned} \dot{x}(t) &= \begin{bmatrix} -\alpha(1+m_1) & \alpha & 0 \\ 1 & -1 & 1 \\ 0 & -\beta & 0 \end{bmatrix} x(t) \\ &+ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} u_2(t), \quad |x_1(t)| \leq 1, \end{aligned} \quad (7)$$

$$\begin{aligned} \dot{x}(t) &= \begin{bmatrix} -\alpha(1+m_0) & \alpha & 0 \\ 1 & -1 & 1 \\ 0 & -\beta & 0 \end{bmatrix} x(t) \\ &+ \begin{bmatrix} \alpha(m_1-m_0) & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} u_3(t), \\ &x_1(t) < -1. \end{aligned} \quad (8)$$

With $\alpha = 9$, $\beta = 14\frac{2}{7}$, $m_0 = -5/7$, $m_1 = -8/7$, $u_1(t) = u_3(t) = (1, 0, 0)^\top$ and $u_2(t) = (0, 0, 0)^\top$, the chaotic trajectory of the circuit, $x_{ci}(t)$, is shown in Figs. 2–4, where the initial point $(-0.1, -0.1, -0.1)^\top$ was used in the simulation. In these figures, the approximate limit cycle $\bar{x}_i(t)$ is calculated by (14) and the continuous controller used is given by (15), as will be seen below. Note that $B_2 u_2(t) = \bar{B}_2 \bar{u}_2(t)$ where $\bar{B}_2 = \text{diag}(0, 1, 1)$ and $\bar{u}_2(t) = (1, 0, 0)^\top$.

3. Continuous Linear Quadratic Tracker for Chua's Circuit

Consider a controllable and observable continuous-time system represented by the following state-space system, which is applicable to each subsystem

Fig. 2. $x_{oi}(t)$: The chaotic trajectory of Chua's circuit, projected onto the x_1 - x_2 plane; $x_{ci}(t)$: The trajectory of Chua's circuit, projected onto the x_1 - x_2 plane, after continuous-time feedback control is applied; $\bar{x}_i(t)$: The trajectory of the approximate limit cycle, projected onto the x_1 - x_2 plane, as the target for tracking.

Fig. 3. $x_{oi}(t)$: The chaotic trajectory of Chua's circuit, projected onto the x_3 - x_2 plane; $x_{ci}(t)$: The trajectory of Chua's circuit, projected onto the x_3 - x_2 plane, after continuous-time feedback control is applied; $\bar{x}_i(t)$: The trajectory of the approximate limit cycle, projected onto the x_3 - x_2 plane, as the target for tracking.

Fig. 4. $x_{oi}(t)$: The chaotic trajectory of Chua's circuit, projected onto the x_3 - x_1 plane; $x_{ci}(t)$: The trajectory of Chua's circuit, projected onto the x_3 - x_1 plane, after continuous-time feedback control is applied; $\bar{x}_i(t)$: The trajectory of the approximate limit cycle, projected onto the x_3 - x_1 plane, as the target for tracking.

(6)–(8) of the circuit:

$$\begin{cases} \dot{x}_c(t) = Ax_c(t) + Bu_c(t), & x_c(0) = x_0, \\ y_c(t) = Cx_c(t), \end{cases} \quad (9)$$

where $x_c(t) \in R^n$, $u_c(t) \in R^m$, and $y_c(t) \in R^p$. Let $p = m$ here for the tracking purpose. For a reference input, $\Gamma(t)$, the optimal state-feedback control

law [Lewis, 1986] that minimizes the performance index

$$J = \int_0^\infty \{ [Cx_c(t) - \Gamma(t)]^\top Q [Cx_c(t) - \Gamma(t)] + u_c^\top(t) Ru_c(t) \} dt, \quad (10)$$

with $Q \geq 0$ and $R > 0$ for system (9), is given by

$$u_c(t) = -K_c x_c(t) + E_c \Gamma(t), \quad (11)$$

where the analog feedback gain $K_c(t) \in R^{m \times n}$ and the forward gain $E_c \in R^{m \times m}$ are

$$K_c = R^{-1} B^\top P \quad \text{and} \quad (12)$$

$$E_c = -R^{-1} B^\top [(A - BK_c)^{-1}]^\top C^\top Q,$$

and P is the solution of

$$A^\top P + PA - PBR^{-1}B^\top P + C^\top QC = 0. \quad (13)$$

Note that implementation of the tracker for Chua's circuit (6)–(8) can be simplified by carrying out most of the work offline.

It has been mathematically proven via the rigorous Lyapunov stability theory [Chen & Dong, 1993a] that the chaotic orbit of Chua's circuit can be controlled to its unstable limit cycle, where the desired feedback control can be applied to the circuit at any time. Therefore, similar mathematical analysis of the controllability of the circuit by the above optimal linear quadratic tracker is not repeated here. Instead, we only show how and how well the tracker works by showing its control of the chaotic orbit to an approximation of the unstable limit cycle of the circuit (since the exact analytic expression for the cycle does not exist). For simplicity, only the second-order approximation is used here for demonstration.

Let $\bar{\Gamma}(t) = (\bar{x}_1(t), \bar{x}_2(t), \bar{x}_3(t))^\top$ be the unstable limit cycle of the circuit. It can be approximated by the following analytic formula [Chen & Dong, 1993a, 1993b]

$$\begin{aligned} \bar{x}_1(t) &\approx a \cos(\xi) \cos(\omega t) - b \sin(\xi) \sin(\omega t) \\ &\quad + c \cos(\xi) \cos(2\omega t) - d \sin(\xi) \sin(2\omega t); \\ \bar{x}_2(t) &\approx e [a \sin(\xi) \cos(\omega t) - b \cos(\xi) \sin(\omega t)] \\ &\quad + f [c \sin(\xi) \cos(2\omega t) - d \cos(\xi) \sin(2\omega t)]; \\ \dot{\bar{x}}_3(t) &= -\beta \bar{x}_2(t), \end{aligned} \quad (14)$$

with $a = 2.6$, $b = 1.2$, $c = d = 0.2$, $e = 0.6$, $f = 0.3$, $\xi = \pi/18$, $\omega = 1.77$, $t = kT$ for $k = 0, 1, 2, \dots$, and T is the sampling period. This second-order approximate orbit is shown also in Figs. 2–4, where $\bar{x}_3(0) = -6$.

Using the tracker gains (12) with $Q = 2 \times 10^3 I_n$ and $R = I_m$ in (10), the desired continuous-time state-feedback controller for tracking the target $\bar{x}(t) = (\bar{x}_1(t), \bar{x}_2(t), \bar{x}_3(t))^T$, which are implemented in the three continuously connected linear subsystems (6)–(8), respectively, are obtained as

$$\begin{aligned}
 u_1(t) = & - \begin{bmatrix} 44.0697 & 2.6119 & -0.0251 \\ 0.6772 & 45.3601 & -6.4634 \\ -0.0065 & -6.4634 & 44.1055 \end{bmatrix} x(t) \\
 & + \begin{bmatrix} 44.7045 & 0.3037 & -0.0473 \\ -0.2653 & 41.9412 & -7.4385 \\ -0.0195 & 7.0973 & 44.0870 \end{bmatrix} \bar{x}(t), \\
 & x_1(t) > 1, \\
 u_2(t) = & - \begin{bmatrix} 45.8580 & 4.9804 & 0.3433 \\ 4.9804 & 45.9862 & -6.3978 \\ 0.3433 & -6.3978 & 44.1152 \end{bmatrix} x(t) \\
 & + \begin{bmatrix} 44.5328 & -3.7224 & 0.3190 \\ 3.7620 & 41.0857 & -7.3755 \\ 0.2843 & 6.9188 & 44.0956 \end{bmatrix} \bar{x}(t), \\
 & |x_1(t)| \leq 1, \\
 u_3(t) = & - \begin{bmatrix} -44.0697 & -2.6119 & 0.0251 \\ 0.6772 & 45.3601 & -6.4634 \\ -0.0065 & -6.4634 & 44.1055 \end{bmatrix} x(t) \\
 & + \begin{bmatrix} -44.7045 & -0.3037 & 0.0473 \\ -0.2653 & 41.9412 & -7.4385 \\ -0.0195 & 7.0973 & 44.0870 \end{bmatrix} \bar{x}(t), \\
 & x_1(t) < -1.
 \end{aligned} \tag{15}$$

The resulting closed-loop controlled system is then given by

$$\begin{cases} \dot{x}_c(t) = [A - BK_c]x_c(t) + BE_c\bar{\Gamma}(t), \\ y_c(t) = Cx_c(t), \end{cases} \tag{16}$$

where $C = I_p$. From the simulation point of view, one may simulate the time responses of a controlled

system by using the following discretizations [Lewis, 1992]:

$$\begin{cases} x_c(k_f T_f + T_f) = g(A - BK_c, T_f) x_c(k_f T_f) \\ \quad + h(A - BK_c, B, T_f) E_c \Gamma(k_f T_f) \\ y_c(k_f T_f + T_f) = C x_c(k_f T_f + T_f), \end{cases} \tag{17}$$

for $k_f = 0, 1, 2, \dots$, where $g(\cdot)$ and $h(\cdot)$ represent some appropriate functions of $(\cdot)$, and $\Gamma(k_f T_f)$ is to be determined to satisfy the tracking objective, $y_c(k_f T_f + T_f) = \bar{\Gamma}(k_f T_f + T_f)$. For simplicity of notation, let

$$G_c = g(A - BK_c, T_f) = e^{(A - BK_c)T_f}$$

and

$$\begin{aligned} H_c &= h(A - BK_c, B, T_f) \\ &= (e^{(A - BK_c)T_f} - I_n)(A - BK_c)^{-1}B, \end{aligned}$$

in which $(e^{(A - BK_c)T_f} - I_n)(A - BK_c)^{-1}$ as a product is always well defined, as can be easily verified by expanding the exponential term in Taylor series. The same remark applies to similar expressions below, throughout this paper.

Thus, based on the discrete final-value theorem, at the steady-state

$$\begin{aligned} \lim_{k_f \rightarrow \infty} y_c(k_f T_f + T_f) \\ = \lim_{z \rightarrow 1} (1 - z^{-1}) Z\{y_c(k_f T_f + T_f)\}, \end{aligned}$$

where $Z\{\cdot\}$ denotes the Z -transform, we have

$$\Gamma(k_f T_f) = \bar{\Gamma}(k_f T_f + T_f) \tag{18}$$

and

$$E_c = [C(I_n - G_c)^{-1}H_c]^{-1}. \tag{19}$$

Regarding the above discretizations, some remarks are in order. First, the gain E_c obtained in (19) is based on the discrete final-value theorem at steady-state, but the gain E_c shown in (12) is derived based on the optimal solution at all times. Second, when the desired gain K_c is used and yet the desired $\Gamma(k_f T_f)$ in (18) is taken as $\Gamma(k_f T_f) = \bar{\Gamma}(k_f T_f)$, then it will yield $y_c(k_f T_f + T_f) = \bar{\Gamma}(k_f T_f)$ which is a T_f -shift of the correct result. Therefore, for a large sampling period T_f , the simulated system output $y_c(k_f T_f)$ will be far away from the

actual output of the given system — even if the controller gain K_c is well designed based on the tracking objective. Notably, for a chaotic system the output responses often have rapid switchings and runnings, so that the difference between $\bar{\Gamma}(k_f T_f)$ and $\bar{\Gamma}(k_f T_f + T_f)$ is significantly large. As a result, the control performance is dramatically degraded, regardless of the rigorous design of the original continuous-time tracker. This is especially true for sampled-data feedback control systems, where the digital controller is held by a zero-order-hold device for a relatively long period of time. This explains why the sampling time chosen by Yang and Chua [1998] is very small ($T = 0.005$ s), where no optimality or advanced redesign technique was used. For comparison, the above digital design based on the continuous-time optimal controller showed improvement over the design method of Yang and Chua [1998]: with $T = 0.015$ s, their method fails, but our simulation is successful even for $T = 0.2$ s. These results are shown in Figs. 2–4. In our simulations, digital control gains (33) and (34) given below were used, with $x_c(0) = x_0 = (-0.1, -0.1, -0.1)^\top$. A further improvement of this digital design via digital redesign will be discussed in the next section.

4. Digital Redesign for Ordering Chaos in Chua's Circuit

Consider a controllable and observable continuous-time system described by

$$\begin{cases} \dot{x}_c(t) = A x_c(t) + B u_c(t), & x_c(0) = x_0, \\ y_c(t) = C x_c(t), \end{cases} \quad (20)$$

where $x_c(t) \in R^n$ is the state, $u_c(t) \in R^m$ is the control input, $y_c(t) \in R^p$ is the output, and (A, B, C) are constant matrices of appropriate dimensions. Let the controller be

$$u_c(t) = -K_c x_c(t) + E_c \Gamma(t), \quad (21)$$

where $\Gamma(t)$ is a reference input, which is assumed to be a piecewise-constant signal, and $\Gamma(t) = \Gamma(kT)$ with $T > 0$ being the sampling time, $kT \leq t < (k+1)T$, $k = 0, 1, 2, \dots$. The controlled system is

$$\dot{x}_c(t) = A_c x_c(t) + B E_c \Gamma(t), \quad x_c(0) = x_0, \quad (22)$$

where $A_c = A - B K_c$. The corresponding discrete-time model for $\Gamma(t) = \Gamma(kT)$ with $kT \leq t < (k+1)T$ in (22) is

$$x_c(kT + T) = G_c x_c(kT) + H_c E_c \Gamma(kT), \quad (23)$$

where

$$G_c = e^{A_c T} \quad \text{and} \quad H_c = (G_c - I_n) A_c^{-1} B.$$

Also, let the fast sampled discrete-time model (22) for $T_N = T/N$, where N is a positive integer, and $\Gamma(t) = \Gamma(kT)$ for $kT \leq t < (k+1)T$, be written as

$$\begin{aligned} x_c(kT + iT_N) &= G_c^{(i)} x_c(kT) + H_c^{(i)} \Gamma(kT), \\ i &= 1, 2, \dots, N, \end{aligned} \quad (24)$$

where

$$G_c^{(i)} = (e^{A_c T_N})^i \quad \text{and} \quad H_c^{(i)} = (G_c^{(i)} - I_n) A_c^{-1} B.$$

Moreover, let the state equation of the continuous-time system (20) with a piecewise-constant input, $u_d(t)$, be represented by

$$\begin{cases} \dot{x}_d(t) = A x_d(t) + B u_d(t), & x_d(0) = x_0, \\ y_d(t) = C x_d(t), \end{cases} \quad (25)$$

where $u_d(t) = u_d(kT)$ for $kT \leq t < (k+1)T$. Then, let the digital controller for system (25), with $\Gamma(t) = \Gamma(kT)$ for $kT \leq t < (k+1)T$, be

$$u_d(t) = -K_d x_d(kT) + E_d \Gamma(kT), \quad (26)$$

where $K_d \in R^{m \times n}$ and $E_d \in R^{m \times m}$ are the feedback and feedforward digital gains, respectively.

Thus, the designed closed-loop sampled-data system (25) and (26) is

$$\dot{x}_d(t) = A x_d(t) - B K_d x_d(kT) + B E_d \Gamma(kT). \quad (27)$$

A zero-order-hold device is used for (27). The corresponding discrete-time model of the sampled-data feedback control system (27) is

$$x_d(kT + T) = (G - H K_d) x_d(kT) + H E_d \Gamma(kT), \quad (28)$$

where

$$G = e^{AT} \quad \text{and} \quad H = (G - I_n) A^{-1} B.$$

The process of finding digital controller gains (K_d, H_d) in (26) from the analog gains (K_c, E_c) in (21), so that the closed-loop state $x_d(t)$ in (27) can closely match the closed-loop state in (22), is called the *state-matching digital redesign*.

To match the closed-loop state $x_d(t)$ in (27) to the closed-loop state in (22) at each sampling instant, $t = kT$, it is required that

$$G_c = G - HK_d \quad \text{and} \quad H_c E_c = H E_d, \quad (29)$$

where the $m \times n$ feedback gain K_d and the $m \times m$ feedforward gain E_d are unknown matrices to be determined. To explore the intersampling behavior of the original analogously designed system (22) and the digitally redesigned system (27), we first briefly review the development of the Chebyshev quadrature method [Shieh *et al.*, 1998] as follows.

The discrete-time model of the sampled-data system (22) for $\Gamma(t) = \Gamma(kT)$, with $kT \leq t < (k+1)T$, can be written as

$$\begin{aligned} x_c(kT + T) &= Gx_c(kT) \\ &- \int_{kT}^{kT+T} e^{A(kT+T-\lambda)} BK_c x_c(\lambda) d\lambda \\ &+ H E_c \Gamma(kT). \end{aligned} \quad (30)$$

To reduce the error in the convolution integral over each time interval T , Shieh *et al.* [1998] used a Chebyshev quadrature formula to evaluate the integral term in (30). The general Chebyshev quadrature formula has the form

$$\int_a^b w(\lambda) f(\lambda) d\lambda \approx W \sum_{i=0}^N f(\lambda_i),$$

where $w(\lambda)$ is a constant weighting function defined on $[a, b]$, and W is the weighting factor

$$W = \frac{1}{N+1} \int_a^b w(\lambda) d\lambda,$$

with $f(\lambda_i)$ being the values of $f(\cdot)$ at the partition points $\lambda_i = a + i(b-a)/N$, $i = 0, 1, \dots, N$.

The above formulation can be extended to the matrix convolution integral shown in (30) as follows:

$$\begin{aligned} &\int_{kT}^{kT+T} e^{A(kT+T-\lambda)} BK_c x_c(\lambda) d\lambda \\ &= \frac{1}{N+1} H K_c \sum_{i=0}^N x_c(kT + iT_N), \end{aligned}$$

where the intersampling state, $x_c(kT + iT_N)$, for the piecewise-constant reference input, $\Gamma(t) = \Gamma(kT)$, is shown in (24). Substituting this equation into (30) yields

$$\begin{aligned} x_c(kT + T) &= Gx_c(kT) - H \left(\frac{1}{N+1} K_c \sum_{i=0}^N G_c^{(i)} \right) x_c(kT) \\ &+ H \left(I_m - \frac{1}{N+1} K_c \sum_{i=0}^N H_c^{(i)} \right) E_c \Gamma(kT). \end{aligned} \quad (31)$$

Now, let the closed-loop states $x_c(kT + T)$ and $x_c(kT)$ in (31), which involve the intersampling states $x_c(kT + iT_N)$ for $i = 0, 1, \dots, N$, be equal to the desired closed-loop state $x_d(kT + T)$ and $x_d(kT)$ in (28), respectively. Then we have

$$\begin{aligned} G - HK_d &= G - H \left(\frac{1}{N+1} K_c \sum_{i=0}^N G_c^{(i)} \right), \\ H E_d &= H \left(I_m - \frac{1}{N+1} K_c \sum_{i=0}^N H_c^{(i)} \right) E_c. \end{aligned} \quad (32)$$

Fig. 5. (a) Continuous-time control system; (b) Digital control system.

Solving Eq. (32) yields the desired digital control gains, as

$$K_d = \frac{1}{N+1} K_c \sum_{i=0}^N G_c^{(i)} = \frac{1}{N+1} K_c ([G_c^{(1)} - I_n]^{-1} (G_c - I_n) + G_c),$$

$$E_d = \left(I_m - \frac{1}{N+1} K_c \sum_{i=0}^N H_c^{(i)} \right) E_c.$$

To this end, letting $N \rightarrow \infty$ leads to

$$\begin{aligned} K_d &= \lim_{N \rightarrow \infty} \frac{1}{N+1} K_c ([G_c^{(i)} - I_n]^{-1} (G_c - I_n) + G_c) \\ &= K_c \lim_{N \rightarrow \infty} \left[\frac{N+1}{N} \left(A_c T + \frac{1}{2!} \frac{(A_c T)^2}{N} + \dots \right) \right]^{-1} (G_c - I_n) \\ &= K_c (A_c T)^{-1} (G_c - I_n). \end{aligned} \quad (33)$$

and

$$\begin{aligned} E_d &= E_c - \lim_{N \rightarrow \infty} \frac{1}{N+1} K_c \left(\sum_{i=0}^N G_c^{(i)} - (N+1)I_n \right) A_c^{-1} B E_c \\ &= E_c + \left(K_c - \lim_{N \rightarrow \infty} \frac{1}{N+1} K_c \sum_{i=0}^N G_c^{(i)} \right) A_c^{-1} B E_c \\ &= E_c + (K_c - K_d) A_c^{-1} B E_c. \end{aligned} \quad (34)$$

It should be remarked that the integral in (30) uses N intersampling states of the closed-loop system (22) and the accuracy of the integration depends on the number of intersampling states involved. Since infinitely many sampling states ($N \rightarrow \infty$) are used to determine the digital control gains in (28), it can be expected that the digitally redesigned sampled-data state $x_d(t)$ in (27) is able to capture the intersampled behavior of the original continuous-time controlled state $x_c(t)$ in (22), for the piecewise-constant reference input $\Gamma(t) = \Gamma(kT)$, for all $k = 0, 1, 2, \dots$.

Consider the special case of $\text{rank}(B) = n$, which is the dimension of the state $x_c(t)$. An explicit formula, named the full-rank formula, for state matching $x_c(kT) = x_d(kT)$, $k = 0, 1, 2, \dots$, is given by

$$K_d = H^{-1}(G - G_c) \quad \text{and} \quad E_d = H^{-1}(H_c E_c).$$

Based on the observation described above, the K_d in (32) and the E_d in (33) used in this paper work also for the general case where $\text{rank}(B)$ may not equal to n . It captures the intersampled behavior of the original continuous-time controlled state $x_c(t)$ shown in (21). However, the tracker obtained

based on the full-rank formula does not have these two advantages in general.

Before leaving this section to the next section on simulation studies and comparison, another pair of digitally redesigned state-feedback and feed-forward gain matrices $(\hat{K}_d, \hat{E}_d)$, derived in [Shieh et al., 1992], are listed here:

$$K_d = (I_m + K_c H^{(\nu)})^{-1} K_c G^{(\nu)}$$

$$\text{and} \quad E_d = (I_m + K_c H^{(\nu)})^{-1} E_c, \quad (35)$$

where

$$G^{(\nu)} = e^{A\nu T}$$

$$\text{and} \quad H^{(\nu)} = \int_0^T e^{A\nu\tau} B d\tau = (G^{(\nu)} - I_n) A^{-1} B.$$

In these formulas, the choice of the tuning parameter ν in (35) depends on the specified sampling interval T and the desired closeness of the trajectory $x_d(t)$ of the redesigned digital system (27) to $x_c(t)$, the orbit of the original continuous-time system (22). The following performance index is suggested as a design criterion for the selection of this

turning parameter, ν :

$$J(\nu) = \sum_{i=1}^n \int_0^{t_f} |x_{ci}(t) - x_{di}(t)| dt,$$

where t_f is the desired finite terminal time for the minimization.

5. Computer Simulation and Comparison Examples

For comparison purpose, we have chosen a long sampling period, $T = 0.015$ s, for which the sampled-data control method of Yang and Chua [1998], which was discussed in Sec. 3, fails.

With a fairly long sampling time, $T = 0.2$ s, the digitally controlled Chua's circuit is given by the following three continuously connected subsystems:

$$\begin{aligned} \dot{x}_d(t) = & \begin{bmatrix} -\alpha(1+m_0) & \alpha & 0 \\ 1 & -1 & 1 \\ 0 & -\beta & 0 \end{bmatrix} x_d(t) \\ & + \begin{bmatrix} -\alpha(m_1-m_0) & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} u_d(kT), \\ & x_1(t) > 1, \end{aligned} \quad (36)$$

where

$$\begin{aligned} u_d(kT) = & - \begin{bmatrix} 1.2774 & 0.2451 & 0.0414 \\ 0.0287 & 4.8760 & 0.0919 \\ -0.0028 & -1.4973 & 4.7466 \end{bmatrix} x_d(kT) \\ & + \begin{bmatrix} 1.9403 & -1.8871 & 0.0395 \\ -0.9633 & 5.3148 & -0.9046 \\ 0.0096 & 11.5726 & 4.7500 \end{bmatrix} \\ & \times \bar{\Gamma}(kT+T) \\ \dot{x}_d(t) = & \begin{bmatrix} -\alpha(1+m_1) & \alpha & 0 \\ 1 & -1 & 1 \\ 0 & -\beta & 0 \end{bmatrix} x(t) \\ & + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} u_d(kT), \quad |x_1(t)| \leq 1, \end{aligned} \quad (37)$$

where

$$\begin{aligned} u_d(kT) = & - \begin{bmatrix} 5.0589 & 0.9363 & 0.1562 \\ 0.1217 & 4.8884 & 0.0932 \\ 0.1326 & -1.4669 & 4.7529 \end{bmatrix} \\ & \times x_d(kT) \\ & + \begin{bmatrix} 3.8055 & -7.0721 & 0.1507 \\ -0.9039 & 5.1533 & -0.9033 \\ 0.0708 & 11.2164 & 4.7561 \end{bmatrix} \\ & \times \bar{\Gamma}(kT+T) \\ \dot{x}_d(t) = & \begin{bmatrix} -\alpha(1+m_0) & \alpha & 0 \\ 1 & -1 & 1 \\ 0 & -\beta & 0 \end{bmatrix} x(t) \\ & + \begin{bmatrix} \alpha(m_1-m_0) & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} u_d(kT), \\ & x_1(t) < -1, \end{aligned} \quad (38)$$

where

$$\begin{aligned} u_d(kT) = & - \begin{bmatrix} -1.2774 & -0.2451 & -0.0414 \\ 0.0287 & 4.8760 & 0.0919 \\ -0.0028 & -1.4973 & 4.7466 \end{bmatrix} \\ & \times x_d(kT) \\ & + \begin{bmatrix} -1.9403 & 1.8871 & -0.0395 \\ -0.9633 & 5.3148 & -0.9046 \\ 0.0096 & 11.5726 & 4.7500 \end{bmatrix} \\ & \times \bar{\Gamma}(kT+T). \end{aligned}$$

The initial condition used is $x_d(0) = (-0.1, -0.1, -0.1)^\top$, and the reference is the second-order approximation of the unstable limit cycle given in Sec. 3 above. The control performance under this digitally redesigned controller is visualized by Figs. 6–8, while Figs. 9–11 are the control input signals.

For comparison, the simulation results obtained based on the parameter tuning approach (35), with $\nu = 0.9$ determined, has also been verified, which are about the same as those obtained in Figs. 6–8. The main difference is that the tuning method relies on a trial-and-error procedure to find

Fig. 6. $x_{ci}(t)$: The trajectory of Chua's circuit, projected onto the x_1 - x_2 plane, after continuous-time feedback control is applied; $x_{di}(t)$: The trajectory of Chua's circuit, projected onto the x_1 - x_2 plane, after digitally redesigned control is applied, with $T = 0.2$ s.

Fig. 7. $x_{ci}(t)$: The trajectory of Chua's circuit, projected onto the x_3 - x_2 plane, after continuous-time feedback control is applied; $x_{di}(t)$: The trajectory of Chua's circuit, projected onto the x_3 - x_2 plane, after digitally redesigned control is applied, with $T = 0.2$ s.

a good parameter ν , while the redesign method suggested and used in this paper provides a systematic approach.

All the simulation results have shown that the digitally redesigned sampled-data feedback control laws are effective in guiding the chaotic trajectories of the chaotic Chua's circuit to the intended (approximation of) unstable limit cycle, even when the sampling period is very long ($T = 0.2$ s in these simulations).

Fig. 8. $x_{ci}(t)$: The trajectory of Chua's circuit, projected onto the x_3 - x_1 plane, after continuous-time feedback control is applied; $x_{di}(t)$: The trajectory of Chua's circuit, projected onto the x_3 - x_1 plane, after digitally redesigned control is applied, with $T = 0.2$ s.

Fig. 9. $u_{c1}(t)$: Control signal in continuous-time feedback control; $u_{d1}(t)$: Control signal in digitally redesigned feedback control; $u_{d1}(0) \approx 1/12 u_{c1}(0)$ which overcomes the saturation problem.

6. Conclusions

An approach to ordering the chaotic Chua's circuit via digital redesign of an existing analog controller, using a relatively long sampling period, has been suggested and simulated in this paper with comparisons. It has been shown that when the sampling period is relatively long, some existing sampled-data feedback control method fails, but the digitally redesigned method used in this paper

Fig. 10. $u_{c2}(t)$: Control signal in continuous-time feedback control; $u_{d2}(t)$: Control signal in digitally redesigned feedback control; $u_{d2}(0) \approx 1/12 u_{c2}(0)$ which overcomes the saturation problem.

Fig. 11. $u_{c3}(t)$: Control signal in continuous-time feedback control; $u_{d3}(t)$: Control signal in digitally redesigned feedback control; $u_{d3}(0) \approx 1/9 u_{c3}(0)$ which overcomes the saturation problem.

succeeds very well. The main reason is that during a long sampling period, a chaotic orbit usually has some abrupt qualitative changes, so the controller has to be effective enough to handle such significant variations or sensitivity of chaos. Our digital redesign method has proven successful for Chua's chaotic circuit in this paper. It is our hope that this digital redesign method can also be applied to other continuous-time chaotic dynamical systems.

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