PERIODIC ORBITS AND BIFURCATIONS IN ONE-DIMENSIONAL ARRAYS OF CHUA'S CIRCUITS

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ABSTRACT

The global dynamic behavior of one-dimensional (1D) arrays of nonlinear oscillators is investigated through a method based on the application of some spatio-temporal spectral techniques. As a case-study 1D arrays of Chua's circuits are considered. The method consists in the following three fundamental steps: a) the set of all stable and unstable limit cycles is determined via the describing function technique; b) an accurate characterization of each limit cycle is obtained through a spatio-temporal harmonic balance (HB) technique, that exploits as input parameter the single harmonic approximation, provided by the describing function technique; c) limit cycle stability and bifurcations are studied by computing the Floquet's multipliers, via a HB based technique.

1. INTRODUCTION

Arrays of nonlinear oscillators are cellular nonlinear networks (CNNs) that present interesting potential applications for solving several computational problems. They are also useful for modelling stationary and wave phenomena in many disciplines, ranging from physics to biology (see [1]).

Such arrays are described by large systems of nonlinear locally coupled differential equations and they can exhibit a complex dynamic behavior. A complete study of their dynamics would require to classify, for given sets of parameters, all the attractors and possibly to estimate their domains of attraction.

Time-domain numerical simulation has allowed to discover several spatio-temporal dynamic phenomena in nonlinear dynamic arrays, but it is not suitable for discovering and classifying all the attractors of a high-dimensional network. In fact the global dynamic analysis, through the sole numerical simulation, would require to identify for each choice of network parameters all sets of initial conditions that converge to different attractors. This would be a formidable, and practically impossible task. In addition time-domain numerical simulation is unsuitable for detecting unstable invariant limit sets (like unstable periodic orbits) that could play a role in the description of the dynamic behavior.

Recently, some harmonic balance (HB) based techniques have been applied to the study of the dynamic behavior of space-invariant cellular nonlinear networks (see [2]-[9]). In particular in [8, 9] it has been shown that a suitable extension of the describing function technique allows to predict and classify some significant spatio-temporal phenomena.

In this paper we present a method, based on spatio-temporal spectral techniques, that allows to investigate the global dynamic behavior of arrays of nonlinear oscillators. It is illustrated by considering as case-study a one-dimensional (1D) array composed by N Chua's circuit [10]. The proposed method consists of three fundamental steps. The first one is the application of the describing function (DF) technique, that consists in the following two sub-steps: a) each admissible limit cycle is represented by means of three parameters: the amplitude and the phase of the first harmonic and a constant term; b) the set of all stable and unstable periodic limit cycles is determined by finding all the solutions of a system of $3N$ nonlinear equations. We remark that such a number of equations is reasonably small and can be numerically solved even for large arrays; in addition the space-invariant structure allows us to analytically derive several results. The second step is the accurate characterization of each limit cycle through a spatio-temporal HB technique, that exploits as input parameter the single harmonic approximation, provided by the describing function technique. Finally the third step is the stability and bifurcation analysis through the computation of limit cycle Floquet's multipliers, via the HB technique presented in [11].

With respect to the spectral and time-domain techniques presented in other works (see [1]-[9]) the pro-

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posed method presents at least two advantages: a) it allows us to identify all the periodic (stable and unstable) invariant limit sets, even for networks composed by a large number of cells; b) it makes possible a rigorous and accurate bifurcation analysis, that is prevented by the sole use of the DF technique [8, 9].

2. ONE-DIMENSIONAL ARRAY OF CHUA'S CIRCUITS

We consider a network composed by a finite number (N) of Chua's circuits and described by the following system of normalized equations reported in [12]:

$$\begin{aligned} \dot{x}_k &= \alpha[y_k - x_k - n(x_k)] + d_1 x_{k-1} + d_2 x_{k+1} - 2d x_k \\ \dot{y}_k &= x_k - y_k + z_k \\ \dot{z}_k &= -\beta\, y_k \end{aligned} \tag{1}$$

The parameters α and β are defined in [10] whereas d_1 and d_2 represent the coupling coefficients, that are assumed to be positive; $n(x_k)$ denotes the well known nonlinear memoryless resistance of the Chua's diode (see [10]).

For the sake of the simplicity we fix the boundary conditions to $x_0(t) = x_{N+1}(t) = 0$. We also assume, according to [13], that the function $n(x_k)$ admits of the following cubic approximation, that guarantees that each uncoupled cell possesses three equilibrium points, located at $x_k = 0,\ \pm 1.5$

$$n(x_k) = -\frac{8}{7}x_k + \frac{4}{63}x_k^3 \tag{2}$$

According to [14], the parameters α and β are chosen in such a way that, in absence of coupling, the k-th cell exhibits the following invariant limit sets (for example $\alpha = 8$ and $\beta = 15$):

- three unstable equilibrium points (denoted by $\boldsymbol{P}_k^+$, $\boldsymbol{P}_k^-$, $\boldsymbol{P}_k^0$ and corresponding to $x_k = \pm 1.5$ and to $x_k = 0$ respectively);
- two asymmetric stable limit cycles (denoted by $\boldsymbol{A}_k^+$ and $\boldsymbol{A}_k^-$) mainly lying in the regions $x_k > 1$ and $x_k < -1$ respectively;
- one stable symmetric limit cycle (denoted by $\boldsymbol{S}_k^s$);
- one unstable symmetric limit cycle (denoted by $\boldsymbol{S}_k^u$).

3. ANALYSIS OF THE GLOBAL DYNAMICS

As pointed out in the Introduction, the first step for the analysis of the global dynamics is the application of the describing function technique to system (1).

We represent the state $x_k(t)$ through a bias term and a single temporal harmonic with suitable amplitudes and phases:

$$x_k(t) \approx \hat{x}_k(t) = A_k + B_k \sin\left(\hat{\omega}t + \sum_{j=1}^{k-1}\eta_j\right) \tag{3}$$

where A_k denotes the bias, B_k the amplitudes of the first harmonic and η_i indicates the phase shift between $\hat{x}_{i+1}(t)$ and $\hat{x}_i(t)$.

Then we compute the output of the nonlinear function $n(\cdot)$ reported in (2) when the input is (3). It is obtained:

$$\begin{aligned} n(\hat{x}_k(t)) &= n_{0k}(A_k, B_k) \\ &+ n_{1k}(A_k, B_k)\sin\left(\hat{\omega}t + \textstyle\sum_{j=1}^{k-1}\eta_j\right) \\ &+ n_{2k}(A_k, B_k)\cos\left[2\left(\hat{\omega}t + \textstyle\sum_{j=1}^{k-1}\eta_j\right)\right] \\ &+ n_{3k}(A_k, B_k)\sin\left[3\left(\hat{\omega}t + \textstyle\sum_{j=1}^{k-1}\eta_j\right)\right] \end{aligned} \tag{4}$$

where:

$$\begin{aligned} n_{0k}(A_k, B_k) &= A_k\left(-\tfrac{8}{7} + \tfrac{4}{63}A_k^2 + \tfrac{2}{21}B_k^2\right) \\ n_{1k}(A_k, B_k) &= B_k\left(-\tfrac{8}{7} + \tfrac{4}{21}A_k^2 + \tfrac{1}{21}B_k^2\right) \\ n_{2k}(A_k, B_k) &= -\tfrac{2}{21}A_k B_k^2 \\ n_{3k}(A_k, B_k) &= -\tfrac{1}{63}B_k^3 \end{aligned} \tag{5}$$

We proceed as follows. For each k we transform (1) into a scalar third order differential equation, only involving $x_k(t)$ and $n(x_k(t))$. Then by substituting expressions (3) and (5) for $x_k(t)$ and $n(x_k(t))$ and by neglecting the higher order harmonics we derive a set of $3N$ nonlinear algebraic equations in the $3N$ unknowns, $A_1, ..., A_N, B_1, ..., B_N, \eta_1, ..., \eta_{N-1}, \hat{\omega}$. The set of $3N$ equations is reported below. Due zero boundary conditions, it is assumed $A_0 = B_0 = A_{N+1} = B_{N+1} = 0$.

$$\begin{aligned} L(0)A_k &= -\,\alpha\, n_{0k}(A_k, B_k) + d_1\, A_{k-1} \\ &\quad + d_2\, A_{k+1} \end{aligned} \tag{6}$$

$$\begin{aligned} \mathbf{Re}[L(j\hat{\omega})]B_k &= -\alpha\, n_{1k}(A_k, B_k) + d_1 B_{k-1}\cos(\eta_{k-1}) \\ &\quad + d_2\, B_{k+1}\cos(\eta_k) \end{aligned} \tag{7}$$

$$\begin{aligned} \mathbf{Im}[L(j\hat{\omega})]B_k &= -\, d_1\, B_{k-1}\, \sin(\eta_{k-1}) \\ &\quad + d_2\, B_{k+1}\, \sin(\eta_k) \qquad 1 \le k \le N \end{aligned} \tag{8}$$

$$L(s) = \frac{s^3 + s^2(1+2d+\alpha) + s(\beta+2d) + (2d+\alpha)\beta}{s^2+s+\beta} \quad (9)$$

The space-invariant structure of equations (6)-(8) allows us to analytically derive the phase shifts η_k and the frequency $\hat{\omega}$.

In fact $\sin\eta_k$ can be expressed through some algebraic manipulations as a function of A_k, B_k and $\hat{\omega}$:

$$\sin\eta_k = \frac{\mathbf{Im}[L(j\hat{\omega})]}{d_2 B_k B_{k+1}} \sum_{l=1}^{l=k} \left(\frac{d_1}{d_2}\right)^{k-l} B_l^2; \quad 1 \le k \le N-1 \quad (10)$$

Then by substituting (10) (with $k = N-1$) into (8) (with $k = N$ and $B_{N+1} = 0$) the following equation is obtained:

$$\mathbf{Im}[L(j\hat{\omega})] \left(\sum_{l=1}^{l=N} \left(\frac{d_1}{d_2}\right)^{N-l} B_l^2\right) = 0 \quad (11)$$

Since $d_1/d_2 > 0$, the above equation is satisfied if and only if

$$\mathbf{Im}[L(j\hat{\omega})] = 0 \quad (12)$$

Owing to (12), from equations (10) and (8) (with $k = N$ and $B_{N+1} = 0$) we derive that only two values of η_k are possible:

$$\eta_k = \begin{cases} 0 \\ \pi \end{cases} \quad 1 \le k \le N-1 \quad (13)$$

Equation (12) also allows us to determine the set of possible oscillation frequencies and reveals that such frequencies do not depend on the number of cells of the network.

By using equations (12) and (13), system (6)-(8) can be simplified and the total number of unknown can be reduced to $2N$, i.e. $A_1, ..., A_N, B_1, ..., B_N$.

An accurate analysis of equations (6)-(8) permits to identify the whole set of periodic invariant limit sets, for small couplings. The results can be summarized as follows. For each one of the 2^{N-1} admissible phase shifts $\{\eta_1, ..., \eta_{N-1}\}$, $\eta_k \in \{0, \pi\}$, the system (1) exhibits:

- 2^N asymmetric limit cycles $\{\boldsymbol{A}_1^{\pm}, \boldsymbol{A}_2^{\pm}, ... \boldsymbol{A}_{N-1}^{\pm}, \boldsymbol{A}_N^{\pm}\}$; they correspond to numerical solutions of (6)-(8) with $\text{sign}(A_1) = \pm 1$, $\text{sign}(A_2) = \pm 1$, ... $\text{sign}(A_{N-1}) = \pm 1$, $\text{sign}(A_N) = \pm 1$;
- one symmetric limit cycles $\{\boldsymbol{S}_1^s, \boldsymbol{S}_2^s, ... \boldsymbol{S}_{N-1}^s, \boldsymbol{S}_N^s\}$; they correspond to numerical solutions of (6)-(8) with $A_k = 0$, $\forall k$.
- one symmetric limit cycles $\{\boldsymbol{S}_1^u, \boldsymbol{S}_2^u, ... \boldsymbol{S}_{N-1}^u, \boldsymbol{S}_N^u\}$; they also correspond to numerical solutions of (6)-(8) with $A_k = 0$, $\forall k$.

The second step of the analysis consists in the accurate characterization of each one of the above limit cycles, through the application of the HB technique, as introduced in [15]. We remark that such an application is possible for two reasons: a) the HB algorithm exploits as initial conditions the approximate numerical solutions provided by the describing function technique: this guarantees the convergence of the numerical algorithm to the solution corresponding to the proper limit cycle; b) the adopted implementation, i.e. that described in [15], is fast and efficient even for a large number of harmonics.

The third step of the analysis is the computation of limit cycle Floquet's multipliers, for detecting the most significant bifurcations. We have exploited the HB based technique described in [11] and for the sake of simplicity we have assumed $d_1 = d_2 = d$.

We have considered a chain of 12 Chua's circuit. Due to symmetric coupling the actual number of limit cycles to be analyzed can be considerably reduced; in addition, the space invariant structure permits to give a rather simple description of the most significant bifurcation processes, that we expect to be valid also for larger networks. The main results can be summarized as follows.

1. For small coupling d there are $2^N \times 2^{N-1}$ asymmetric limit cycles and $2 \times 2^{N-1}$ symmetric limit cycles.
   (a) There are 2^N stable asymmetric limit cycles, i.e. those corresponding to $\{\boldsymbol{A}_1^{\pm}, \boldsymbol{A}_2^{\pm}, ... \boldsymbol{A}_{N-1}^{\pm}, \boldsymbol{A}_N^{\pm}\}$, with all the phase shifts equal to π ($\eta_k = \pi$, $i \le k \le N-1$).
   (b) The other $2^N \times (2^{N-1} - 1)$ asymmetric limit cycles are unstable; they present as many Floquet's multipliers $|\mu| > 1$ as the number of phase shifts $\eta_k = 0$.
   (c) There is only one stable symmetric limit cycle, i.e. that corresponding to $\{\boldsymbol{S}_1^s, \boldsymbol{S}_2^s, ... \boldsymbol{S}_{N-1}^s, \boldsymbol{S}_N^s\}$, with all the phase shifts equal to zero ($\eta_k = 0$, $k \le i \le N-1$).
   (d) There are $2^{N-1} - 1$ unstable symmetric limit cycles, corresponding to $\{\boldsymbol{S}_1^s, \boldsymbol{S}_2^s, ... \boldsymbol{S}_{N-1}^s, \boldsymbol{S}_N^s\}$, with at least one phase shift equal to π ($\exists$ k: $\eta_k = \pi$). They exhibit as many Floquet's multipliers $|\mu| > 1$ as the number of phase shifts $\eta_k = \pi$.
   (e) There are 2^{N-1} unstable symmetric limit cycles, corresponding to $\{\boldsymbol{S}_1^u, \boldsymbol{S}_2^u, ...$

$S^u_{N-1}, S^u_N\}$. The number of Floquet's multipliers with $|\mu| > 1$ can be computed as N plus the number of phase shifts $\eta_k = 0$.

2. By increasing the coupling d, both the stable and the unstable asymmetric limit cycle disappear through either tangent or Naimark-Sacker bifurcations.

3. By increasing the coupling d, the stable symmetric limit cycle does not undergo any bifurcations and remains stable; some unstable symmetric cycles remains unstable, but do not disappears; some others disappear through a collision with stable asymmetric limit cycles (tangent bifurcation).

The explicit computation of the Floquet's multipliers and the detailed analysis of the above bifurcation process is reported in [16].

4. CONCLUSION

We have proposed a method based on spatio-temporal spectral techniques for the global dynamic analysis of 1D arrays of oscillators. As a case study we have considered a 1D array of Chua's circuit. The method consists in the following three main steps: 1) the set of all stable and unstable limit cycles are determined through the solution of a rather simple system of equations, that is obtained through the application of the describing function technique; 2) an accurate characterization of each limit cycle is obtained through the HB technique, by exploiting as input parameter the single harmonic approximation, yielded by the describing function technique; c) limit cycle stability and bifurcations are investigated by computing the Floquet's multipliers, via an accurate HB based technique.

The main advantage of the proposed method is the joint use of the describing function technique, that allows us to identify all periodic invariant limit sets, and of the HB technique that permits an accurate and rigorous bifurcation analysis.

5. REFERENCES

[1] *IEEE, Trans. Circuits and Syst.-I. Special issue on nonlinear waves, patterns and spatio-temporal chaos in dynamic arrays*, vol. 42, October 1995.

[2] K. R. Crounse and L. O. Chua, "Methods for image processing and pattern formation in cellular neural network: a tutorial," *IEEE Trans. Circuits Syst.-I*, vol. 42, pp. 583-601, October 1995.

[3] G. Setti, P. Thiran, and C. Serpico, "An approach to information processing in 1-dimensional cellular neural networks - Part II: Global propagation," *IEEE, Trans. Circuits Syst.-I*, vol. 45, July 1998.

[4] L. Goras and L. O. Chua, "Turing Patterns in CNNs: equations and behaviors," *IEEE Trans. Circuits Syst.-I*, vol. 42, pp. 612-626, October 1995.

[5] P. P. Civalleri and M. Gilli, "A spectral approach to the study of propagation phenomena in CNNs," *Int. J. Circuit Theory and Applicat.*, vol. 24, pp. 37-47, January-February 1996.

[6] M. Gilli, "Investigation of chaos in large arrays of Chua's circuits via a spectral technique," *IEEE Trans. Circuits Syst.-I*, vol. 42, no. 10, pp. 802-806, October 1995.

[7] M. Gilli and P. P. Civalleri, "A harmonic balance approach for the qualitative analysis of limit cycles occurring in CNNs," *IEEE 4-th International Workshop on Cellular Neural Networks and their Applications*, pp. 133-138, Seville, June 1996.

[8] P. P. Civalleri and M. Gilli, "Analysis of periodic and chaotic oscillations in one-dimensional arrays of Chua's circuits," *1997 European Conference on Circuit Theory and Design*, pp. 353-358, Budapest, September 1997.

[9] M. Gilli and P. P Civalleri, "A HB technique for the classification of periodic and chaotic attractors in one-dimensional arrays of Chua's circuits," *IEEE International symposium on nonlinear theory and its applications*, Dresda, September 2000.

[10] R. N. Madan, *Chua's circuit: A paradigm for chaos*, World Scientific Series on Nonlinear Science. Series B. Vol. 1. (World Scientific, Singapore).

[11] F. Bonani and M. Gilli, "Analysis of stability and bifurcations of limit cycles in Chua's circuit through the harmonic balance approach", *IEEE Trans. Circuits Syst-I*, vol. 46, no. 8, pp. 881-890, August 1999.

[12] G. V. Osipov, and V. D. Shalfeev, "Chaos and structures in a chain of mutually-coupled Chua's circuits,"*IEEE Trans. Circuits Syst.-I*, vol. 42, no. 10, pp. 693-699, October 1995.

[13] R. Genesio, and A. Tesi, "Distortion control of chaotic systems: The Chua's circuit," *Journal of Circuits Systems and Computers*, vol. 3, no. 1, pp. 151-171, March 1993.

[14] A. I. Khibnik, D. Roose, and L. O. Chua, "On periodic orbits and homoclinic bifurcations in Chua's circuit with a smooth nonlinearity," *Journal of Circuits, Systems and Computers*, pp. 145-178, 1993.

[15] K. S. Kundert, A. Sangiovanni-Vincentelli, J. K. White, *Steady-state methods for simulating analog and microwave circuits*, Boston: Kluwer Academic Publisher, 1990.

[16] M. Gilli, P. Checco, and M. Biey, "A spatio-temporal spectral technique for the global dynamic analysis of nonlinear dynamic arrays," *Politecnico di Torino, Internal Report*, September 2002.