A Chaos-Based Communication Scheme via Robust Asymptotic Feedback

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Abstract—A communication scheme based on chaos synchronization via feedback is presented. The main idea is to construct an augmented dynamical system from the synchronization error system, which is itself uncertain. Dynamic output feedback is applied to perform synchronization in spite of transmitter/receiver mismatches. In this way, the transmitted message (which can be analog or digital) can be recovered. Two illustrative examples are presented. 1) The Chua oscillator is used to show that the message signal is recovered in spite of parametric transmitter/receiver mismatches. 2) Two second-order driven oscillators are presented to show that the message can be recovered in spite of a strictly different model, which results in different master/slave dynamics.

Index Terms—Chaos-based transmission, secure communication.

I. INTRODUCTION

RECENTLY, synchronization of chaotic systems and its application to secure communication has received much attention. Since 1990, when master-slave synchronization was introduced by Pecora and Carroll [1], transmission of signals via a chaotic carrier has been widely studied. The main goal has been to mask the message signal via the transmitter which is a chaotic system [2]. The security level of the transmitter/receiver scheme can accommodate message signals given by a triangle wave, period doubled signal, and digital signal which means that n -periodic signals can be masked by chaotic carrier (see [2]–[4] for more details). Moreover, chaotic and voice signals can be transmitted via chaos-based schemes. Chaos-based secure communication has been tested via the power spectrum with promising results [3].

Several chaos-based communication schemes have been published. Among these, two basic configurations can be identified. 1) An approach consisting of the addition of the message signal to the chaotic carrier (transmitted signal) that is sent to the receiver. That is, the master system (the so-called drive system or transmitter) comprises the full-state model whereas the slave system (the so-called response system or receiver) is composed of a reduced model (homogeneous synchronization, see, for instance, [2] and [4]). 2) Another transmitter/receiver design is based on the full-state model of the driving and response systems [2]. That is, both drive and response systems are represented by dynamical systems of the same order. The homogeneous-synchronization configuration has been recently

addressed via parameter modulation. Kocarev and Parlitz have proposed a generalization of these approaches which extends the capabilities for constructing synchronized systems. Their approach enables the message signal to be integrated as a driving signal. However, the message signal can be recovered only under ideal conditions [5].

On the other hand, several synchronization schemes have also been recently developed. Monotone synchronization is a feedback scheme based on high-gain control [6]. Monotone synchronization has application to secure communication. However, it is based on high-gain feedback which is very sensitive to noise.

The synchronization of nonidentical chaotic systems and its usage in transmitter/receiver configurations is an interesting problem. There are two main objectives to be considered here. The first one is to test the security level of communications based on synchronization of chaotic systems. The point is, can a response system unmask a message signal in spite of the drive system being unknown?. The synchronization of two strictly different chaotic systems is possible [7]. Hence, a response system whose dynamical model is not similar to the drive system can reconstruct the message signal. The latter objective is related with the dynamic reconstruction of the master dynamics (drive system). This problem can be addressed from the control theory point of view, where two basic ideas can be identified.

1) *Construct an Observer:* A dynamical system (which is called an *observer*) is constructed from the transmitter model in such a way that it yields the dynamical reconstruction of the transmitter states. In this way, the observer results in a response system which can be synchronized in spite of unknown initial conditions of the drive system [8], [9]. A standard approach to solve the observer problem is to use a receiver that is a copy of the transmitter (with unknown initial conditions). The copy of the transmitter is modified with a term depending on the difference between the drive and response system [10]. Such an additional term serves to attenuate the dynamics of the synchronization error states. Therefore, this procedure requires *a priori* knowledge of the drive-system model. Moreover, although the procedure could be shown to be successful in many instances, no global synchronization can be claimed. However, the main idea behind the observer construction is fundamentally deterministic. That is, according to the Takens theorem [11], there are certain conditions for which the dynamics of a system can be embedded in a finite-dimensional space. In this sense, the deterministic systems have an *observability property*. The observability property means that the history of the transmitted signal contains all the information required to reconstruct the states of the drive system. In such a case, it is said that the drive system is *observable* [10].

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2) *Design a Feedback Control Scheme:* The main idea here is to design a control loop in the receiver system. The feedback steers the states of the synchronization-error system to the origin, i.e., the states of the response system *track* the states of the drive system. The feedback results in a coupling force between the transmitter and receiver in such a way that the discrepancy between the drive and response systems is controlled. In this sense, the synchronization problem can be interpreted as a chaos suppression problem. Since the 1990s, chaos suppression has been intensively studied. Some relevant references can be mentioned [12]–[15]. It has been shown that, under feedback control, chaos control can be attained [13], [16]. This means that the dynamics of the synchronization error can be led to the origin in spite of the drive and response models being nonidentical (and even if they are strictly different [8]). Since chaos suppression can be achieved under uncertain vector fields and unknown initial conditions, the feedback yields chaos synchronization. As a consequence, this interpretation can also be applied to secure communication. Nevertheless, a minimum phase condition is often assumed. Although this condition can be satisfied in many situations, minimum phase is a strong assumption.

In this paper, the chaos-synchronization problem and its application to communication is addressed from a control theory perspective. In particular, the synchronization problem is interpreted as a stabilization one. The goal is to stabilize, at the origin, the discrepancy between the drive and response systems. Discrepancy is defined as the dynamical differences between the drive and response systems and includes: 1) model mismatches, which means that the model of the drive system might not be the same as that of the response system; 2) unknown initial conditions, which implies that the time series of the transmitter cannot be equal to that for the receiver; and 3) parametric uncertainty, which means that the receiver circuit could be constructed with inaccuracies.

In order to design the stabilizing feedback, the discrepancy system is transformed into a canonical form via geometrical tools [17]. The main idea behind our proposal is, departing from the discrepancy system, to construct an extended nonlinear system which should be dynamically equivalent to the canonical representation. In this way, the discrepancy is lumped into a nonlinear function, which is rewritten into the extended nonlinear system as a state variable. By using the results reported by Teel and Praly [18], an observer can be constructed to get an estimated value of the lumping nonlinear function via the augmented state variable. In fact, this procedure has been used to perform synchronization of strictly different chaotic oscillators [7]. In this paper, the procedure presented in [7] is generalized and the minimum phase assumption is considered.

II. PROBLEM STATEMENT

Consider the following nonlinear system:

$$\begin{aligned}\dot{x}_M &= f_M(x_M; \pi_M) + g_M(x_M)S_m \\ y_M &= h(x_M)\end{aligned}\quad (1)$$

where

$$\begin{aligned}x_M &\in \mathbb{R}^n && \text{state vector;} \\ \pi_M &\in \mathbb{R}^P && \text{parameter set;}\end{aligned}$$

$$S_m \in \mathbb{R}$$

bounded known function and denotes a message signal (which is injected into the nonlinear system);

$$f_M : \mathbb{R}^n \rightarrow \mathbb{R}^n$$

smooth vector field;

$$g_M : \mathbb{R}^n \rightarrow \mathbb{R}^n$$

smooth vector field;

$$y_M \in \mathbb{R}, h(x_M)$$

smooth function which determines the *transmitted* states.

Remark 1: The message signal S_m can be interpreted as an input signal, i.e., there exist a signal S_m with suitable frequency and amplitude such that the dynamics of system (1) is chaotic. Note that the message signal can be represented by: 1) A modulated parameter of the drive system. In such a case, it is said that the message signal is *demodulated* by the response system; or 2) A smooth time function which is not a parameter of the drive system. In such a case, the message signal is *decoded* by the response system. $\square$

Now, let us consider the following nonlinear system

$$\begin{aligned}\dot{x}_S &= f_S(x_S; \pi_S) + g_S(x_S)u \\ y_S &= h(x_S)\end{aligned}\quad (2)$$

where

$$x_S \in \mathbb{R}^n$$

state vector;

$$\pi_S \in \mathbb{R}^q$$

parameter set;

$$f_S : \mathbb{R}^n \rightarrow \mathbb{R}^n$$

smooth vector field;

$$g_S : \mathbb{R}^n \rightarrow \mathbb{R}^n$$

smooth vector field

$$u \in \mathbb{R}$$

demodulated signal (Control command);

$$y_S \in \mathbb{R}, h(x_S)$$

smooth functions which determine the *comparison* states.

The comparison states are chosen in such a way that they are the same as the transmitted states, i.e., the output of the response system y_S is chosen equal to the transmitted state y_M which does not mean that the channel is perfect. Indeed, it means that the same state at drive and response system are available for feedback. In this sense, it implies that receiver and transmitter have agreement in communication. Henceforth, the security level of the communication scheme is fixed by comparison states.

Remark 2: The nonlinear systems (1) and (2) have the same order. From the synchronization (chaos theory) point of view, this assumption is strong. However, it is a standard consideration for the signal-transmission case. In addition, note that the vector field $g_S(x_S)$ represents the way the input u is entered into system (2). $\square$

By defining $x_i := x_{i,M} - x_{i,S}$, the following uncertain dynamical system can be obtained:

$$\begin{aligned}\dot{x} &= \Delta f(x; \pi) + g_M(x_M)S_m - g_S(x_S)u \\ y &= h(x)\end{aligned}\quad (3)$$

where

$$x \in \mathbb{R}^n$$

state vector;

$$x \in \mathbb{R}^n, \pi \in \mathbb{R}^r$$

parameter set;

$$\Delta f : \mathbb{R}^n \rightarrow \mathbb{R}^n$$

vector field defined by $\Delta f(x, \pi) := f_M(x_M; \pi_M) - f_S(x_S; \pi_S)$;

$$h(x)$$

smooth function which determines the difference between the *transmitted* and *comparison* states.

Note that models of the drive and response systems are assumed nonidentical, hence $\Delta f(x, \pi)$ is an uncertain vector field. Finally, the term $g_M(x_M)S_m$ is unknown.

Definition 1 (Practical Synchronizability): It is said that the chaotic systems (1) and (2) are practically synchronized if, for a compact set of initial conditions, $X_0 \subseteq \mathbb{R}^n$, and for each pair of compact sets (U_P, U_{PS}) , which are neighborhoods of the origin with $U_P \subset U_{PS}$, there exists a locally Lipschitz function, $u = a_1\|x\|$, where a_1 is a positive constant, such that the solution of $x(t) = (x_M(t) - x_S(t)) \in U_{PS} \subset \mathbb{R}^n$ for system (3), with initial conditions $x(0) \in X_0$, belongs to U_P for all $t \geq 0$. $\square$

Remark 3: Since the trajectories $x_M(t), x_S(t)$ of the uncontrolled systems (1) and (2) are chaotic, the differences $x_i(t) := x_{i,M}(t) - x_{i,S}(t)$ belong to any chaotic attractor [7]. That is, trajectories $x_i(t)$ starting at the initial condition $x_i(0) := (x_{i,M}(0) - x_{i,S}(0)) \in X_0$ converge to the set U_P which is not necessarily equal to X_0 . Moreover, if systems (1) and (2) can be synchronized then the trajectories $x(t)$ will converge to any neighborhood U_P of the origin which is contained in the attractor set U_{PS} . Finally, Definition 1 includes several kinds of synchronization: 1) Complete exact synchronization (where $\|x_S(t) - x_M(t)\| \equiv 0$ for all $t \geq 0$); 2) Complete inexact synchronization (where $\|x_S(t) - x_M(t)\| \approx 0$ for all $t \geq 0$), iii) Partial synchronization (where at least for one state $x_i(t)$, for any $i \leq n, \|x_M(t) - x_S(t)\| \neq 0$) and iv) Almost synchronization (where only the phase of the drive system is similar to the response system but the amplitude is different). $\square$

Lemma 1 (Isidori [17]): Let us assume that there exists a smallest integer ρ such that: 1) $\mathcal{L}_g \mathcal{L}_f^i h(x) = 0, i = 1, 2, \dots, \rho - 2$ and (ii) $\mathcal{L}_g \mathcal{L}_f^{\rho-1} h(x) \neq 0$, where $\mathcal{L}_f^\rho h(x) := \mathcal{L}_f^\rho(\mathcal{L}_f^{\rho-1} h(x)), \mathcal{L}_f^0 h(x) := h(x)$ and, by definition, $\mathcal{L}_f h(x) := (\partial h(x)/\partial x)f(x)$ is the Lie derivative of $h(x)$ along $f(x)$. This implies that there exists a coordinate transformation $(z, \nu) = \Phi(x)$ such that system (3) can be globally transformed into the canonical form

$$\begin{aligned} \dot{z}_i &= z_{i+1}, & i &= 1, 2, \dots, \rho - 1 \\ \dot{z}_\rho &= \alpha(z, \nu) + \beta(z, \nu)u \\ \dot{\nu} &= \zeta(z, \nu) \\ y &= z_1 \end{aligned} \quad (4)$$

where $\alpha(z, \nu) = \mathcal{L}_f^\rho h(x), \beta(z, \nu) = \mathcal{L}_g \mathcal{L}_f^{\rho-1} h(x), \nu \in \mathbb{R}^{n-\rho}$ denotes the unobservable states and $\zeta(z, \nu) = [\Phi_{\rho+1}, \Phi_{\rho+2}, \dots, \Phi_n]$ is such that the coordinate transformation $\Phi(x) = [h(x), \mathcal{L}_f h(x), \dots, \mathcal{L}_f^{\rho-1} h(x), \Phi_{\rho+1}, \Phi_{\rho+2}, \dots, \Phi_n]$ is globally invertible. Therefore, if system (4) is minimum phase, i.e., $\dot{\nu} = \zeta(0, \nu)$ is stable, there exists a state feedback controller $u(x) := -(\mathcal{L}_f^\rho h(x) + v(x))/\mathcal{L}_g \mathcal{L}_f^{\rho-1} h(x)$, where $v = K^T \Phi^{-1}(z)$ and $K > 0$ are such that system (3) is asymptotically stable at the origin. $\square$

Remark 4: The geometrical control theory is well developed and classical books have been published [17], [19]. Thus, the relative degree ρ is obtained from the following algorithm. Let $y = h(x)$ be the system output where $h(x)$ is a smooth function. Differentiating the output with respect to time, we obtain $\dot{y} = (\partial h(x)/\partial x)\dot{x}$, where $\dot{x} = f(x) + g(x)u$ which implies that, by the definition of the Lie derivative, $\dot{y} = \mathcal{L}_f h(x) + \mathcal{L}_g h(x)u$.

If $\mathcal{L}_g h(x) = 0$ (which can be interpreted as the control input u does not affects the system output y) for any $x \in \mathbb{R}^n$, one must differentiate the system output again to get $\ddot{y} = \mathcal{L}_f^2 h(x) + \mathcal{L}_g \mathcal{L}_f h(x)u$. Once again, if $\mathcal{L}_g \mathcal{L}_f h(x) = 0$ for any $x \in \mathbb{R}^n$ the system output should be differentiated with respect to time until conditions 1) and 2) are satisfied. After that, one can define the coordinate transformation $z_{i+1} := \mathcal{L}_f^i h(x) = y^{(i)}$ to get system (4). $\square$

We have made the following assumptions.

- Assumption A.1) The order of the drive and response system is the same.
- Assumption A.2) There are uncertain model mismatches between the transmitter and the receiver; i.e., $f_M(x_M) \neq f_S(x_S)$ which implies that the discrepancy model is unknown.
- Assumption A.3) $g_S(x_S) \neq g_M(x_M)$ where $g_S(x_S)$ is known and bounded away from zero. In addition $\text{sign}(g_S(x_S)) = \text{sign}(g_M(x_M))$.
- Assumption A.4) System (3) is minimum phase, i.e., the subsystem of the zero dynamics, $\dot{\nu} = \zeta(0, \nu)$ where $\nu \in \mathbb{R}^{n-\rho}$, is asymptotically stable.

Some comments about the above assumptions are in order. Assumptions (A.1) and (A.2) are realistic situations. Indeed transmitter and receiver oscillators are often designed with same order and model discrepancy can be expected due to differences between actual values in components of the circuit. In extreme cases, the model of the transmitter could be strictly different than the receiver. However, recovering of the message signal is desired even in the extreme case. Assumption A.3) implies that, if exact synchronization between drive and response system is attained, one has that $f_M(x_M) - f_S(x_S) = 0$ and $x_M(t) - x_S(t) \equiv 0$ for all $t \geq 0$ then $g_S(x_S)S_m \equiv g_M(x_M)u$ for all $t \geq 0$, i.e., $\sigma g_S(x_S) = g_M(x_M)$ for all $t \geq 0$, where $\sigma = u/S_m$. According to Assumption A.3) $\text{sign}(g_S(x_S)) = \text{sign}(g_M(x_M))$, hence $\text{sign}(\sigma) > 0$. Therefore, the message signal S_m will be recovered with a factor $\|\sigma\|$. The minimum phase supposition is stronger condition which implies that the uncontrollable states $\nu \in \mathbb{R}^{n-\rho}$ of the discrepancy system are asymptotically stable. Fortunately, most chaotic oscillators satisfy this assumption. Now, under the above assumptions, the synchronization problem becomes: *Is there any smooth function $u = u(x)$ such that the uncertain nonlinear system (4) is asymptotically stable at the origin and the message signal can be decoded?* In this sense the synchronization problem has been interpreted as the chaos suppression problem, i.e., if $x \rightarrow 0$ then $x_S \rightarrow x_M$.

III. MAIN RESULTS

Definition 2 (Practical Semiglobal Stabilizability): Let us consider the affine system $\dot{z} = f(z) + g(z)u$, where $z \in \mathbb{R}^n$ and $f(z)$ and $g(z)$ are smooth vector fields. The equilibrium $z^* = 0$ of the affine system is said to be semiglobally practically stabilizable by dynamic output feedback if, for each pair of compact sets (U_P, U_{PS}) , which are neighborhoods of the origin with $U_P \subset U_{PS}$, there exists: 1) a locally Lipschitz function $u = u(z)$ and (ii) a pair of compact sets (U_{Po}, U_{PSo}) such that

the solution of the affine system $z(t) \in \mathbb{R}^n$, with initial conditions in the basin of attraction in $U_{PS} \times U_{PSO}$, converges to the set $U_P \times U_{PO}$ for all $t \geq 0$. $\square$

For the sake of clarity in presentation, we should point out that the practical synchronizability (Definition 1) and practical semiglobal stabilizability (Definition 2) are very close notions. Indeed, from the viewpoint of the framework theory presented in this paper, the practical synchronizability problem is a stabilization problem. The idea is to lead the states of the discrepancy system $x(t) \in \mathbb{R}^n$ from the initial condition $x(t=0)$ in a subset X_0 of $\mathbb{R}^n$ to an arbitrarily small subset $U_P \subset U_{PS}$ containing the origin. Several kinds of synchronization can be found where feedback control is used [20], Definitions 1 and 2 comprise all the kinds of synchronization reported in recent publications.

Theorem 1: Let us assume that there exists a coordinate transformation $z = \Phi(x)$ such that the uncertain nonlinear system (3) can be transformed into the canonical form (4). Now suppose that there is an estimate $\beta_E(z)$, defined by the Lie derivative along $g_S(x_S)$ of the uncertain function $\beta(z, \nu) := \mathcal{L}_g \mathcal{L}_f^{\rho-1} h(x)$, such that $\text{sign}(\beta_E(z)) = \text{sign}(\beta(z, \nu))$. Let us define $\delta(z, \nu) = \beta(z, \nu) - \beta_E(z)$, $\Theta(z, \nu, u) := \alpha(z, \nu) + \delta(z, \nu)u$ and $\eta = \Theta(z, \nu, u)$. Then there exists an invariant manifold such that the nonlinear system (4) can be rewritten in the following form:

$$\begin{aligned} \dot{z}_i &= z_{i+1}, \quad i = 1, 2, \dots, \rho - 1 \\ \dot{z}_\rho &= \eta + \beta_E(z)u \end{aligned} \quad (5.1)$$

$$\begin{aligned} \dot{\eta} &= \Gamma(z, \nu, \eta, u, \dot{u}) \\ \dot{\nu} &= \zeta(z, \nu) \\ y &= z_1 \end{aligned} \quad (5.2)$$

where $\Gamma(z, \nu, \eta, \dot{u}) = \sum_{i=1}^{\rho-1} [z_{i+1} \partial_i \Theta(z, \nu, u)] + [\eta + \beta_E(z)] \partial_\rho \Theta(z, \nu, u) + \delta(z, \nu) \dot{u} + \partial_\nu \Theta(z, \nu, u) \zeta(z, \nu)$.

Proof: By definition, $\delta(z, \nu) = \beta(z, \nu) - \beta_E(z)$, $\Theta(z, \nu, u) = \alpha(z, \nu) + \delta(z, \nu)u$ and $\eta = \Theta(z, \nu, u)$. This implies that there exists a manifold $\Psi(z, \eta, \nu, u) = \eta - \Theta(z, \nu, u)$ such that $d\Psi(z, \eta, \nu, u)/dt = 0$ (i.e., it is a time invariant manifold with boundary) for the initial condition $\Psi_0 = 0$. Since by definition $\Psi(z, \eta, \nu, u) = \eta - \Theta(z, \nu, u)$ and $d\Psi(z, \eta, \nu, u)/dt = 0$, it means that $\Psi(z, \eta, \nu, u)$ is the first integral of the nonlinear system (5). Therefore the solution of system (4) is a projection of the solution of system (5). Hence system (5) is dynamically equivalent to system (4). $\square$

Remark 5: Mismatches between the drive and response system have been lumped into a nonlinear function $\Theta(z, \nu, u)$ which has been interpreted as an augmented state variable η to obtain system (5). In this way, η represents the uncertain functions $\alpha(z, \nu)$ and $\beta(z, \nu)$ which include the transmitter/receiver discrepancy. Then, if one is able to stabilize system (5) around the origin via output feedback, then the trajectories of system (4) will converge to a neighborhood containing the origin. Finally, estimate value the $\beta_E(z)$ can be computed from the Lie derivative of the output function $h(x)$ along the vector field $g_S(x_S)$. $\square$

Theorem 2: Systems (1) and (2) are practically synchronized if the equilibrium point $(z^*, \eta^*, \nu^*) = (0, 0, 0)$ of system (4) is semiglobally practically stabilizable.

Lemma 2: Consider the following state-feedback: $u = -(\eta + K^T z)/\beta_E(z)$, where the $K'_i s, i = 1, 2, \dots, \rho$, are coefficients of the Hurwitz polynomial $P_\rho(s) = s^\rho + K_1 s^{\rho-1} + \dots + K_{\rho-1} s + K_\rho$. Under state feedback, the solution $(z(t), \eta(t), \nu(t))$ converges to the origin if the initial conditions satisfy $\Psi_0 = 0$.

Proof: Suppose that the initial conditions satisfy $\Psi_0 = 0$. Thus since $\Psi(z, \eta, \nu, u)$ is an invariant manifold, $\eta = \Theta(z, \nu, u)$ for all $t \geq 0$. Then, we have that $\eta = \alpha(z, \nu) + \delta(z, \nu)u$, where $\delta(z, \nu) = \beta(z, \nu) - \beta_E(z)$. Combining η with the state feedback controller, we have that $u = -(\alpha(z, \nu) + K^T z)/\beta(z, \nu)$. Since $\Delta f(x, \pi)$, $g_M(x_M)$ and $g_S(x_S)$ are smooth vector fields, $\alpha(z, \nu)$ is a smooth function of its arguments. Consequently, the augmented state $\eta(t)$ and the state feedback controller $u = u(z)$ are bounded. On the other hand, convergence of the states $(z(t), \eta(t))$ follows from the fact that the subsystem (5.1) is in cascade form and the corresponding characteristic polynomial is Hurwitz under a state feedback controller. In addition, if $(z(t), \eta(t)) = 0$ and the subsystem $\dot{\nu} = \zeta(0, \nu)$ converges to the origin (minimum phase assumption), the state feedback is a practical stabilizer of system (5). $\square$

Remark 6: Note that the controller in Lemma 2 requires full information about the state of system (5). In this sense the following comments are in order: (i) Lemma 2 implies that the closed-loop system (i.e., the discrepancy system under control actions) can be written as $\dot{z} = Az$, where the matrix $A \in \mathbb{R}^{n \times n}$ is Hurwitz; that is it has all its eigenvalues in the open left-hand complex plane; (ii) the augmented state η is not available for feedback. This fact is obvious because η represents, by definition, the mismatches between drive and response systems; (iii) it is desired that only one state is transmitted in chaos-based communication schemes, then only one state $x_i = x_{i,M} - x_{i,S}$ is available for feedback from *on-line* measurements. Consequently, estimated values of the states (z, η) are required for practical implementation. According to recent results, estimated values of partial feedback can be obtained from robust estimation procedures (see for instance [18]). $\square$

Remark 7: The state feedback $u = u(z(t), \eta(t))$ requires a priori knowledge about the augmented state $\eta(t)$. According to Assumption A.2, that is an unrealistic situation. Hence, an estimated value of the uncertainties is desired. We are interested in a dynamic output feedback of the form

$$\begin{aligned} \dot{\hat{z}}_i &= \hat{z}_{i+1} + L^i \kappa_i (z_1 - \hat{z}_1), \quad i = 1, 2, \dots, \rho - 1 \\ \dot{\hat{z}}_\rho &= \hat{\eta} + \beta_E(\hat{z})u + L^\rho \kappa_\rho (z_1 - \hat{z}_1) \end{aligned} \quad (6.1)$$

$$\begin{aligned} \dot{\hat{\eta}} &= L^{\rho+1} \kappa_{\rho+1} (z_1 - \hat{z}_1) \\ u &= \text{Sat} \left\{ \frac{-1}{\beta_E(\hat{z})} [\hat{\eta} + K^T \hat{z}] \right\} \end{aligned} \quad (6.2)$$

where $\text{Sat}\{\cdot\} : \mathbb{R}^n \rightarrow S \subset \mathbb{R}^n$, S is a bounded set and $(\hat{z}, \hat{\eta})$ are estimated values of (z, η) , respectively. The $\kappa'_j s, j = 1, 2, \dots, \rho + 1$, and $K'_i s, i = 1, 2, \dots, \rho$, are chosen in such a way that they are coefficients of Hurwitz polynomials and $L > 0$ is the high-gain estimation parameter. The dynamic

output feedback (6) is able to suppress chaos with uncertain vector fields [7] and thus, in principle, it can be used to unmask signals via chaotic synchronization. $\square$

Lemma 3: Under actions of the dynamic output feedback (6), the origin of the nonlinear system (5) is semiglobally practically stabilizable.

Proof: Let $e \in \mathbb{R}^{\rho+1}$ be a vector whose components are defined by: $e_j := L^{\rho+1-j}(z_j - \hat{z}_j)$, $j = 1, 2, \dots, \rho$, $e_{\rho+1} := \eta - \hat{\eta}$. In this way the closed-loop system is obtained by substituting controller (6) and the dynamics of the above defined estimation error into Eq. (5). Thus the closed-loop system becomes

$$\dot{z} = Z(z, \eta, e; L) \quad (7.1)$$

$$\dot{\eta} = \Gamma(z, \eta, e, u, \dot{u}) \quad (7.2)$$

$$\dot{\nu} = \zeta(z, \nu) \quad (7.3)$$

$$\dot{e} = LA'(\kappa; r)e + \varphi_2(z, \eta, e; L)B' \quad (7.4)$$

where $z \in \mathbb{R}^\rho$, $\nu \in \mathbb{R}^{n-\rho}$, $Z(z, \eta, e; L) = Az + [\eta + \beta_E(\hat{z})u]B$, where B is the ρ -dimensional vector $[0, \dots, 0, 1]^T$, the nonlinear $\beta_E(\hat{z})$ is given as above, $e = (e', e_{\rho+1}) \in \mathbb{R}^{\rho+1}$ denotes the control error, i.e., the difference between state $x(t)$ minus the reference $r = 0$ (which is given by origin) and $A \in \mathbb{R}^{\rho \times \rho}$ is a companion matrix given by

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \ddots & 0 & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 1 \\ 0 & 0 & \cdots & 0 & 0 \end{bmatrix}.$$

The feedback function takes the form $u = u((z - N(L)e'), (\eta - e_{\rho+1}))$, where $e' := [e_1, \dots, e_\rho]^T$, $N(L) = \text{diag}(L^{-\rho}, L^{-\rho+1}, \dots, L^{-1})$ and $\Gamma(z, \nu, \eta, u, \dot{u})$ is given as above. Note that since the saturation function is bounded, there exists a continuous function $\gamma(|e|)$ such that $|Z(z, \eta, e; L) - Z(z, \eta, 0; L)| < \gamma(|e|)$. In addition, note that since $\eta = \alpha(z, \nu) + \delta(z, \nu)u$ and $u = \text{Sat}\{(\hat{\eta} - K^T \hat{z})/\beta_E(\hat{z})\}$, one can obtain the contraction $\eta = F(z, \eta, u, e; L)$ (which can be computed from the first integral of system (5), i.e., $\eta = \int \Gamma(z, \nu, \eta, u, \dot{u}) d\sigma$). Then, according to the Contraction Mapping Theorem, the state η can be expressed globally and uniquely as a function of the coordinates (z, e) . On the other hand, the matrix $A'(\kappa) \in \mathbb{R}^{(\rho+1) \times (\rho+1)}$ is the companion matrix of the estimator (6.1) and is given by

$$A'(\kappa; r) = \begin{bmatrix} -\kappa_1 & 1 & 0 & \cdots & 0 \\ -\kappa_2 & 0 & 1 & \cdots & 0 \\ \vdots & 0 & 0 & \ddots & \vdots \\ -\kappa_\rho & \vdots & \ddots & \ddots & 1 \\ -r(t)\kappa_{\rho+1} & 0 & \cdots & 0 & 0 \end{bmatrix}.$$

B' is a $(\rho + 1)$ -dimensional vector given by $[0, \dots, 0, 1]^T$ and $r(t) = [\beta_E(\hat{z}) - \text{Sat}'\{\cdot\}\delta(z, \nu)/\beta_E(\hat{z})]$ where $\text{Sat}'\{\cdot\}$ is the time derivative of the saturation function. The nonlinear function $\varphi_2(z, \eta, e; L) = \Gamma(z, \nu, \eta, u, \dot{u}) - L\kappa_{\rho+1}e_1$ is a continuous and bounded function.

Now, since the polynomial $s^{\rho+1} + \kappa_1 s^\rho + \dots + \kappa_{\rho-1}s + \kappa_{\rho+1} = 0$ is Hurwitz (see Remark 6), there exists a positive number, ε , satisfying $0 < \varepsilon < 1$ (see Theorem 2.7 in [21]) such

that the nominal system $\dot{e} = A'(\kappa; r(t))e$ is quadratically stable for $|1 - r(t)| < \varepsilon$. This implies that the Lyapunov equation $A'^T(\kappa; r(t))P + PA(\kappa; r(t)) = -I$ has a positive-definite solution. Since the nonlinear function $\varphi_2(z, \eta, e; L)$ is bounded, the subsystem (7.4) is quadratically asymptotically stable.

From this and the boundedness of $Z(z, \eta, e; L)$ and $\varphi_2(z, \eta, e; L)$, we conclude that (applying Theorem 3 in [22]) given a compact set of initial conditions $X_0 \subset \mathbb{R}$ containing the origin, there exists an upper bound U^{MAX} with $|\text{Sat}\{\cdot\}| \leq U^{\text{MAX}}$ and a high-gain estimation parameter L such that X_0 is contained in the attraction basin $U_{\text{PS}} \times U_{\text{PS}_0}$. Hence system (7) is semiglobally practically stable. $\square$

Proof of Theorem 2: Suppose that the nonlinear system (5) is semiglobally practically stable, that is $(z, \nu, \eta) \rightarrow 0$. Then, by the minimum phase assumption and Theorem 1, i.e., the solution of system (4) is a projection of system (5), one has that the states $(z, \nu) \rightarrow 0$ via module $\prod \cdot (z, \eta, \nu)$ (module $\prod \cdot (z, \eta, \nu)$ is the projection of system (5) onto system (4)). In addition, according to Lemma 1, the coordinate transformation $z = \phi(x)$ is globally defined. Consequently as $\prod \cdot (z, \eta, \nu) \rightarrow 0$ as $(z, \nu) \rightarrow 0$. Therefore $x(t) \rightarrow 0$, which implies that $x_S(t) \rightarrow x_M(t)$ for all $t \geq 0$. $\square$

Corollary 1: The message signal S_m is demodulated by the dynamic output feedback (6) if and only if systems (1) and (2) are completely practically synchronized.

Proof: (I) According to Theorem 2, there exists an upper bound U^{MAX} with $|\text{Sat}\{\cdot\}| \leq U^{\text{MAX}}$ and a high-gain estimation parameter L such that systems (1) and (2) are practically synchronized via the dynamic output feedback (6). Then, from system (3) and Assumption (A.3) one gets that $u/S_m \approx \Delta f(x, \pi)g_M(x_M)/g_S(x_S)$. This means that $\|\Delta f(x, \pi)\| \approx 0$ (complete synchronization) $\Rightarrow \|u - s_m\| \approx 0$. (II) Suppose that $u = s_m$. Then, from system (3) and Assumption (A.3) one has that at the equilibrium $x^*(t) = 0$, $\Delta f(x, \pi)g_M(x_M)/g_S(x_S) \approx u - s_m = 0$. Since $g_M(x_M)$ and $g_S(x_S)$ are bounded away from zero, the above condition implies that $\Delta f(x, \pi) \approx 0$. $\square$

IV. ILLUSTRATIVE EXAMPLES

We present two examples in this section. The first example consists of two third-order systems whose model is similar but parameter values are different. The aim is to show that the message signal can be recovered in spite of parametric variations and to illustrate that the chaotic minimum-phase assumption is satisfied. In addition, we have assumed that the transmission channels are equal, i.e., $g_S(x_S) = g_M(x_M)$. This assumption was considered in order to isolate parametric differences. The second example consists of two strictly different oscillators. Here, the goal is to show that the message signal can be acceptably recovered in spite of model differences between transmitter and receiver. We chose two second-order driven oscillators to illustrate this case. Thus, the drive system is given by the Duffing equation whereas the response system is given by van der Pol oscillator. In the second example, we have assumed that transmission channels are unequal, i.e., $g_S(x_S) \neq g_M(x_M)$.

Signal Transmission in spite of parametric variations. Suppose that the transmitted and recovered signal are carried out by

same channel, i.e., $g_S(x_S) = g_M(x_M) = (1, 0, 0)^T$. Chua circuit has been chosen to illustrate the proposed decoding scheme. This oscillator is widely studied as a carrier for secure communication schemes. The drive system can be written in dimensionless form as follows:

$$\begin{aligned}\dot{x}_{1,M} &= \pi_{1,M}[x_{2,M} - x_{1,M} - f(x_{1,M})] + S_m \\ \dot{x}_{2,M} &= x_{1,M} - x_{2,M} + x_{3,M} \\ \dot{x}_{3,M} &= -\pi_{2,M}x_{2,M}\end{aligned}\quad (8)$$

where $f(x_{1,M}) = \pi_{3,M}x_{1,M} + 1/2(\pi_{4,M} - \pi_{3,M})(|x_{1,M} + 1| - |x_{1,M} - 1|)$. Suppose that the same configuration is used as a response system. However, assume that there are differences between the electronic devices (for instance, the nonlinear resistance). That is, the parameter values of the receiver are different than the transmitter. In this way the response system becomes

$$\begin{aligned}\dot{x}_{1,S} &= \pi_{1,S}[x_{2,S} - x_{1,S} - f(x_{1,S})] + u \\ \dot{x}_{2,S} &= x_{1,S} - x_{2,S} + x_{3,S} \\ \dot{x}_{3,S} &= -\pi_{2,S}x_{2,S}\end{aligned}\quad (9)$$

where $f(x_{1,S}) = \pi_{3,S}x_{1,S} + 1/2(\pi_{4,S} - \pi_{3,S})(|x_{1,S} + 1| - |x_{1,S} - 1|)$, with $\pi_S \neq \pi_M$.

From the differences $x_i = x_{i,M} - x_{i,S}$, the discrepancy system (4) can be obtained as follows: $\dot{x}_1 = \Delta f_1 + s_m - u$, $\dot{x}_2 = \Delta f_2$, $\dot{x}_3 = \Delta f_3$. Now defining the transmitted state by $y_M = x_{1,M}$ and the receiver output by $y_S = x_{1,S}$, one has that $h(x) = x_1$. This implies that the smallest integer such that: (i) $\mathcal{L}_g \mathcal{L}_f^i h(x) = 0$, $i = 1, 2, \dots, \rho - 2$ and (ii) $\mathcal{L}_g \mathcal{L}_f^\rho h(x) \neq 0$ is $\rho = 1$. In this way the coordinate transformation is globally defined and becomes: $z_1 = x_1$, $\nu_1 = x_2$ and $\nu_2 = x_3$. Then the discrepancy system can be rewritten as

$$\begin{aligned}\dot{z}_1 &= \Delta f_1 + s_m - u \\ \dot{\nu}_1 &= \Delta f_2 \\ \dot{\nu}_2 &= \Delta f_3 \\ \dot{y} &= z_1\end{aligned}\quad (10)$$

where Δf_i are unknown functions and y denotes the output of the discrepancy system.

In order to illustrate that system (10) satisfies the minimum phase assumption, one can show that $\Delta f_2 = z_1 - \nu_2 + \nu_3$ and $\Delta f_3 = -2\gamma_2 x_2 + \delta_2$, where $\gamma_2 = 1/2(\pi_{2,M} + \pi_{2,S})$ and $\delta_2 = \pi_{2,M}x_{2,S} + \pi_{2,S}x_{2,M}$. Now, γ_2 and δ_2 are uncertain; however it is clear that $\gamma_2 > 0$ and δ_2 is bounded. As $z_1 \rightarrow 0$ (zero dynamics), one has that: $\dot{\nu} = C\nu + \Delta_2$, where $\Delta_2 = [0 \ \delta_2]^T$ and

$$C = \begin{bmatrix} -1 & 1 \\ -2\gamma_2 & 0 \end{bmatrix}.$$

Hence since $\gamma_2 > 0$ and Δ_2 is bounded, the zero dynamics subsystem $\dot{\nu} = C\nu + \Delta_2$ is asymptotically stable. That is, the discrepancy between systems (8) and (9) is a minimum phase system.

Since Assumptions (A.1)–(A.4) are satisfied, the augmented state can be defined as: $\eta = \Delta f_1 + S_m$. Then, system (5) can

Fig. 1. Complete exact synchronization of the drive and response systems via the dynamic output feedback (11). States of the drive system (solid line) are tracked by states of the response system (dashed line).

be constructed and the system for the dynamic output feedback (6) is given by

$$\begin{aligned}\dot{\hat{z}}_1 &= \hat{\eta} - u + L\kappa_1(z_1 - \hat{z}_1) \\ \dot{\hat{\eta}}_1 &= L^2\kappa_2(z_1 - \hat{z}_1) \\ u &= \text{Sat}\{\hat{\eta} + K_1\hat{z}_1\}\end{aligned}\quad (11)$$

where

$$\text{Sat}\{\cdot\} = \begin{cases} U^{\text{MAX}}, & \text{if } u > U^{\text{MAX}} \\ \hat{\eta} + K_1\hat{z}_1, & \text{if } -U^{\text{MAX}} \leq u \leq U^{\text{MAX}} \\ -U^{\text{MAX}}, & \text{if } u < -U^{\text{MAX}} \end{cases}$$

where $U^{\text{MAX}} = 7.5$ was arbitrarily chosen. The control gain was chosen to be $K_1 = 7.0$. The estimation constants, κ_1 and κ_2 , were chosen is such way that the polynomial $s^2 + \kappa_1 s + \kappa_2 = 0$ has its roots at -1.0 . The high gain parameter $L = 100.0$. The initial conditions were arbitrarily chosen as follows: $x_M(0) = (0.1, -0.5, 0.5)$, $x_S(0) = (0.6, 0.4, -0.8)$ and $(\hat{z}_1(0), \hat{\eta}(0)) = (0.1, -0.2)$.

Fig. 1 shows the synchronization of the drive and response system. Here there is no message signal ($S_m = 0$). The feedback was turned on at $t = 30$ s. Note that the dynamic output feedback (11) yields complete synchronization. Indeed, an equal drive and response model was used in this numerical simulation. Consequently, the trajectories of the discrepancy system converges exactly to the origin (complete exact synchronization). Fig. 2 shows the performance of the proposed scheme. Here the dynamic output feedback was turned on at $t = 0$ and the message signal was transmitted for $t \geq 10$ s. The message signal was chosen to be a periodic function $S_m = \sin(\omega t)$ where $\omega = 10$ rad/s. The frequency was chosen

Fig. 2. An analog signal was transmitted. The output feedback (6) unmasks the message signal in spite of transmitter/receiver mismatches: The response system parameters were modified by: (a) 5%; (b) 10%; (c) 15%.

such that the dynamic behavior of the drive system is chaotic. The proposed scheme displays certain robustness margin against parametric variations. The parameters of the response system were varied 5% in Fig. 2(a) and (b) while in Fig. 2(c) and (d), a variation of 10% was induced. The message signal is decoded with acceptable accuracy in both cases. In fact, even if parameters of the response system are varied at 15%, the message signal can be acceptably decoded (see Fig. 2(e) and (f)). Note that since mismatches between the drive and response system are present, the trajectories of the discrepancy system converge to an arbitrarily small neighborhood containing the origin (complete practical synchronization). The same performance was found when the message signal consists of a digital word. The transmitted signal consisted of the digital word “message”. Fig. 3 shows the results for several parameters mismatches. The message signal was transmitted at 5 rad/s. Once again, the drive and response systems are completely practically synchronized and the message signal is acceptably recovered.

In order to obtain a measure of the performance, we consider a performance index defined by $I := T \int_0^T |S_m(\sigma) - u(\sigma)| d\sigma$, where T is a constant. Fig. 4 shows the index for several parameter values of the controller (11). The message signal was chosen in such way that the transmitter behaves chaotically, hence $S_m = 0.2 \sin(10t)$. The initial conditions of the drive system were chosen as $x_M(0) = (0.1, -0.5, 0.5)$ while the initial conditions of the response system were chosen to be $(0.2, -0.3, -0.1)$. The controller (11) was activated at $t = 0$ and the estimation constants were chosen to be $\kappa_1 = 2$ and $\kappa_1 = 1$. The message signal S_m was entered at $t = 0$. Here $\% \Delta \alpha := \pi_{1,S} = \pi_{1,M}(1 + \Delta \pi_{1,M})$ and $\% \Delta \beta := \pi_{3,S} =$

Fig. 3. The digital signal transmitted was chosen as the word “message”. The output feedback (6) recovers the message signal in spite of transmitter/receiver mismatches: The response system parameters were modified by: (a) 5%; (b) 10%; (c) 15%.

Fig. 4. Performance index versus parametric differences of drive and response systems. An analog signal was transmitted. Note that as the differences tends to zero as the recovery improves.

$\pi_{3,M}(1 + \Delta \pi_{3,M})$. Thus, for example, $\% \Delta \alpha = -10$ means that $\pi_{1,S} = 0.9\pi_{1,M}$ and $\% \Delta \beta = 5$ means that $\pi_{3,S} = 1.05\pi_{3,M}$.

The intervals of $\% \Delta \alpha$ and $\% \Delta \beta$ were chosen in such a way that systems (8) and (9) behave chaotically. Note that $\% \Delta \alpha \rightarrow 0$ and $\% \Delta \beta \rightarrow 0$ as the index I decreases. This implies that the message signal is recovered with acceptable accuracy.

Signal transmission in spite of strictly different model. The goal of this example is to illustrate that the message signal can be recovered: 1) in spite of different model for the drive and response systems, which is the extreme case of transmitter/receiver mismatch; 2) external perturbations by oscillatory signals which can be interpreted as noise and (iii) the transmission channels are different, i.e., $g_S(x_S) \neq g_M(x_M)$. Thus, we chose the Duffing equation as the transmitter and the van der Pol oscillator as receiver. The drive system becomes

$$\dot{x}_M = f_M(x_M; \pi_M) + g_M(x_M)S_m + \tau_M(t) \quad (12)$$

where $f_M(x_M) = (x_{2,M}, x_{1,M} + x_{1,M}^3 - 0.15x_{2,M})^T$, $g_M(x_M) = (0, 2)^T$, $\tau_M(t) = (0, 0.275 \cos(1.1t))^T$ and the message signal is given by $S_m = 0.25 \sin(2t)$. The response system is given by

$$\dot{x}_S = f_S(x_S; \pi_S) + g_S(x_S)u + \tau_S(t) \quad (13)$$

where $f_S(x_S) = (x_{2,S}, 0.01(1 - x_{1,S})x_{2,S} - x_{1,S})^T$, $g_S(x_S) = (0, 1)^T$, $\tau_S(t) = (0, 1.273 \cos(0.3t))^T$. If $h_1(x) = x_{1,M}$ and $h_2(x) = x_{1,S}$ are the comparison states, and by defining $x_i = x_{i,M} - x_{i,S}$, $i = 1, 2$, one gets the following discrepancy system

$$\begin{aligned} \dot{x} &= \Delta f(x; \pi) + g_M(x_M)S_m - g_S(x_S)u + \Delta \tau(t) \\ y &= x_1 \end{aligned} \quad (14)$$

where y is the output of the discrepancy system, $\Delta f(x; \pi) = f_M(x_M) - f_S(x_S)$ and $\Delta \tau(t) = \tau_M(t) - \tau_S(t)$. Thus the coordinate transformation is given by $z_1 = x_1$ and $z_2 = \dot{x}_2$. In such a way, system (14) is transformed into $\dot{z}_1 = z_2$ and $\dot{z}_2 = \alpha(z, t; \pi) + \beta(z)u$, where $\alpha(z, t; \pi) = \Delta f(z; \pi) + g_M(x_M)S_m + \Delta \tau(t)$ and $\beta(z) = g_S(x_S)$. Note that system (14) is *fully linearizable*, i.e., there is no unobservable states v in the discrepancy system because the relative degree $\rho = n$. Now, defining $\eta := \alpha(z)$, system (5) can be constructed and the feedback (6) becomes

$$\begin{aligned} \dot{\hat{z}}_1 &= \hat{z}_2 + L\kappa_1(z_1 - \hat{z}_1) \\ \dot{\hat{z}}_2 &= \hat{\eta} + g_S(x_S)u + L^2\kappa_2(z_1 - \hat{z}_1) \\ \dot{\hat{\eta}} &= L^3\kappa_3(z_1 - \hat{z}_1) \\ u &= -\hat{\eta} + K_1\hat{z}_1 + K_2\hat{z}_2 \end{aligned} \quad (15)$$

where $L = 5$, $\kappa = (3.0, 3.0, 1.0)^T$ and $K = (25.0, 5.0)$. In addition, note that if exact synchronization of system (12) and (13) is attained under external perturbations, $\Delta f(x; \pi) = 0$ and $x_M - x_S \equiv 0$ for all $t \geq 0$. This implies that $u g_S(x_S) = S_m(g_M(x_M) + \Delta \tau(t))$. This means that a filter can be required to recover the message signal. We designed a low pass filter whose transfer function is given by $G(s) = K_r/(\tau_r s + 1)$ where u_r is the filtered signal, τ_r is the characteristic time, K_r is an amplification factor. Parameters of the filter were arbitrarily chosen as $K_r = 0.5$ and $\tau_r = 3.9$. Fig. 5 shows the performance of the feedback (15). Fig. 5(a) and (b) shows the dynamical evolution of the discrepancy states. Systems (12) and (13) are syn-

Fig. 5. Recovering a message signal when the drive-system model is strictly different for the response system model. Simulations were carried out in the presence of an external perturbing signals. A lowpass filter was used and the message signal is acceptably recovered.

chronized. Moreover, the message signal is acceptably recovered with delay, which is induced by the filter (see Fig. 5(c)).

V. CONCLUSION

A chaos-based transmission scheme via synchronization of chaotic systems was presented in this paper. The synchronization problem was addressed as one of chaos suppression. To this end, a dynamical system was constructed which comprises the discrepancy between the drive and response systems. The proposed scheme consists of a dynamic output feedback which performs the suppression of chaos on the discrepancy system. The dynamic output feedback is designed by means of the following algorithm: 1) uncertainties in the discrepancy system are lumped into a nonlinear function and (ii) the lumping nonlinear function is interpreted as an augmented state. In particular, we have chosen a dynamic output feedback based on high-gain estimation and a *linearizer*-like controller. However, the proposed scheme is not restricted to this class of controller. Indeed, adaptive control or robust control theory can be used to design a dynamic output feedback which should render the chaos suppression of the augmented system of the discrepancy system.

In addition, the proposed scheme allows for message signal recovery in spite of parametric variations and strictly different model of transmitter and receiver. Since the drive and response systems are nonlinear, the transmitter/receiver mismatches can yield additive (or multiplicative) uncertainties. This class of uncertainties can be dealt with H_∞ control theory [23], which is

beyond the aim of this paper. However, results in this direction will be reported as soon as available.

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