Higher Order Spectral Analysis of Chua's Circuit

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***Abstract*—Higher order spectral analysis is used to investigate nonlinearities in time series of voltages measured from a realization of Chua's circuit. For period-doubled limit cycles, quadratic and cubic nonlinear interactions result in phase coupling and energy exchange between increasing numbers of triads and quartets of Fourier components as the nonlinearity of the system is increased. For circuit parameters that result in a chaotic, Rössler-type attractor, bicoherence and tricoherence spectra indicate that both quadratic and cubic nonlinear interactions are important to the dynamics. When the circuit exhibits a double-scroll chaotic attractor the bispectrum is zero, but the tricoherences are high, consistent with the importance of higher-than-second order nonlinear interactions during chaos associated with the double scroll.**

I. Introduction

SINCE THEIR introduction 30 years ago [4], higher order spectral techniques, which isolate nonlinear interactions between the Fourier components of a time series, have been used to study many nonlinear systems (see [2] for a recent review). The purpose of the present study is to investigate with higher order spectra the nonlinear interactions between triads and quartets of Fourier components of voltages measured in the Chua circuit as it undergoes a period-doubling cascade to chaos.

The auto bispectrum is formally defined as the Fourier transform of the third-order correlation function of the time series [4]. The discrete bispectrum, appropriate for sampled data, is [5], [7]

$$B(f_1, f_2) = E[A_{f_1} A_{f_2} A^*_{f_1+f_2}] \qquad (1)$$

where A_{f_i} is the complex Fourier component of the time series at frequency f_i, asterisk is complex conjugation, and $E[\cdot]$ the expected-value, or average, operator.

From (1), the bispectrum is zero if the average triple product of Fourier coefficients is zero. This occurs if the Fourier components are independent of each other, i.e., for the random phase relationships between Fourier modes in a linear process, such as a time series with Gaussian statistics. It is convenient to recast the bispectrum into its normalized (by the power at each of the three frequencies in the triad) magnitude, called the squared bicoherence, $b^2(f_1, f_2)$, which represents the fraction of the power of the triad of Fourier components $(f_1, f_2, f_1 + f_2)$ that is owing to quadratic coupling [7].

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Similar to the bispectrum, the auto trispectrum is formally defined as the Fourier transform of the fourth-order correlation, and the discrete trispectrum is

$$T(f_1, f_2, f_3) = E[A_{f_1} A_{f_2} A^*_{f_1+f_2+f_3}]. \qquad (2)$$

The normalized magnitude of the trispectrum is called the squared tricoherence, $t^2(f_1, f_2, f_3)$ and is a measure of the fraction of the power of the quartet of Fourier components $(f_1, f_2, f_3, f_1 + f_2 + f_3)$ that is owing to cubic nonlinear interactions. Further details, additional references, and tutorial examples of bispectra and trispectra can be found in [2].

II. Higher Order Spectra of Chua's Circuit

Chua's circuit is described by a set of three ordinary differential equations [8], [1], [11]

$$\begin{aligned} C_1 \frac{dv_{C_1}}{dt} &= G(v_{C_2} - v_{C_1}) - g(v_{C_1}) \\ C_2 \frac{dv_{C_2}}{dt} &= G(v_{C_1} - v_{C_2}) + i_L \\ L \frac{di_L}{dt} &= -v_{C_2} \end{aligned} \qquad (3)$$

where v_{C_1} and v_{C_2} are the voltages across capacitors C_1 and C_2 respectively, and i_L is the current flowing upwards through the inductor. G denotes the conductance of a nonlinear resistor and $g(\cdot)$ is a piecewise-linear function relating the current in the resistor $(g(v_R))$ to the voltage (v_R)

$$g(v_R) = m_0 v_R + 0.5(m_1 - m_0)(|v_R + B_p| - |v_R - B_p|). \qquad (4)$$

The slope of the current versus voltage curve changes from m_0 to m_1 when the voltage changes in absolute value from greater than B_p to less than B_p. The implementation of Chua's circuit used in the present study is discussed in [6], [3]. The nonlinearity of the system is increased by increasing the capacitance C_1, while the other components remain fixed (here, $C_2 = 178.5$ nF, $R = 1.001$ kΩ, $L = 12.44$ mH, $B_p = 1$ V, $m_0 = -0.712$, and $m_1 = -1.14$). Five time series of the voltage waveform v_{C_1}, including a period-doubling sequence and two chaotic states (Rössler [10] and double-scroll attractors [1], [9]), were examined.

The harmonic structure of the limit cycles is clearly displayed in the power spectra (Fig. 1). For period-1 motion, the spectrum is dominated by a primary spectral peak with $f = 2.5$ kHz, and its higher harmonics. As C_1 is increased, the subharmonic ($f = 1.25$ kHz) is excited (period-2 motion, Fig. 1(b)) and, owing to nonlinear interactions, the spectrum contains peaks at frequencies corresponding to sum interactions of the subharmonic, the primary, and their harmonics.

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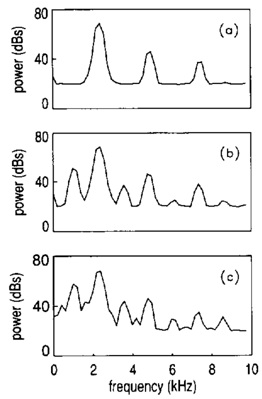

Fig. 1. Power spectra of voltage v_{c_1} measured from Chua's circuit. (a) Period-1 limit cycle, (b) period-2 limit cycle, and (c) period-4 limit cycle. The corresponding values of the capacitance, C_1 are 17.2, 17.5, and 16.9 nF, respectively. The units of power are arbitrary. (See [2] for data processing details.)

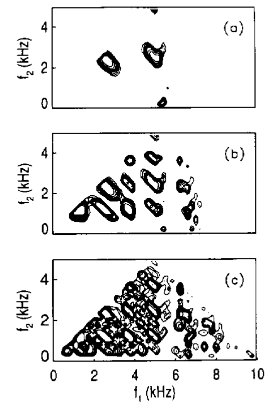

Fig. 2. Bicoherence spectra of voltage v_{c_1} measured from Chua's circuit. (a) Period-1 limit cycle, (b) period-2 limit cycle, and (c) period-4 limit cycle. Contours indicate quadratic phase coupling between motions with frequencies f_1, f_2, $f_1 + f_2$. The minimum contour plotted is $b = 0.2$, with additional contours every 0.1.

As C_1 is increased further, another period-doubling occurs ($f = 0.625$ kHz is excited, period-4 motion), and the power spectrum (Fig. 1(c)) contains many peaks, corresponding to the two subharmonics, the primary, and their combination tones.

The quadratic and cubic interactions between triads and quartets of Fourier components for the limit cycles of Chua's circuit are isolated by bicoherence and tricoherence spectra, respectively (Figs. 2 and 3). For the period-1 case, the bicoherence spectrum clearly shows the quadratic coupling between motions at the primary spectral peak frequency and its harmonics ($f_1 = 2.5$, $f_2 = 2.5$ and $f_1 = 5.0$, $f_2 = 2.5$ kHz, Fig. 2(a)). The high bicoherence values associated with $f_2 =$ 2.5 kHz (Fig. 2(a)) indicate nonlinear energy transfer from the primary to higher-frequency components. The quadratic interactions are restricted to triads of Fourier components that include the primary and its harmonics. Cubic interactions also occur for period-1 motion, and the tricoherence spectrum indicates that there is strong coupling among the quartet of components consisting of $f_1 = f_2 = f_3 = 2.5$ and $f_4 = 7.5$ kHz (Fig. 3(a)), as well as between the primary and higher harmonics (e.g., $f_1 = 5$, $f_2 = 2.5$, $f_3 = 2.5$, $f_4 = 10$ kHz, see fig. 11 of [2]).

The power spectrum for the period-2 case (Fig. 1(b)) shows narrow peaks between the harmonics of the primary peak. The corresponding bicoherence spectrum (Fig. 2(b)) shows the coupling between motions at the primary peak frequency ($f = 2.5$ kHz), its harmonics, the period-doubled frequency (subharmonic, $f = 1.25$ kHz), and its harmonics. Quadratic interactions between oscillations at the primary and the period-doubled subharmonic are transferring energy into higher harmonics. Similarly, there are strong cubic interactions between the subharmonic and the primary (Fig. 3(b)), as well as between the subharmonic and higher frequency motions (not shown, see [2]).

For period-4 motion an additional subharmonic is excited ($f = 0.625$ kHz), and the power (Fig. 1(c)), bicoherence (Fig. 2(c)), and tricoherence (Fig. 3(c)) spectra contain peaks

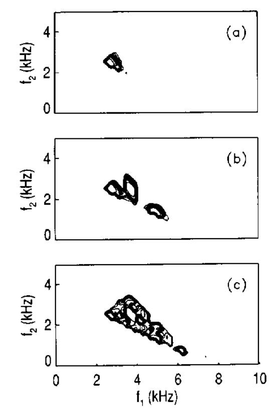

Fig. 3. Tricoherence spectra of voltage v_{c_1} measured from Chua's circuit. (a) Period-1 limit cycle, (b) period-2 limit cycle, and (c) period-4 limit cycle. In each panel, contours indicate cubic phase coupling between motions with frequencies f_1, f_2, f_3, $f_4 = f_1 + f_2 + f_3$, where $f_4 = 7.5$ kHz. The minimum contour plotted is $t = 0.2$ with additional contours every 0.1.

associated with the primary ($f = 2.5$ kHz), both subharmonics ($f = 1.25$ and $f = 0.625$ kHz), and all their combination tones. The higher order spectra indicate that these interactions are quadratic and cubic, and delineate precisely which Fourier components are interacting with each other.

Time series corresponding to the Rössler attractor exhibit similarities to, and differences from, the period-doubling sequences. The Rössler-like attractor is chaotic and has a fairly broad power spectrum (Fig. 4(a)) with only remnants of the sharp primary and harmonic peaks of the period-doubled cases (Fig. 1). However, as indicated by the bicoherence (Fig. 4(b)) and tricoherence (Fig. 5) spectra, both quadratic and cubic interactions are still important. Motions corresponding to the remnant of the primary peak ($f_2 = 2.5$ kHz) are quadratically coupled to both higher frequencies (horizontal band of contours at $f_2 = 2.5$ kHz in Fig. 4(b)) and to lower frequencies (vertical band of contours at $f_1 = 2.5$ kHz).

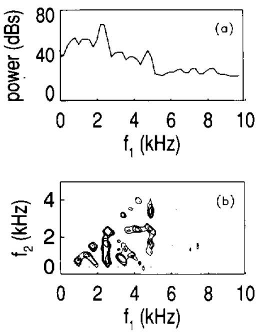

Fig. 4. (a) Power spectrum and (b) contours of bicoherence of voltage v_{c_1} measured from Chua's circuit when the system exhibits a Rössler-like attractor. $C_1 = 16.6$ nF. The units of power are arbitrary and the minimum contour plotted is $b = 0.6$ with additional contours every 0.1.

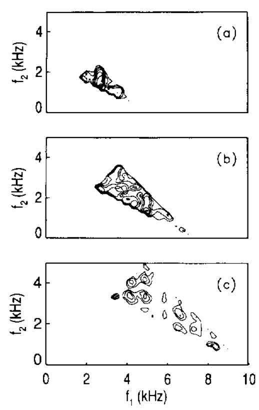

Fig. 5. Tricoherence spectra of voltage v_{c_1} measured from Chua's circuit when the system exhibits a Rössler-like attractor. The constant sum frequencies are (a) $f_4 = 5.0$, (b) $f_4 = 7.5$, and (c) $f_4 = 10.0$ kHz. The minimum contour plotted is $t = 0.2$ with additional contours every 0.1.

Weaker quadratic interactions occur between motions at the harmonics ($f = 5.0$ and $f = 7.5$ kHz) of the primary spectral peak and both higher and lower frequency motions. Bicoherences are statistically significant for many frequency triads, indicating that quadratic nonlinear interactions occur between nearly all the frequency components of the Rössler system. Cubic nonlinear interactions (Fig. 5), although individually weaker than the quadratic interactions, are also important in the Rössler attractor and occur between nearly all the components of the system.

Unlike the period-doubled and Rössler attractors, the double scroll is not dominated by quadratic interactions. The double-scroll attractor is characterized by a very broad power spectrum (not shown, see [2]), with only a vestige of the primary peak and much more energetic low-frequency motions. There are no statistically significant bicoherences, indicating that the nonlinearities for the double scroll are not quadratic. Higher-than-second-order spectra are required to isolate the individual interacting Fourier components for the double scroll. The tricoherence spectrum (not shown, see [2]) demonstrates that many of the Fourier components of the double-scroll system are cubically coupled to each other, and further suggests that interactions involving low frequency components are important to the dynamics of the double scroll (i.e., there are many cubically coupled triads involving low frequencies). Similar importance of nonlinear interactions involving very low frequency components has been observed in other chaotic systems.

III. Conclusions

As Chua's circuit undergoes a period-doubling sequence to chaos, quadratic and cubic nonlinear interactions couple, and transfer energy between, triads and quartets of the Fourier components of the system. The individual interactions are isolated by bicoherence and tricoherence spectra, which show increasing numbers of phase coupled components as the nonlinearity of the system is increased. For period-doubled limit cycles, motions at the frequency corresponding to the power spectral peak frequency and the many combinations of subharmonics and super harmonics are nonlinearly coupled. When the system becomes chaotic the power spectrum broadens, but in the case of the Rössler attractor, quadratic and cubic nonlinear interactions remain important. Motions at the primary frequency and its harmonics are coupled to motions at many other frequencies. For the double-scroll attractor the system no longer contains quadratic nonlinear interactions, and bicoherence values are essentially zero. However, tricoherence spectra demonstrate the continued importance of cubic nonlinear interactions.

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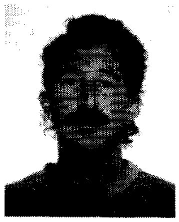

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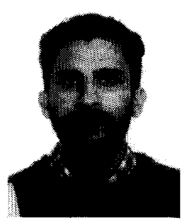

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