Generation of Homoclinic Oscillation in Coupled Chua's Oscillators

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***Abstract*-An experimental method of generating homoclinic oscillation using two nonidentical Chua's oscillators coupled in unidirectional mode is described here. Homoclinic oscillation is obtained at the response oscillator in the weak coupling limit of phase synchronization. Different phase locking phenomena of homoclinic oscillation with external periodic pulse have been observed when the frequency of the pulse is close to the natural frequency of the homoclinic oscillation or its sub-harmonics.**

***Index terms*- Homoclinic orbit, coupled Chua's oscillator, limit cycle, phase synchronization.**

I. INTRODUCTION

Generation of homoclinic chaos of Shil'nikov type [1] in laser has been reported in [2] and its phase synchronization (PS) with sinusoidal forcing has been confirmed in [3-4]. Its possible application in information encoding in the interspike interval [5] of homoclinic chaos has also been explored recently for the purpose of secure communication. Encoding [5] uses PS [6] of message signal in the time sequence of spiking homoclinic chaos. The shape of spikes may be changed by channel noise but the time sequence of spike trains is not disturbed. Complete synchronization (CS) in master-slave coupled chaotic oscillators, as proposed [7] earlier for secure communication, is susceptible to channel noise [8] that causes loss of synchronization. Thus homoclinic chaos has advantage over other deterministic form of chaos in communication applications, particularly, in the context of chaotic pulse position modulation scheme (CPPM) [9]. Moreover, the dynamics of interspike intervals of chaotic signal is a recent thrust area [10] in chaotic dynamics. The question still remains unanswered how sensory system of neuron assembly encodes external information in the form of interspike intervals. The spiking train of homoclinic chaos has similarity with the spiking neurons [11,12] in response to external stimuli. Studies on PS of homoclinic chaos with external stimulation may help understanding how neurons communicate information with each other. In this context, we attempted an experiment with two coupled Chua's oscillators for the generation of homoclinic oscillation as reported in this paper. Homoclinic orbit is a bounded dynamical trajectory asymptotic to equilibrium point of a model flow in both forward and backward time. The homoclinic oscillation is defined here as repeated cycles of homoclinic trajectory, which tends to one of the mirror symmetric equilibrium of Chua's oscillator and moves spirally away from it.

Numerical methods [13] are available for locating homoclinic orbit, which relied on either *continuation* of a limit cycle to large period as it approaches homoclinic orbit or using numerical integration of *shooting* orbits in the stable and unstable manifolds of the equilibrium and then computing a distance between them. Further improvements on these methods have been included in the numerical tool HOMCONT [14], but it is quite involving in rigors of numerical algorithm. Experimental studies on imperfect homoclinic bifurcation using van der Pol oscillator has also been found in literature [15]. It is often found [16,17] in coupled neural oscillators that two limit cycle oscillations lead to PS in the weaker coupling limit when the limit cycles are close to the homoclinic bifurcation. Homoclinic bifurcation has been established in [18] as a mechanism of PS. Two coupled Chua's oscillators in drive-response mode can, indeed, generate homoclinic oscillation in the weaker coupling limit of PS as shown in our experiment. The homoclinic orbit at the response repeats its trajectory with a time period of the limit cycle oscillation of the driver. Experimental evidence of phase locking on homoclinic oscillation to external periodic pulses is also presented. In the next section, the experimental setup of generating homoclinic oscillation is elaborated. The mechanism of homoclinic oscillation in coupled Chua's Oscillator is explained in section III. In section IV, phase locking of homoclinic oscillation with external periodic pulse is described with a conclusion in section V.

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II. EXPERIMENTAL SET UP

Existence of homoclinic orbits has been proved [19] numerically in Chua's circuit as a rigorous proof of chaos in such system. Homoclinic orbit has also been identified numerically as biasymptotic to a saddle focus in 3-D space of Chua's circuit [1] and in Colpitts oscillator [20], but they are found to exist for a moderate amount of time only before the trajectory veers away from the homoclinic orbit. The experimental circuit with two unidirectionally coupled nonidentical Chua's oscillators, as shown in Fig.1, can generate homoclinic oscillation, which repeats its homoclinic cycles for long period of time. The parameters of the driver oscillator are selected for single scroll oscillations in the period-adding bifurcation regime [21]. In this regime, the driver has periodic windows of single scroll limit cycles (period-3, period-4, period-5 even higher period) of large amplitude that crosses from either (depending upon the initial point) of the mirror symmetric outer regions to the inner region near origin [19,22] as shown in Figs.2-3. This large amplitude of limit cycle is found as an important criterion for generation of homoclinic oscillation in the response. The amplitude of the limit cycle oscillation of the driver is not large enough, in the period-doubling bifurcation regime, to cross to the inner region. Before coupling, the response oscillator is kept in non-oscillating state (stable focus) with appropriate choices of circuit parameters. For strong coupling, the response oscillator as forced by the limit cycle oscillation of the driver, produces limit cycle oscillation in complete synchrony [23] with the driver. The amplitudes and phases of coupled oscillators are correlated in this CS regime. As the coupling strength is made weaker and weaker, successive synchronization regimes of lag synchronization (LS), intermittent lag synchronization (ILS) and PS have been observed [23,24]. The amplitudes of the coupled oscillators are uncorrelated in PS regime but the phases still remain correlated. In this PS regime, for a critical coupling strength, the response trajectory spirals away from the equilibrium as driven out by the limit cycle oscillation of the driver. But it tends to equilibrium at the instant the trajectory of the limit cycle oscillation of the driver crosses from mirror symmetric outer region to the inner region as deep into the negative region near origin. It is possible only when the driver oscillation is large enough. The mechanism of homoclinic oscillation is given in detail in the next section. The homoclinic

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orbits in 3-D space as shown in Fig.4 are reconstructed using embedding technique. Finally, a phase velocity measure is used to confirm that the response trajectory reaches the equilibrium. The phase velocity V_ϕ is calculated from measured time series using embedding technique as given by

$$V_\phi = \sqrt{{v_{t-\tau}}^2 + {v_{t+\tau}}^2 + {v_{t+3\tau}}^2} \qquad (1)$$

$$where\ v_t = \frac{dV_{C3}(t)}{dt}$$

$V_{C3}(t)$ is the node voltage measured at capacitor C_3 in response oscillator, τ is an appropriately chosen delay time. The phase velocity of a trajectory is zero at equilibrium. This is taken as a basis whether a trajectory tends to equilibrium or is close to it. The response trajectory is a limit cycle close to equilibrium for coupling strength both higher and lower than a critical coupling as shown in Fig.5. The coupling strength acts as control parameter once the driver parameters are fixed in the period-adding bifurcation regime. The homoclinic oscillation shows intermittent instabilities because of instabilities in the driver due to unavoidable noise or parameter fluctuations, which can be stabilized by small periodic pulse forcing. A digital oscilloscope (TEK, TDS220) is used for measurements with its software (WaveStar) for data acquisition.

III. MECHANISM OF HOMOCLINIC OSCILLATION

The uncoupled response oscillator at rest has three equilibria [19, 24], one unstable focus at origin (in the inner region D_0) and two mirror symmetric stable foci (in the outer regions D_1 and D_{-1}). The unstable focus has one real positive eigen value and two complex conjugate eigen values with real negative parts. The eigen values of the coupled oscillator (period-4) are given in Table 1.

Table 1. Eigen values [R_d=1516Ω, R_r=1750Ω and R_c=10.96kΩ]

Circuit	Inner region	Outer region
Driver	4.483, -1.291 ± 3.098i	-4.348, 0.045 ± 2.921i
Response	4.424, -1.234 ± 3.548i	-4.324, -0.094 ±3.406i

The stable focus of the response has one real negative eigen value and two complex conjugate eigen values with negative real parts. If an external periodic pulse is forced, a spiking trajectory moves away from one of the mirror symmetric stable focus (depending upon the initial point) at the start of the pulse. At the end of the pulse, the spiking trajectory spirally tends to the equilibrium due to the real negative part of the complex eigen value of the stable focus until the pulse is repeated again. Instead of forcing a periodic pulse, another self oscillating Chua's oscillator as driver is coupled in unidirectional mode to the response oscillator for the generation of homoclinic oscillation as discussed in section II. The driver parameters are selected for single scroll oscillation near period-4, period-5 limit cycles, when each of the mirror symmetric equilibria is a saddle focus with one real negative and two complex conjugate eigen values with positive parts (Table 1). The limit cycle trajectory (period-4, period-5 or higher orbits) of the driver spirals away around the equilibrium (say, in D_{-1} region) and moves to the inner region D_0. In this inner region, the trajectory is folded back by the real unstable eigenvector of inner region D_0 and ultimately comes back along the real eigenvector of D_1 region as shown in Fig.2. Alternatively, for period-4 and higher periodic orbits, the driver trajectory may (depending upon the initial point) expands spirally from the other outer region (D_1) to the D_0 region and even crosses to the negative region near origin (Fig.3). Subsequently, the trajectory of the driver twisted around the unstable real eigenvector of D_0 region and comes back to its original outer region (D_1). Forced by the period-4 or period-5 oscillation of the driver, the response oscillator follows the driver with a time lag for weaker coupling [23,24] until the driver trajectory reaches zero. At this instant, the response trajectory folded back fast to the equilibrium in outer region along its real eigenvector with negative eigenvalue as shown in Fig.6a. When the driver grows spirally again to repeat its periodic cycle, the response trajectory is forced spirally away from the equilibrium and repeats its homoclinic cycle. Thus the coupled circuit generates a homocilinic oscillation at the response when the driver and response are phase locked. The phase difference between the driver and response remains bounded in time as shown in the phase difference plot in Fig.6b.

IV. PHASE LOCKING WITH EXTERNAL PULSE

Forcing a periodic pulse at the driver capacitor C_1 (Fig.1), the instabilities in homiclinic oscillation due to parameter fluctuations have been stabilized in the response oscillator. The homoclinic oscillation readjusts its period to different stable homoclinic oscillations as forced by the periodic pulse. The amplitude of external forcing pulse is selected small enough while the pulse period is chosen as close to the time period of homoclinic oscillation. The external pulse is phase locked to the homoclinic oscillations with different locking ratios. For the selected parameters, the natural period of the Chua's oscillator (driver) is $T_n=1/f_n$=390μs, f_n is the natural frequency of homoclinic oscillation. When the pulse period is $T_p \cong T_n \pm T_s$ (close to T_n within a small bound of period. $T_s \cong$14μs), 1:1 phase locking has been observed (Fig.7a). Homoclinic oscillation is a period-5 oscillation, hence the period of a homoclinic cycle is $T_h=5T_n$=1950μs while the forcing pulse has an amplitude of 248mV and time period T_p=1936μs ($T_s \cong$14μs). For pulse period T_p=1152μs and amplitude 336mV, the homoclinic oscillation is a stable period-6 orbit (Fig.7b), whose time period is given by $T_h=6T_n$= 2340μs. The pulse is 1:2 phase locked to the period-6 homoclinic orbit; $T_h=2T_p+T_s$ =2340μs ($T_s \cong$36μs). For pulse amplitude of 420mV and T_p=762μs, the homoclinic oscillation is stable at perioid-6 orbit (Fig.7c), whose time period is given by $T_h=6T_n$=2340μs. The pulse is 1:3 phase locked to the period-6 homoclinic orbit; $T_h=3T_p+T_s$ =2340μs ($T_s \cong$54μs). If the difference in time period between forced pulse and homoclinic oscillation is larger beyond a bounded limit (say, ~54μs or more), phase slips occur and finally loses phase locking. Details are not shown here due space limitations. The pulse amplitude is measured using the digital oscilloscope at source. Actual amplitude of the pulse at the capacitor C_1 is much less since the pulse source V_s is connected to the capacitor with a series resistance R_s=25kΩ.

V. CONCLUSION

An experiment using two Chua's oscillators coupled in the PS regime is described, which can generate homoclinic oscillation for a long time. A phase velocity measure is used to confirm that the 3-D homoclinic trajectory reaches equilibrium for a critical coupling. Phase locking of homoclinic oscillation with external periodic pulse is possible and the locking ratio depends upon the frequencies of the driver and the forcing pulse. An aperiodic pulse train can be phase locked to the homoclinic oscillation if the variation in pulse intervals remain within a smaller bound. Message can be encoded in the varying time intervals of the aperiodic pulse, which problem is undertaken for our future work.

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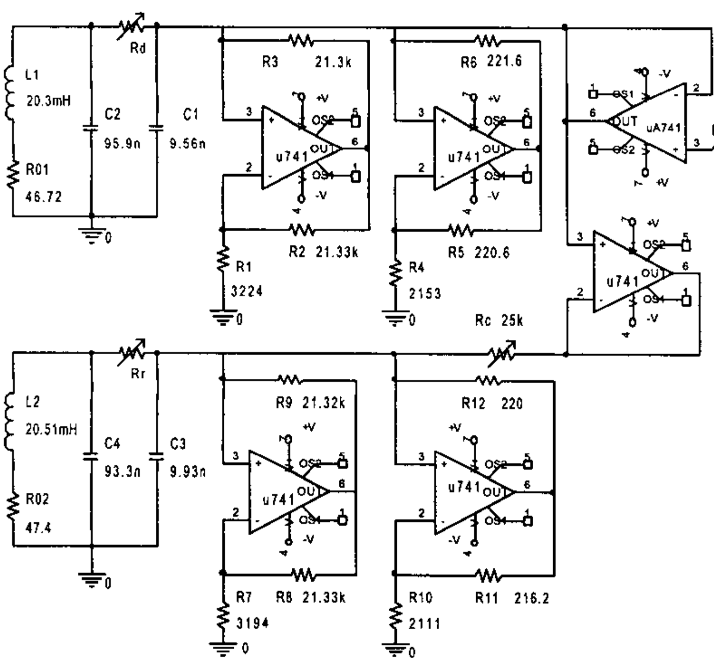

Fig.1. Two Chua's oscillators coupled in drive-response mode: R_d and R_r are the control parameters for different regimes of oscillation in the driver and response respectively. R_c decides the coupling strength, which is strong for lower value and vice versa. V_s is the external pulse source.

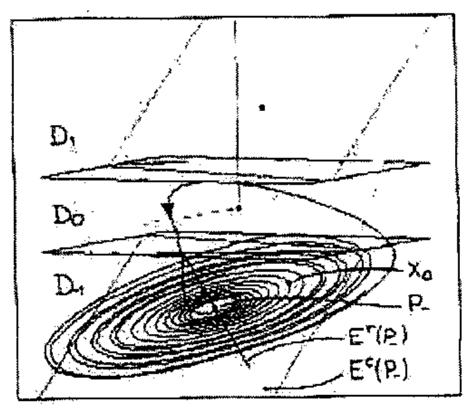

Fig.2. Three eigen planes of Chua's oscillator, two in mirror symmetric outer regions D_1, D_{-1} and one in the inner region D_0. $E^r(P_-)$ is the eigenvector and $E^c(P_-)$ is eigenplane corresponding to real and complex eigen values at euilibrium P_- in outer region D_{-1} [reproduced from Ref.22].

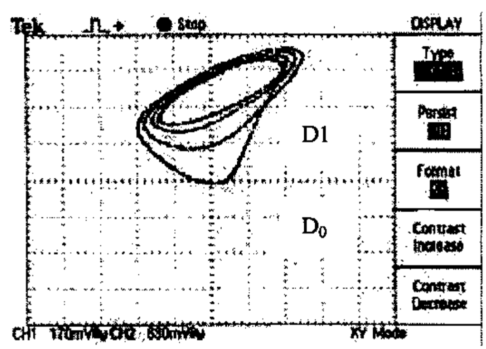

Fig.3. Phase portrait of period-4 orbit of the driver for R_d=1516Ω, oscilloscope display of capacitor voltages V_{C2} (CH1:170mV/div) and V_{C1}(CH2:690mV) with DC coupling to capacitor nodes.

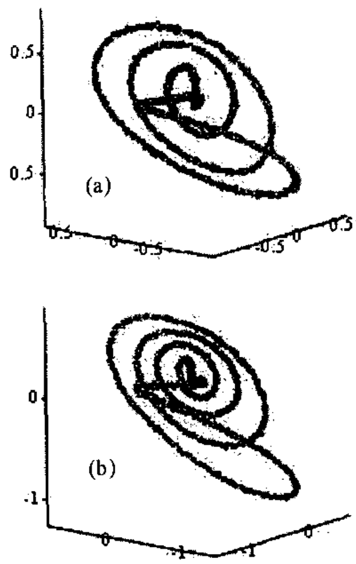

Fig.4. Homoclinic trajectory in 3-D space reconstructed using embedding technique: (a) period-4 homoclinic orbit, R_d=1516Ω, R_r=1750Ω, R_c=10.96kΩ, (b) period-5 homoclinic orbit, R_d=1499Ω, R_r=1879Ω, R_c=18.36kΩ.

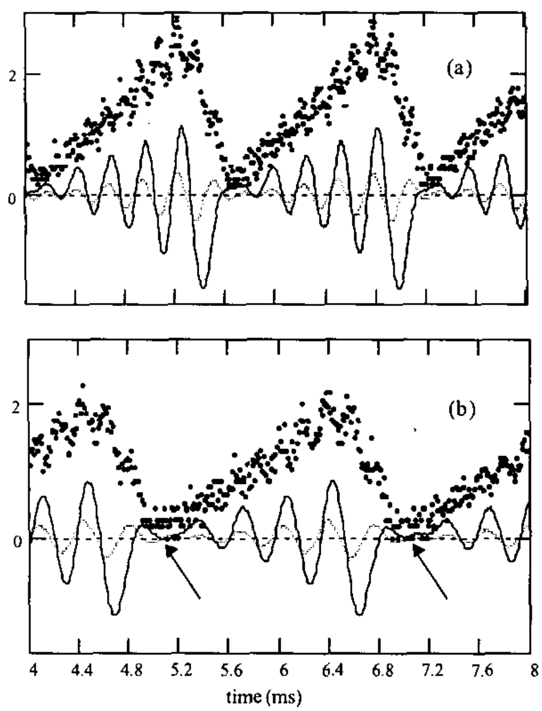

Fig.5. Phase velocity V_ϕ (in cluster of dots) varies periodically with minima [period-5 homoclinic cycle, R_d=1499Ω, R_r=1879Ω]. For coupling stronger than critical coupling (a) R_c=12.14kΩ, phase velocity minima are close to zero indicating the trajectory is a limit cycle close to equilibrium. The minima (at 5.13ms and 7ms) are exactly zero for critical coupling (b) R_c=18.36kΩ, when homoclinic trajectory reaches equilibrium. Measured voltages V_{C3} (bold trace) and V_{C4} (dotted trace) reach zero simultaneously when phase velocity is zero as indicated by arrows. Voltages are measured wit AC coupling of oscilloscope when the equilibrium is set at origin.

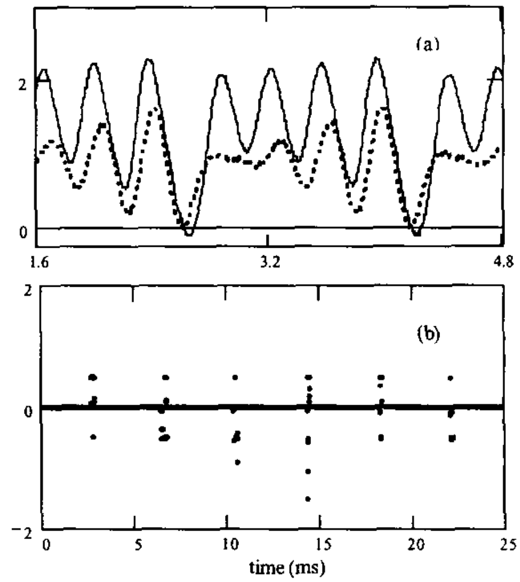

Fig.6. Phase synchronization of driver (bold line) and response (dotted line) signals for period-4 limit cycle: R_d=1516Ω, R_r=1750Ω, R_c=10.96kΩ. (a) response is phase locked at the instant the driver crosses zero, (b) phase difference of driver and response as calculated by using Hilbert transform on scalar signal [23] is bounded in time. Few errors in phase estimation as $\pm\pi$ jumps are seen as the driver signal changes in sign of the slope while crossing from positive to negative region near origin.

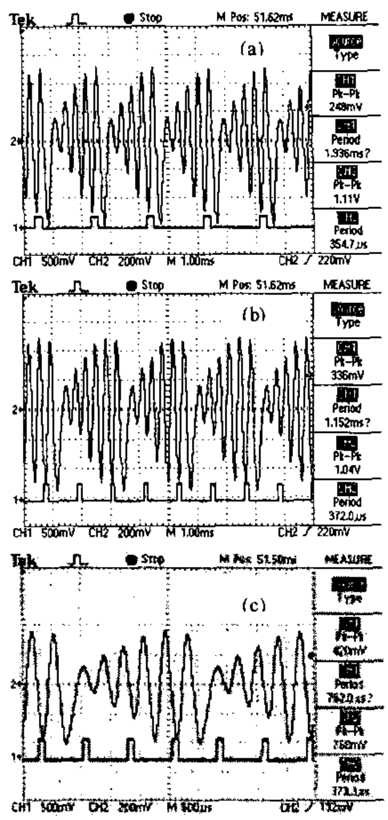

Fig.7 Phase locking with external pulse (a) 1:1 locking, pulse amplitude=248mV, T_P=1936us (b) 1:2 locking, pulse amplitude=336mV, T_P=1152us (c) 1:3 locking, pulse amplitude=420mV, T_P=762us