Ordering Chaos of Chua's Circuit

— A Feedback Control Approach

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Abstract. *Ordering or controlling chaos has in recent years attracted special attention from a number of research groups in the areas of nonlinear dynamics and control. The present authors have developed a unified linear feedback control methodology for this purpose, which works well for many general chaotic systems. It has been shown that under certain conditions a chaotic trajectory can be guided to one of the unstable limit cycles of the dynamic system, provided that an appropriate feedback control is applied. This control technique is refined and applied to the well-known Chua's circuit in this paper, giving simplier implementation with even better results. A simple sufficient condition for the design of such a linear feedback controller is given, and the numerical simulation as well as the physical implementation of the designed feedback control configuration are both illustrated.*

1 Introduction

Chua's circuit is a well-known electronic system, which displays very rich and typical bifurcation and chaotic phenomena such as double scroll, dual double scroll, double hook, etc. The circuit itself is quite simple: it consists of only one inductor (L), two capacitors (C_1, C_2), one linear resistor (G) and one piecewise-linear resistor (g) [1-8]. Its dynamics can be described by

$$\left\{\begin{array}{rcl} C_1\dot{v}_{C_1} & = & G(v_{C_2} - v_{C_1}) - g(v_{C_1}) \\ C_2\dot{v}_{C_2} & = & G(v_{C_1} - v_{C_2}) + i_L \\ L\dot{i}_L & = & -v_{C_2} \end{array}\right. \tag{1}$$

where v_{C_1} and v_{C_2} are the voltages across C_1 and C_2, respectively, i_L the current through the inductor L, and the v-i characteristic of the nonlinear resistor is

$$\begin{array}{rcl} g(v_{C_1}) & = & g(v_{C_1}; m_0, m_1) \\ & = & m_0 v_{C_1} + \frac{1}{2}(m_1 - m_0)(|v_{C_1} + 1| - |v_{C_1} - 1|) \end{array}$$

with $m_0 < 0$ and $m_1 < 0$ being some appropriately chosen constants [2].

Since the inception of Chua's circuit in early 1980's, much attention has been devoted to the investigations of the dynamical, analytical, experimental, or implemental aspects of the circuit and its associate canonical circuit family [1-8]. To our knowledge, however, very little is known as how to introduce order into the chaotic responses of the circuit, except perhaps for a paper by the present authors [23].

It has been known [2] that Chua's circuit has an unstable saddle-type limit cycle existing outside the double scroll attractor: its Poincaré map is stable in one direction but unstable in another direction. This limit cycle cannot be observed from the oscilloscope measuring the circuit, nor be obtained by ordinary numerical integration techniques. This unstable limit cycle has nevertheless been verified from different points of view [2]. We hereafter will discuss a technique of designing a simple linear feedback control law which allows such an unstable circuit response to emerge from its chaotic state (double scroll) of the trajectory, and to approach and finally reach the inherently unstable limit cycle of the circuit.

Different algorithms for controlling chaotic systems have been developed in recent years [9-22]. Most of the methods proposed so far can be divided into two major classes. One is controlling chaos by varying or perturbing some key parameters of the system in certain skillful way. However, perturbing the system's parameters actually changes the main characteristic of the original dynamic system, and the control of such key parameters usually depends on many issues such as the specific system under investigation and the experience of the designer. Besides, very often the system under consideration is required to operate at a particular set of parameters and therefore any variation of parameters is not allowed. The other major type of methods employs conventional control engineering techniques. These methods do not necessarily resort to feedback, but the approach taken by the present authors [20-24] provides a stabilizing technique using feedback control, which turns out to be very efficient and has indeed a unified theme in the sense that we are not only able to describe a general method which can be applied to different chaotic systems but also able to drive a chaotic trajectory to both unstable equilibrium points and unstable (even multi-periodic) limit cycles. One of the advantages of this approach is that no system parameter needs to be adjusted directly, and controller is added to the original system "from outside" whose effect will vanish immediately whenever it is being disconnected from the system. This general feedback control strategy has been successfully applied to many typical nonlinear chaotic systems such as the discrete-time Hénon [22] and Lozi [21] systems and the continuous-time Duffing [20] system. While all the feedback controllers used in [20-23] are linear, the design of nonlinear feedback controller is also feasible as shown in [24].

The control technique developed in [23] is refined and applied to the well-known Chua's circuit in this paper, giving simpler implementation with even better results. A simple sufficient condition for the design of such a linear feedback controller is given, and the numerical simulation as well as the physical implementation of the designed feedback control configuration are both illustrated.

2 Control of Chua's Circuit

2.1 Dynamics of Chua's Circuit

To facilitate our presentation, the circuit equation, Eq. (1), of the Chua's circuit is first reformulated into the following dynamically equivalent state equation [2]:

$$\left\{\begin{array}{rcl} \dot{x} & = & p\,[-x + y - f(x)] \\ \dot{y} & = & x - y + z \\ \dot{z} & = & -qy \end{array}\right. \tag{2}$$

where $p = \frac{C_2}{C_1} > 0$ and $q = \frac{C_2}{LG^2} > 0$ are two main bifurcation parameters of the circuit, and corresponding to $g(v_{C_1})$ in Eq. (1), $f(x)$ is represented by

$$\begin{array}{rcl} f(x) & = & g(x; m_0', m_1') \\ & = & m_0'x + \frac{1}{2}(m_1' - m_0')(|x+1| - |x-1|)\,, \end{array}$$

or by a 3-segment piecewise-linear function

$$f(x) = \left\{\begin{array}{ll} m_0'x + m_1' - m_0' & x \geq 1 \\ m_1'x & |x| \leq 1 \\ m_0'x - m_1' + m_0' & x \leq -1 \end{array}\right.$$

where $m_0' = \frac{m_0}{G} < 0$ and $m_1' = \frac{m_1}{G} < 0$.

For consistency, we use parameter values $p = 9$, $q = 14\frac{2}{7}$, $m_0' = -\frac{5}{7}$, and $m_1' = -\frac{8}{7}$, as did in [2] and in many other papers studying Chua's circuit. We observe a double scroll (strange attractor) and the aforementioned unstable saddle-type periodic orbit, which are both

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shown in Fig. 1, where the initial point $(-0.1, -0.1, -0.1)$ was used for the strange attractor.

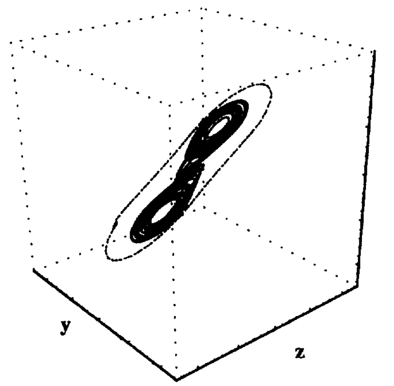

Figure 1. A double scroll attractor and a saddle-type periodic orbit of Chua's circuit.

2.2 Design of Linear Feedback Controllers

Denoting by $(\bar{x}(t), \bar{y}(t), \bar{z}(t))$ one of the unstable limit cycles of the circuit represented by Eq. (2), our goal is to control the system trajectory such that for any given $\varepsilon > 0$, there exists some $T_\varepsilon > t_0$ for which

$$|x(t) - \bar{x}(t)| \le \varepsilon, \quad |y(t) - \bar{y}(t)| \le \varepsilon, \quad \text{and}$$
$$|z(t) - \bar{z}(t)| \le \varepsilon \qquad \text{for all} \quad t \ge T_\varepsilon .$$

We have the following main result on this controllability of the circuit:

Theorem 1 *Let $(\bar{x}, \bar{y}, \bar{z})$ be the unstable limit cycle of Chua's circuit described by Eq. (2). Then, the chaotic trajectory (x, y, z) of the circuit can be driven to reach this limit cycle by a linear feedback control of the form*

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = -K \begin{bmatrix} x-\bar{x} \\ y-\bar{y} \\ z-\bar{z} \end{bmatrix} = -\begin{bmatrix} 0 & 0 & 0 \\ 0 & K_{22} & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x-\bar{x} \\ y-\bar{y} \\ z-\bar{z} \end{bmatrix} \tag{3}$$

provided that

$$0 < \frac{1}{m_0' + 1} \le K_{22} \le -\frac{1}{m_1' + 1}.$$

The closed-loop feedback control configuration of the system is shown in Fig. 2.

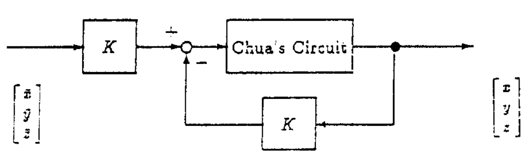

Figure 2. Feedback Control Configuration for Chua's Circuit

We would like to emphasize that Theorem 1 only provides a sufficient condition, where the condition on K_{22} is not always necessary as shown by the example given in the next section. We also remark that for the sake of simplicity we have set all the elements but K_{22} to zero in the above feedback gain matrix K, while more nonzero elements in K usually means more degrees of freedom in the design and the fine-tuning of the controller. The control input of the form (3) is among the simplest linear feedback controllers that one can use in Chua's circuit.

We now give a proof of the theorem.

Proof. Note that the controlled Chua's circuit can be written as

$$\left\{ \begin{array}{lcl} \dot{x} & = & p\,[-x + y - f(x)] \\ \dot{y} & = & x - y + z - K_{22}(y - \bar{y}) \\ \dot{z} & = & -qy\,. \end{array} \right. \tag{4}$$

and that the limit cycle $(\bar{x}, \bar{y}, \bar{z}) = (\bar{x}(t), \bar{y}(t), \bar{z}(t))$ is itself a solution of the circuit, i.e.,

$$\left\{ \begin{array}{lcl} \dot{\bar{x}} & = & p\,[-\bar{x} + \bar{y} - f(\bar{x})] \\ \dot{\bar{y}} & = & \bar{x} - \bar{y} + \bar{z} \\ \dot{\bar{z}} & = & -q\bar{y}\,. \end{array} \right. \tag{5}$$

Subtracting (5) from (4), with the notation $X = x - \bar{x}$, $Y = y - \bar{y}$, $Z = z - \bar{z}$ and $\tilde{f}(x, \bar{x}) = f(x) - f(\bar{x})$, we obtain

$$\left\{ \begin{array}{lcl} \dot{X} & = & p\,[-X + Y - \tilde{f}(x, \bar{x})] \\ \dot{Y} & = & X - Y + Z - K_{22}Y \\ \dot{Z} & = & -qY \end{array} \right. \tag{6}$$

where

$$\tilde{f}(x, \bar{x}) = \left\{ \begin{array}{ll} m_0'(x - \bar{x}) & x \ge 1, \bar{x} \ge 1 \\ m_0'x - m_1'\bar{x} + m_1' - m_0' & x \ge 1, -1 \le \bar{x} \le 1 \\ m_0'(x - \bar{x}) + 2(m_1' - m_0') & x \ge 1, \bar{x} \le -1 \\ m_1'x - m_0'\bar{x} - m_1' + m_0' & -1 \le x \le 1, \bar{x} \ge 1 \\ m_1'(x - \bar{x}) & -1 \le x, \bar{x} \le 1 \\ m_1'x - m_0'\bar{x} + m_1' - m_0' & -1 \le x \le 1, \bar{x} \le -1 \\ m_0'(x - \bar{x}) - 2(m_1' - m_0') & x \le -1, \bar{x} \ge 1 \\ m_0'x - m_1'\bar{x} - m_1' + m_0' & x \le -1, -1 \le \bar{x} \le 1 \\ m_0'(x - \bar{x}) & x \le -1, \bar{x} \le -1 \end{array} \right.$$

with $m_1' < m_0' < 0$.

We define the Lyapunov function for Eq. (6) by

$$V(X, Y, Z) \triangleq \frac{q}{2}X^2 + \frac{pq}{2}Y^2 + \frac{p}{2}Z^2.$$

It is clear that $V(0,0,0) = 0$ and that $V(X, Y, Z) > 0$ when at least one of X, Y, and Z is not zero. Furthermore, since $p,\ q > 0$, we have

$$\begin{array}{rcl} \dot{V} & = & qX\dot{X} + pqY\dot{Y} + pZ\dot{Z} \\ & = & qX[-pX + pY - p\tilde{f}(x, \bar{x})] \\ & & +pqY(X - Y + Z - K_{22}Y) + pZ(-qY) \\ & = & -pq[X^2 + Y^2 + K_{22}Y^2 - 2XY + X\tilde{f}(x, \bar{x})] \\ & = & -pq\{X^2 + Y^2 + K_{22}[Y^2 - \frac{2}{K_{22}}XY \\ & & +\frac{1}{K_{22}^2}X^2 - \frac{1}{K_{22}^2}X^2 + \frac{1}{K_{22}}X\tilde{f}(x, \bar{x})]\} \\ & = & -pq\{X^2 + Y^2 + K_{22}(Y - \frac{1}{K_{22}}X)^2 \\ & & +[X\tilde{f}(x, \bar{x}) - \frac{1}{K_{22}}X^2]\} \\ & = & -pq\{Y^2 + K_{22}(Y - \frac{1}{K_{22}}X)^2 \\ & & +[X\tilde{f}(x, \bar{x}) + (1 - \frac{1}{K_{22}})X^2]\} \\ & \le & 0 \end{array}$$

for all X, Y and Z, if

$$\left\{ \begin{array}{rcl} K_{22} & > & 0 \\ X\tilde{f}(x, \bar{x}) + (1 - \frac{1}{K_{22}})X^2 & \ge & 0 \end{array} \right. \tag{7}$$

or

$$K_{22}X\tilde{f}(x, \bar{x}) + (K_{22} - 1)X^2 \ge 0\,. \tag{8}$$

for all x and $\bar{x}$.

We now look at each every one of the possible cases in detail, namely:

Cases 1 and 9. $x, \bar{x} \ge 1$ or $x, \bar{x} \le -1$: $\tilde{f} = m_0'X$. Obviously, inequality

(8) is satisfied if

$$K_{22}m_0'X^2 + (K_{22}-1)X^2 = (K_{22}m_0' + K_{22} - 1)X^2 \geq 0,$$

or $K_{22} \geq \frac{1}{m_0'+1}$.

Case 2. $x \geq 1,\ -1 \leq \bar{x} \leq 1$: Note that $\tilde{f} = m_1'X + (m_0' - m_1')(x-1)$ and $X \geq 0,\ x-1 \geq 0,\ m_0' - m_1' > 0$, and we have (8) if

$$\begin{aligned} & K_{22}X[m_1'X + (m_0'-m_1')(x-1)] + (K_{22}-1)X^2 \\ = \ & (K_{22}m_1' + K_{22} - 1)X^2 + K_{22}X(m_0'-m_1')(x-1) \\ \geq \ & 0, \end{aligned}$$

or $K_{22}m_1' + K_{22} - 1 \geq 0$ and $K_{22} \geq 0$. That is, $K_{22} \leq -\frac{1}{m_1'+1}$ and $K_{22} \geq 0$.

All the other cases listed below can be discussed in a similar manner, which we omit for brevity:

Case 3. $x \geq 1,\ \bar{x} \leq -1$ or $X \geq 2$:
$\tilde{f} = m_0'X + 2(m_1' - m_0')$.

Case 4. $-1 \leq x \leq 1,\ \bar{x} \geq 1$ or $-2 \leq X \leq 0$:
$\tilde{f} = m_1'X + (m_0' - m_1')(1 - \bar{x})$.

Case 5. $-1 \leq x \leq 1,\ -1 \leq \bar{x} \leq 1$ or $-2 \leq X \leq 2$:
$\tilde{f} = m_1'X$.

Case 6. $-1 \leq x \leq 1,\ \bar{x} \leq -1$ or $X \geq 0$:
$\tilde{f} = m_1'X + (m_1' - x_0')(\bar{x}+1)$.

Case 7. $x \leq -1, \bar{x} \geq 1$:
$\tilde{f} = m_0'X + 2(m_0' - m_1')$.

Case 8. $x \leq -1,\ -1 \leq \bar{x} \leq 1$ or $X \leq 0$:
$\tilde{f} = m_0'X + (m_0' - m_1')(\bar{x} - 1)$.

Combining all the conditions just derived results in

$$0 < \frac{1}{m_0'+1} \leq K_{22} \leq -\frac{1}{m_1'+1},$$

which guarantees inequality (7).

Hence, if the condition stated in the theorem is satisfied, then the equilibrium point $(0,0,0)$ of the controlled circuit (Eq. (6)) is globally asymptotically stable, so that

$$|X| \to 0\,, \quad |Y| \to 0 \text{ and } \quad |Z| \to 0 \qquad \text{as} \quad t \to \infty.$$

That is, starting feedback control at any time, we have

$$\lim_{t\to\infty} |x(t) - \bar{x}(t)| = 0\,, \quad \lim_{t\to\infty} |y(t) - \bar{y}(t)| = 0\,, \quad \text{and}$$

$$\lim_{t\to\infty} |z(t) - \bar{z}(t)| = 0\,.$$

3 Numerical Results

It has just been mathematically proved that the chaotic trajectory of Chua's circuit can be controlled to its unstable saddle-type limit cycle, where the designed feedback control can be initiated at any time. To see the performance of the linear feedback controller designed in the last section in the control of Chua's circuit and to demonstrate the control process, we show how to drive the chaotic trajectory of Fig. 1 to the saddle-type periodic orbit of the same figure.

For simplicity, we only use a second-order approximant of the unstable limit cycle (since an exact analytic expression for the limit cycle does not exist). Let $(\bar{x}, \bar{y}, \bar{z})$ be such an unstable limit cycle of Chua's circuit, and approximate it with the following second-order formulas

$$\left\{ \begin{array}{rcl} \bar{x}(t) & \approx & a\cos\alpha\cos(\omega t) - b\sin\alpha\sin(\omega t) \\ & & +c\cos\alpha\cos(2\omega t) - d\sin\alpha\sin(2\omega t) \\ \bar{y}(t) & \approx & e[a\sin\alpha\cos(\omega t) - b\cos\alpha\sin(\omega t)] \\ & & +f[c\sin\alpha\cos(2\omega t) - d\cos\alpha\sin(2\omega t)] \\ \dot{\bar{z}}(t) & = & q\bar{y}(t) \end{array} \right.$$

where $a = 2.6$, $b = 1.2$, $c = d = 0.2$, $e = 0.6$, $f = 0.3$, $\alpha = \frac{\pi}{18}$, and $\omega = 1.77$. Applying the control law (Eq. (3)) to the system, we have

$$\left\{ \begin{array}{rclcl} \dot{x} & = & p\,[-x+y-f(x)] & = & p\,[-x+y-f(x)] \\ \dot{y} & = & x-y+z+u_2 & = & x-y+z-K_{22}(y-\bar{y}) \\ \dot{z} & = & -qy & = & -qy \end{array} \right.$$

The chaotic trajectories of the circuit, before and after this control law is applied, are shown in Fig. 3. Note that we have chosen the feedback gain $K_{22} = 2.0$ in this simulation. It can be seen from these figures that the designed linear feedback control law is very effective in directing the trajectory away from the double scroll attractor and then driving it to approach (and finally) reach the target periodic orbit.

4 Conclusions

In this paper, we have discussed a conventional feedback control approach for ordering or controlling Chua's circuit. The linear feedback controller used is perhaps the simplest possible in the design. The feedback control scheme and the associate sufficient condition have been mathematically justified in Section 2. The effectiveness of the control is demonstrated by the computer simulation results shown in Section 3.

We would like to point out that among the most appealing features of the proposed additive feedback control strategy is its efficiency and unified manner. Also worth mentioning is that this technique does not require varying or adjusting any system parameter, which actually will alter the original system, so that the control effect can be eliminated immediately whenever the controller is being disconnected. Depending on particular applications, the additive feedback controller can be applied to, or disconnected from, the chaotic system at any time.

Finally, for those more electronically inclined readers, we would like to remark that introducing $-K_{22}(y-\bar{y})$ to Eq. (2) corresponds to adding $-K_{22}G(v_{C_2} - \bar{v}_{C_2})$ to Eq. (1). And this, in turn, implies that an additional linear resistor, $R' = \frac{1}{G'} = \frac{1}{K_{22}G}$, and an appropriate periodic-signal generator, $v_{sg} = \bar{v}_{C_2}$, are being added to the original physical system, as illustrated in Fig. 4. These two additional components are both easy to realize in the circuitry.

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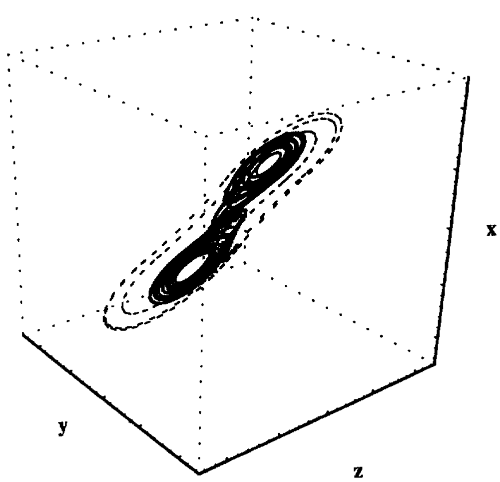

Figure 3.2. A Chua's circuit solution trajectory (before and when a feedback control is being applied) in the x-y-z space

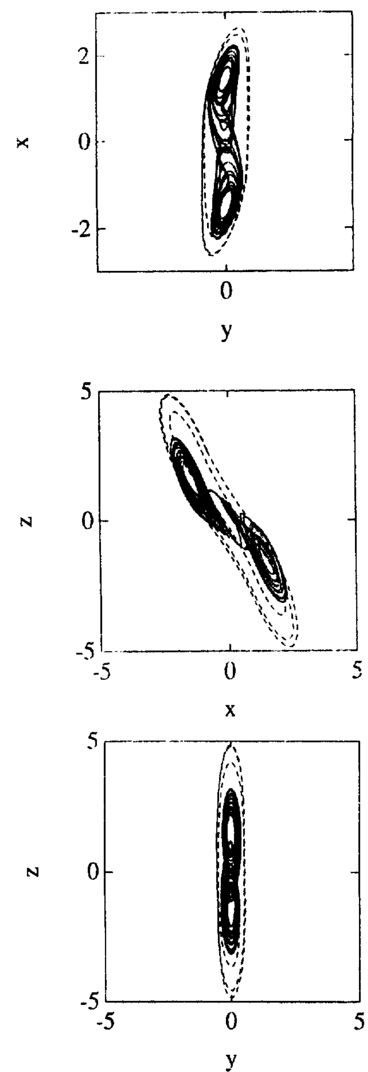

Figure 3.1. A Chua's circuit solution trajectory (before and when a feedback control is being applied) projected onto y-x, x-z, y-z planes

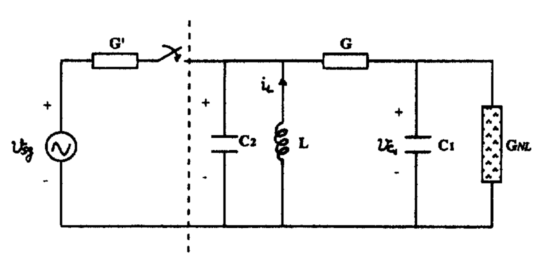

Figure 4. Circuit realization of feedback control of Chua's circuit