A HARMONIC BALANCE APPROACH TO BIFURCATION ANALYSIS OF LIMIT CYCLES

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ABSTRACT

This work provides an application of the Harmonic Balance (HB) technique to the stability and bifurcation analysis of limit cycles of dynamical systems amenable to be expressed in Lur'e system form. A numerically efficient spectral approach is exploited to evaluate the system limit cycle with an arbitrary number of harmonic components, which are then exploted to perform a linearized, small-change stability analysis of the limit cycle itself. On the basis of a Mittag-Leffer expansion of the determinant of the infinite matrix representing the linearized system, the limit cycle Floquet multipliers (FM) are evaluated and exploited to perform the bifurcation analysis. As an example of application, parameter space bifurcation conditions for classical Chua's circuit are thoroughly examined.

1. INTRODUCTION

Harmonic balance (HB) is a classical technique for studying limit cycles in nonlinear dynamic systems [1], that has been successfully applied to the design of electronic oscillators and of microwave circuits [2, 3]. Most of the applications of HB techniques concern the steady-state behaviour of systems presenting a single periodic attractor, i.e. a limit cycle that attracts all the system trajectories.

In the last few years much interest has been dedicated to systems with complex dynamics, i.e. exhibiting several attractors and bifurcation phenomena [4, 5]: in fact complex systems are useful for modeling a large variety of phenomena originating from chemistry, biology and ecology. Moreover it has been found that neural networks and simple electronic oscillators may exhibit a complex dynamics as well [6]. The global dynamic behaviour of nonlinear systems presenting several attractors is normally studied through time-domain techniques, that exploit rather complicated geometrical concepts [7] and cannot be simply applied to the design of electronic circuits.

Recently, some extensions of the HB technique have been proposed for the study of the global dynamic behaviour and of bifurcation processes in complex systems.

In [8] the describing function technique (i.e., HB with a single harmonic) has been exploited for predicting chaos and several bifurcation phenomena. Such a technique provides simple and useful analytical results, but is not able to accurately predict all the complex dynamic phenomena occurring in nonlinear circuits (see for example the various fold, flip and homoclinic bifurcations of Chua's circuit shown in [9]). In [10] a method, based on the HB technique, for detecting fold and flip bifurcations was proposed. In [11] the local stability of limit cycles is analysed through the application of Nyquist's theorem. In [12] bifurcations are studied through a spectral technique based on the introduction of measuring probes into the circuit. In [13] a spectral technique is exploited for studying flip bifurcations in time-delayed systems.

The above approaches have the disadvantage of not providing a method for computing the limit cycle Floquet multipliers (FMs), the simplest tool for studying the stability of a limit cycle and for detecting its bifurcations [7].

In this work, we propose a method for studying limit cycle bifurcations, based on the evaluation of the FMs, through a HB approach. Firstly limit cycles are detected by using the HB technique as introduced in [14], which is fast and efficient even for a large number of harmonics. Then the FMs are expressed as the roots of an algebraic equation of degree equal to the order of the system; such an equation is derived through a suitable extension of the technique proposed in [15]. Finally stability and bifurcation conditions are easily established in terms of the coefficients of the above algebraic equation.

As a case-study, we consider Chua's circuit [6], a complex dynamical system that presents a rich dynamic behaviour, and we show that our method is able to accurately identify all its significant bifurcations. We remark that, through the HB approach developed in this paper, other dynamical systems (e.g. Duffing's and Rossler's equations) can be investigated.

2. THEORY

Let us consider an autonomous nonlinear feedback system admitting the Lur'e representation [8]:

$$L(D)x(t) + n[x(t)] = 0 \tag{1}$$

where $D = \mathrm{d}/\mathrm{d}t$ is the time derivative, $L(\cdot)$ is a linear operator and $n(\cdot)$ is a nonlinear function. Both the linear operator and the nonlinear function can depend on several parameters. Since the HB technique assumes a periodic solution to take place, $x(t)$ can be expanded as a superposition of harmonics (Fourier series):

$$x(t) = \sum_{k=0}^{\infty} x_k(t),$$

where

$$\begin{cases} x_0(t) &= A_0 \\ x_k(t) &= A_k \cos(k\omega t) + B_k \sin(k\omega t). \end{cases} \tag{2}$$

For computational purposes, the Fourier series has to be truncated to a suitable degree N high enough to represent accurately the solution $x(t)$, thereby obtaining:

$$x(t) = \sum_{k=0}^{N} x_k(t) \tag{3}$$

where the unknown variables to be determined through the HB technique are the $2N+1$ spectral coefficients A_k and B_k, and the solution period $T = 2\pi/\omega$. Such $2N+2$ independent equations are obtained [14] sampling the dynamic equation (1) in $2N+1$ time samples uniformly spaced within the period-wide range $]0, T]$:

$$t_k = \frac{T}{2N+1}k \qquad k = 1, \dots, 2N+1 \tag{4}$$

and imposing one of the coefficients (e.g., B_1) of the fundamental harmonic to be zero. According to [14], the dynamic equation (1) is converted in $2N+2$ (nonlinear) algebraic equations involving the aforementioned unknowns:

$$\begin{cases} \underline{\underline{\Omega}}(\omega)\,\underline{X} + \underline{N}\left(\underline{\underline{\Gamma}}^{-1}\,\underline{X}\right) = 0 \\ B_1 = 0 \end{cases} \tag{5}$$

where $\underline{X} = [A_0\ A_1\ B_1 \dots A_N\ B_N]^{\mathrm{T}}$, matrix $\underline{\underline{\Omega}}$ is given by

$$\underline{\underline{\Omega}}(\omega) = \begin{bmatrix} L(0) & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & R_1 & I_1 & 0 & 0 & \dots & 0 & 0 \\ 0 & -I_1 & R_1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & R_2 & I_2 & \dots & 0 & 0 \\ 0 & 0 & 0 & -I_2 & R_2 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \dots & R_N & I_N \\ 0 & 0 & 0 & 0 & 0 & \dots & -I_N & R_N \end{bmatrix} \tag{6}$$

and $R_k = \mathrm{Re}\{L(\mathrm{j}k\omega)\}$, $I_k = \mathrm{Im}\{L(\mathrm{j}k\omega)\}$, $k = 1, \dots, N$. Furthermore, matrix $\underline{\underline{\Gamma}}^{-1}$ is defined as:

$$\underline{\underline{\Gamma}}^{-1} = \begin{bmatrix} 1 & \gamma^{c}{}_{1,1} & \gamma^{s}{}_{1,1} & \cdots & \gamma^{c}{}_{1,N} & \gamma^{s}{}_{1,N} \\ 1 & \gamma^{c}{}_{2,1} & \gamma^{s}{}_{2,1} & \cdots & \gamma^{c}{}_{2,N} & \gamma^{s}{}_{2,N} \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 1 & \gamma^{c}{}_{2N+1,1} & \gamma^{s}{}_{2N+1,1} & \cdots & \gamma^{c}{}_{2N+1,N} & \gamma^{s}{}_{2N+1,N} \end{bmatrix} \tag{7}$$

where:

$$\gamma^{c}{}_{p,q} = \cos(q\omega t_p) = \cos\left(\frac{q2\pi p}{2N+1}\right)$$
$$\gamma^{s}{}_{p,q} = \sin(q\omega t_p) = \sin\left(\frac{q2\pi p}{2N+1}\right)$$

Finally, $\underline{N}\left(\underline{\underline{\Gamma}}^{-1}\,\underline{X}\right) = \underline{\underline{\Gamma}}\,\underline{n}\left(\underline{\underline{\Gamma}}^{-1}\,\underline{X}\right)$ and $\underline{n}\left(\underline{\underline{\Gamma}}^{-1}\,\underline{X}\right)$ denotes the vector of time samples of the nonlinear function $n(\cdot)$.

In this work we shall show that the FMs can be computed as the roots of an algebraic equation derived through an extension of the technique proposed in [15] for the evaluation of Hill's determinant. Then, conditions for fold and flip bifurcations are expressed as simple constraints among the coefficients of the above equation.

As a first step we consider a small perturbation $\tilde{x}(t)$ of the limit cycle $x(t)$ which has to satisfy the variational equation:

$$L(D)\tilde{x}(t) + g(t)\tilde{x}(t) = 0 \tag{8}$$

where

$$g(t) = \left.\frac{\mathrm{d}n(\zeta)}{\mathrm{d}\zeta}\right|_{\zeta = x(t)} \tag{9}$$

which describes a linear periodic time-varying system, whose solution can be expressed as [16]:

$$\tilde{x}(t) = \sum_{i=1}^{M} H_i v_i(t) \exp(\lambda_i t) \tag{10}$$

where M is the order of the dynamical system, H_i are suitable constants depending on the initial conditions and $v_i(t)$ are periodic functions of period T; finally, λ_i are constant eigenvalues from which the FMs are easily determined as $\exp(\lambda_i T)$.

In order to determine the eigenvalues λ_i we substitute the generic eigenfunction $v(t)\exp(\lambda t)$ into (8):

$$L(D)v(t)\exp(\lambda t) + g(t)v(t)\exp(\lambda t) = 0. \tag{11}$$

Notice that, although $x(t)$ is represented in (3) by means of a finite number of harmonics, the periodic function $g(t)$ is in general expressed by an infinite number of harmonics, since it is the derivative of a nonlinear function evaluated in $x(t)$:

$$g(t) = \sum_{k=-\infty}^{\infty} G_k \exp(jk\omega t) \tag{12}$$

For the sake of simplicity we restrict our attention to the case of a linear block $L(s) = P(s)/Q(s)$, with deg(P(s)) = deg(Q(s)) +1 = M = 3, i.e. the case occurring in Chua's circuit. It is possible to prove that the two FMs different from 1 are the roots of the following second-order algebraic equation

$$\mu^2 + a\mu + b = 0 \tag{13}$$

where:

$$\begin{aligned} a &= -\frac{a_{\mathrm{N}}}{a_{\mathrm{D}}} \\ a_{\mathrm{N}} &= c_1\mu_{s1}(\mu_{s2}-1)(\mu_{s3}-1)(\mu_{s2}+\mu_{s3}) \\ &\quad + c_2\mu_{s2}(\mu_{s1}-1)(\mu_{s3}-1)(\mu_{s1}+\mu_{s3}) \\ &\quad + c_3\mu_{s3}(\mu_{s1}-1)(\mu_{s2}-1)(\mu_{s1}+\mu_{s2}) \\ a_{\mathrm{D}} &= \mu_{s1}\mu_{s2}\mu_{s3}[c_1\mu_{s1}(\mu_{s2}-1)(\mu_{s3}-1) \\ &\quad + c_2\mu_{s2}(\mu_{s1}-1)(\mu_{s3}-1) \\ &\quad + c_3\mu_{s3}(\mu_{s1}-1)(\mu_{s2}-1)] \\ b &= \frac{1}{\mu_{s1}\mu_{s2}\mu_{s3}} \\ \mu_{si} &= \exp\left(-2\lambda_{si}\pi/\omega\right) \qquad i = 1,2,3. \end{aligned} \tag{14}$$

The constants c_1, c_2 and c_3 are the determinants of the infinite matrices $\underline{\underline{R}}_i$ ($i = 1,2,3$), which are easily described in terms of the matrix (infinite) diagonals $\underline{D}_q$, where q is an integer running from $-\infty$ to ∞; $q = 0$ means the main diagonal, $q > 0$ denotes an upper diagonal and $q < 0$ the corresponding lower one. The expressions for these diagonals are:

$$\underline{D}_0 = \begin{bmatrix} \vdots \\ 1 \\ \vdots \\ 1 \\ 1 \\ 0 \\ 1 \\ 1 \\ \vdots \\ 1 \\ \vdots \end{bmatrix} \qquad \underline{D}_q = \begin{bmatrix} \vdots \\ G_{-q}^{(-M)}(\lambda_{si}) \\ \vdots \\ G_{-q}^{(-2)}(\lambda_{si}) \\ G_{-q}^{(-1)}(\lambda_{si}) \\ \overline{G}_{-q} \\ G_{-q}^{(1)}(\lambda_{si}) \\ G_{-q}^{(2)}(\lambda_{si}) \\ \vdots \\ G_{-q}^{(-M)}(\lambda_{si}) \\ \vdots \end{bmatrix} \quad q \neq 0 \tag{15}$$

where $G_q^{(p)}(\lambda) = f_p(\lambda) G_q$, $f_p(\lambda) = [L(\lambda + jp\omega) + G_0]^{-1}$, $\overline{G}_q = f_0^i G_q$ and

$$\begin{aligned} f_0^1 &= \frac{(\lambda_{s1} - \lambda_{s4})(\lambda_{s1} - \lambda_{s5})}{\eta(\lambda_{s1} - \lambda_{s2})(\lambda_{s1} - \lambda_{s3})} \\ f_0^2 &= \frac{(\lambda_{s2} - \lambda_{s4})(\lambda_{s2} - \lambda_{s5})}{\eta(\lambda_{s2} - \lambda_{s1})(\lambda_{s2} - \lambda_{s3})} \\ f_0^3 &= \frac{(\lambda_{s3} - \lambda_{s4})(\lambda_{s3} - \lambda_{s5})}{\eta(\lambda_{s3} - \lambda_{s1})(\lambda_{s3} - \lambda_{s2})} \end{aligned} \tag{16}$$

Finally, $(\lambda_{s1}, \lambda_{s2}, \lambda_{s3})$ and $(\lambda_{s4}, \lambda_{s5})$ are poles and zeros, respectively, of function $f_0(\lambda)$; η is the residue of $1/f_0(\lambda)$ for $\lambda = \infty$.

The conditions for fold and period doubling (flip) bifurcations are obtained by simply imposing that one FM equals 1 or -1 respectively, i.e.

$$\begin{aligned} &\text{fold bifurcation} \iff 1 + a + b = 0 \\ &\text{flip bifurcation} \iff 1 - a + b = 0. \end{aligned} \tag{17}$$

We remark that: (a) from a computational standpoint, the constants c_i are evaluated computing the determinant of matrix $\underline{\underline{R}}_i$ truncating the infinite representation of $g(t)$ to a finite number of harmonics, which can be rather small since, owing to the structure of $\underline{\underline{R}}_i$ and to the definition of $f_k(\lambda)$ given above, the infinite determinant of $\underline{\underline{R}}_i$ is rapidly convergent (see [15], Sec. 2.8); (b) the proposed method represents an improvement with respect to the technique shown in [1] where the FMs are detected through the solution of a complex eigenvalue problem and are provided in a form that is not suitable for imposing bifurcation conditions.

3. APPLICATION EXAMPLE

As an example of application, we shall analyse the bifurcation phenomena in a well known dynamical system: classical Chua's circuit [6]. This choice is supported by the complex dynamic behaviour of this system, thereby enabling a sound test for the HB approach to bifurcation analysis. The dynamic equations describing Chua's circuit can be written in terms of the following Lur'e system representation [8]:

$$L(D) = \frac{D^3 + (1+\alpha)D^2 + \beta D + \alpha\beta}{\alpha\,(D^2 + D + \beta)} \tag{18}$$

where α and β are normalized parameters. We assume that the memoryless nonlinear function $n(\cdot)$ be approximated as a cubic nonlinearity [8]:

$$n(x) = -\frac{8}{7}x + \frac{4}{63}x^3. \tag{19}$$

It is possible to show [9] that Chua's circuit, as described by the aforementioned functions, exhibits, for $\alpha < 7$, two stable equilibria symmetric with respect to the origin of the state space. For $\alpha = 7$ a Hopf bifurcation gives rise to two asymmetric limit cycles, which in turn are symmetric with respect to the origin since the nonlinear system is odd. By further increasing α, a fold bifurcation occurs yielding a pair of symmetric limit cycles (one stable and one unstable). We start our investigation for values of α and β (e.g., $\alpha = 8$ and $\beta = 15$) for which the two stable asymmetric limit cycles and the stable symmetric limit cycle coexist (see [9], Sec. 3.2). By solving the HB system for $\alpha = 8$ and $\beta = 15$, the asymmetric and the symmetric limit cycles have been detected. For β held fixed, both cycles have then been continued with respect to the cycle period T; therefore the nonlinear algebraic system (5) is solved with $\underline{X}$ and α as unknowns. The values of α vs. the cycle period T for $\beta = 15$ are shown in Fig. 1, which has been obtained with $N = 21$ harmonics (including DC).

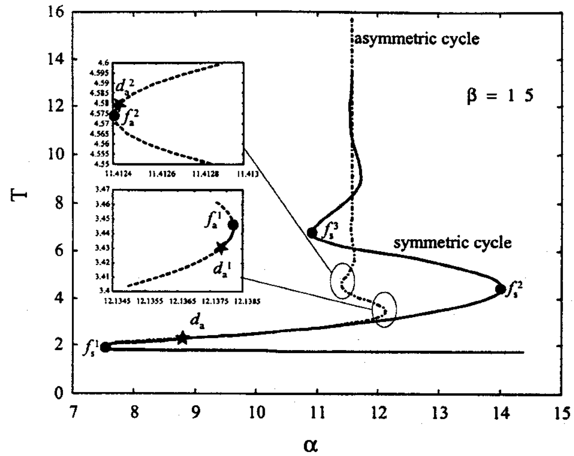

Figure 1: Parameter α vs. cycle period T for both asymmetric and symmetric limit cycles. The HB system was solved with $N = 21$ harmonics and $\beta = 15$.

Moreover, for both symmetric and asymmetric limit cycles and for each value of α and T the FMs have been evaluated as previously discussed. Several bifurcations have been detected and shown in Fig. 1:

- the period-doubling bifurcation d_a of the asymmetric limit cycle, giving rise to the well-known spiral attractor;
- as period T is increased, the asymmetric limit cycle becomes unstable and undergoes a sequence of flip and fold bifurcations. We have reported in Fig. 1 the first four bifurcations encountered by increasing the period T:

- a flip bifurcation (denoted with d_a^1) after which the limit cycle returns to be stable;
- a fold bifurcation (denoted with f_a^1) corresponding to the first vertical tangent of the $T(\alpha)$ curve, after which the limit cycle is again unstable;
- another fold bifurcation (denoted with f_a^2) corresponding to the second vertical tangent of the $T(\alpha)$ curve, after which the limit cycle return to be stable;
- another flip bifurcation (denoted with d_a^2), that causes the limit cycle to be again unstable.

The FMs, for the range of periods giving rise to the above sequence of bifurcations, turn out to be real. They are reported in Fig. 2: it is seen that the HB technique is able to detect all the flip and fold bifurcations even if they arise for very close values of the parameters.

- the fold bifurcation f_s^1 of the symmetric limit cycle, which gives rise to a pair of symmetric cycles, one stable and one unstable;
- as period T is increased, the unstable symmetric limit cycle undergoes a sequence of fold bifurcations for values of α corresponding to a vertical tangent of the $T(\alpha)$ curve. The first two such bifurcations, indicated as f_s^i $i = 2, 3$, are also shown in Fig. 1;
- a strong indication of the existence of a homoclinic orbit, due to the fact that both curves exhibit a vertical asymptote.

We remark that starting from the bifurcation points detected from Fig. 1, bifurcation curves can be obtained by continuation with respect to β, i.e. by solving (5) and condition (17) with $\underline{X}$, T and α as unknowns.

4. CONCLUSIONS

We have presented a HB approach for the study of limit cycle. stability and bifurcations in complex nonlinear systems and circuits. The method is based on the evaluation of the FMs, through the extension of the technique reported in [15] to generic Lur'e systems. The proposed HB technique has been applied to a rather complex nonlinear system, that exhibits a large number of attractors and bifurcation phenomena: Chua's circuit [6]. The most significant flip, fold and homoclinic bifurcations have been accurately detected. We remark that the above HB approach is also suitable for studying distributed systems, which are not described by ordinary differential equations.

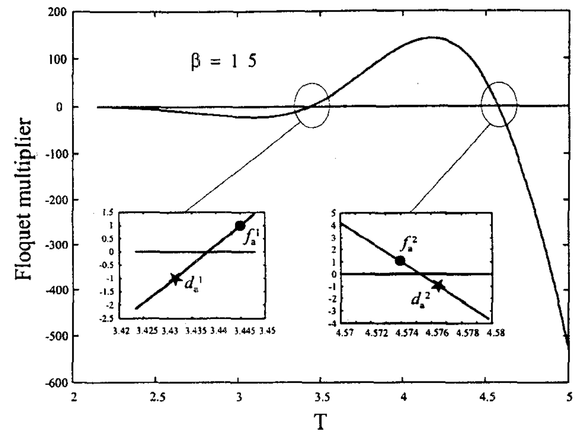

Figure 2: Floquet multipliers for the asymmetric limit cycle as a function of period T for $\beta = 15$.

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