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FIXED POINT STABILITY ANALYSIS OF CHUA'S CIRCUIT: A CASE STUDY WITH A REAL CIRCUIT*

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This short paper discusses the effect of the internal resistance of the inductor in Chua's circuit which is often neglected by many even when actual implementation is intended. Using a fixed point stability analysis it is shown that varying the inductor resistance it is possible to suppress or allow chaotic oscillations. The results reported in this paper have clear consequences for the control of Chua's circuit.

1. Introduction and Statement of the Problem

One of the most popular benchmarks for studying nonlinear oscillations is Chua's circuit.[1–3] This system has been also used as a benchmark for studying several aspects of nonlinear system identification and control.[4–6] One of the reasons for the preference of Chua's circuit in practical applications is that this circuit can be easily built to produce chaotic oscillations.[7,8] The equations governing Chua's circuit are[1]

$$C_1\frac{dv_1}{dt} = \frac{(v_2 - v_1)}{R} - i_{\mathrm{d}}(v_1);\ C_2\frac{dv_2}{dt} = \frac{(v_1 - v_2)}{R} + i_L;\ L\frac{di_L}{dt} = -v_2\,, \tag{1}$$

where v_i is the voltage across capacitor C_i, i_L is the current through the inductor and the current through Chua's diode is given by

$$i_{\mathrm{d}}(v_1) = \begin{cases} m_0 v_1 + B_{\mathrm{p}}(m_0 - m_1) & v_1 < -B_{\mathrm{p}} \\ m_1 v_1 & |v_1| \le B_{\mathrm{p}} \\ m_0 v_1 + B_{\mathrm{p}}(m_1 - m_0) & v_1 > -B_{\mathrm{p}}\,, \end{cases} \tag{2}$$

where B_{p}, m_0 and m_1 are respectively the break point and the inclinations of the piecewise-linear function of Eq. (2).

The following components were used in the implemented circuit: $C_1 = 10 \pm 0.5$ nF, $C_2 = 100 \pm 5$ nF, $L = 18 \pm 2\%$ mH and R is a 2.0 kΩ trimpot. Chua's diode was built using the two-operational-amplifier configuration suggested in Ref. 8 and

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the following parameters were measured: $m_0 = -0.37 \pm 0.04$ mS, $m_1 = -0.68 \pm 0.04$ mS, and $B_p = 1.1 \pm 0.2$ V.

The location and the stability of the fixed points of a system are important in determining the general structure of the flow in phase space and depend on the parameters of the system. This paper discusses the stability analysis performed on Chua's circuit fixed points and shows the effects of the inductor resistance. The results discussed in this paper are believed to be relevant in some control problems of Chua's circuit.

2. Stability Analysis

Expressing Eq. (1) as $\dot{x} = f(x)$, the *fixed points* are defined as the solutions of $\dot{x} = 0$. Equation (1) has three distinct fixed points, namely a *trivial fixed point* and two *nontrivial fixed points* which are symmetrical with respect to the origin and will be denoted by $\{p_n, p_t, -p_n\} \in \mathbb{R}^3$.

Chua's circuit exhibits a wealth of dynamical regimes. For instance, varying the resistor in the range $2.00\,\text{k}\Omega \geq R \geq 1.40$ kΩ, the oscillator goes from a dc equilibrium to a limit cycle passing through a sequence of period-doubling bifurcations and different chaotic attractors.[8] An important point to notice is that *the first bifurcations are characterized by the loss of stability of the nontrivial fixed points* and subsequent bifurcations by the loss of stability of periodic cycles.

An example that illustrates this fact is the behavior shown in Fig. 1. These results were encountered in practice. In this figure, v_2 is plotted against v_1. This procedure yields a projection of the phase space onto the plane $v_2 \times v_1$. The roughly

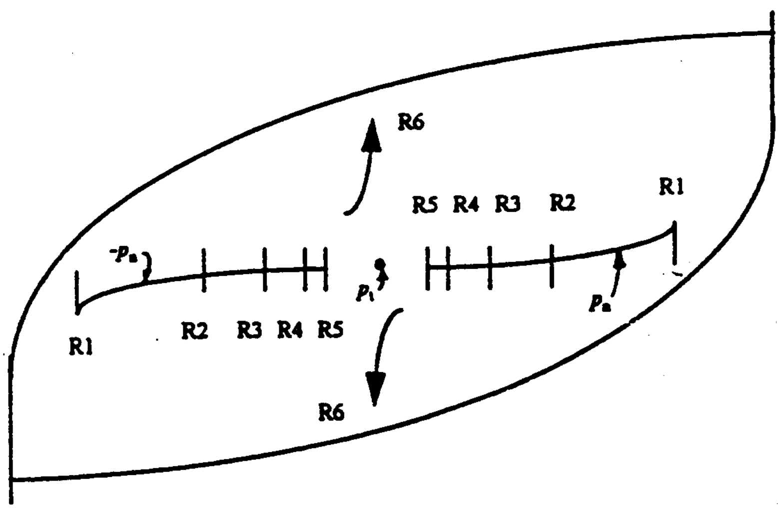


Fig. 1. $v_2 \times v_1$. Bifurcation sequence displayed by Chua's oscillator with large inductor resistance. The only attractors observed were those which coincided with the nontrivial fixed points and the last period-one limit cycle. $\{R_i\}_{i=1}^{6} \approx \{2.00, 1.86, 1.80, 1.75, 1.74, 1.35\,\text{k}\Omega\}$. The dimension of both axes is volts.

horizontal lines in this figure correspond to the location of the system nontrivial fixed points as the trimpot was varied from $R \approx 2.0$ kΩ to $R \approx 1.74$ kΩ. Notice how the nontrivial fixed points just move towards the trivial fixed point but *without losing stability*, thus hindering the appearance of any oscillations. Finally, for $R \approx 1.35$ kΩ the system "reaches" the trivial fixed point which is unstable, thus provoking the limit cycle shown. This scenario revealed that the nontrivial fixed points were always stable. Such fixed points, however, have to be unstable in order to enable oscillations. This suggested that a fixed-point stability analysis should be carried out in terms of one of the circuit parameters, and in the present study the inductor internal resistance was selected. This choice was motivated by the fact that this parameter is not usually considered in the literature and therefore it was not included in the dynamical Eq. (1). All the simulations based on such equations showed that the circuit should be oscillating even if other parameters such as C_1, C_2 and L were varied over reasonably wide ranges.

The analysis outlined above was carried out using the following set of equations

$$C_1\frac{dv_1}{dt} = \frac{(v_2 - v_1)}{R} - i_d(v_1);\; C_2\frac{dv_2}{dt} = \frac{(v_1 - v_2)}{R} + i_L;\; L\frac{di_L}{dt} = -v_2 + r_L i_L\,, \quad (3)$$

which now includes the parameter r_L. The fixed points for this system are located at

$$p_n = \left\{ \frac{-B_p(m_0-m_1)(r_L+R)}{1+(R+r_L)m_0}, \frac{-B_p(m_0-m_1)r_L}{1+(R+r_L)m_0}, \frac{B_p(m_0-m_1)}{1+(R+r_L)m_0} \right\} \text{ for } v_1 < -B_p \quad (4)$$

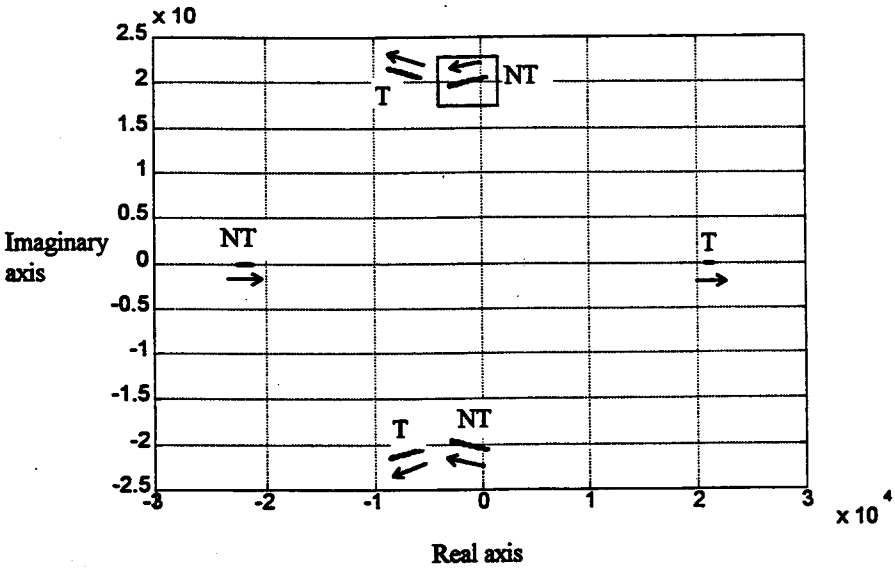
$$p_t = \{0,0,0\} \text{ for } |v_1| \le B_p \quad (5)$$

and $-p_n$ for $v_1 > B_p$. Evaluating the jacobian Df at the nontrivial and trivial fixed points yields, respectively

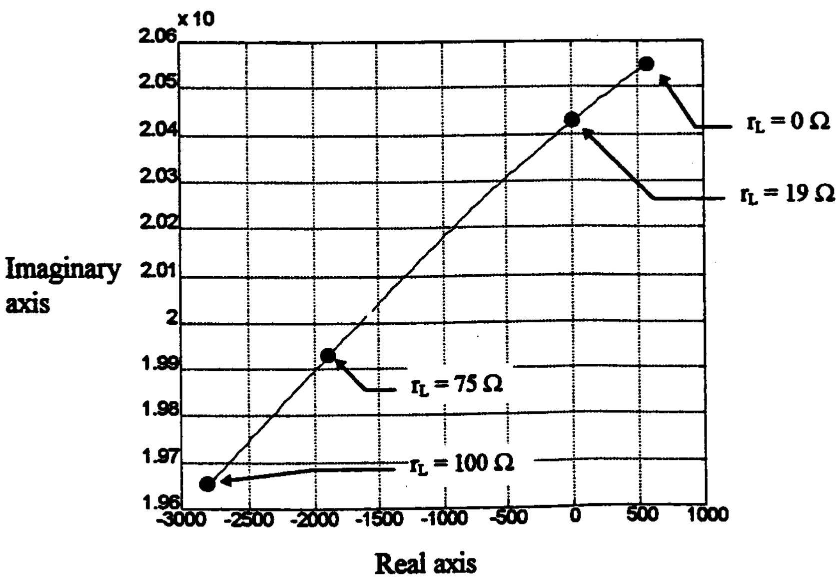
$$Df|_{p_n} = \begin{bmatrix} -\frac{1+Rm_0}{RC_1} & \frac{1}{RC_1} & 0 \\ \frac{1}{RC_2} & -\frac{1}{RC_2} & \frac{1}{C_2} \\ 0 & -\frac{1}{L} & \frac{r_L}{L} \end{bmatrix};\; Df|_{p_t} = \begin{bmatrix} -\frac{1+Rm_1}{RC_1} & \frac{1}{RC_1} & 0 \\ \frac{1}{RC_2} & -\frac{1}{RC_2} & \frac{1}{C_2} \\ 0 & -\frac{1}{L} & \frac{r_L}{L} \end{bmatrix}. \quad (6)$$

In order to assess how r_L affects the stability of the fixed points, the eigenvalues of $Df|_{p_n}$ and $Df|_{p_t}$ were calculated for various values of r_L. The results are shown in Fig. 2.

This figure shows that the trivial fixed point is always unstable for all values of r_L considered. This is in perfect accord with the dynamic behavior shown in Fig. 1. On the other hand, Fig. 2(a) shows how the unstable complex eigenvalues of $Df|_{p_n}$ stabilize as r_L is increased. In fact, as can be seen in Fig. 2(b) for values greater than 19 Ω, all the eigenvalues associated with the nontrivial fixed points are stable.



(a)



(b)

Fig. 2. (a) Root locus of the eigenvalues of $Df|_{p_n}$ and $Df|_{p_t}$ for $0\ \Omega \leq r_L \leq 100\ \Omega$. The arrows indicate the direction in which r_L increases. T and NT indicate the eigenvalues of Df evaluated at the trivial and nontrivial fixed points, respectively. (b) A detailed view of the marked part of (a).

The results reported so far have clear consequences for control of chaos in this circuit.[4,5] Because the inductor internal resistance has a stabilizing effect on the nontrivial fixed points, this fact could be incorporated in the design of a control scheme for this circuit. In particular, a variable resistance element in series with the inductor could be used as the control parameter and could be varied in order to stabilize or unstabilize the nontrivial fixed points.

3. Conclusions

This paper has discussed the stability analysis of Chua's oscillator fixed points. The results reported in this paper seem relevant in two aspects. First, if real implementations are in view, the set of Eq. (3) should be preferred to Eq. (1). This observation seems pertinent because most authors tend to use Eq. (1) *even when real implementations are intended.* Second, the analysis performed revealed that the inductor internal resistance has a stabilizing effect on the nontrivial fixed points of the circuit. Consequently this can be used in designing chaos control schemes for Chua's circuit.

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