CS281A/STAT241A - Homework I

August 28, 2014

This assignment is due at the beginning of class on September 4.

1. A random variable x has probability distribution function

$$P_x(x) = [1 - \exp(-x)]u(x)$$

where $u(x)$ is the unit step function:

$$u(x) = \begin{cases} 1 & x \geq 0 \\ 0 & x < 0 \end{cases}.$$

(a) Calculate $\Pr[x \leq 1]$, $\Pr[x \leq 5]$, and $\Pr[x = 2]$

(b) Compute the probability density function for x

(c) Let y be a random variable obtained from x as follows

$$y = \begin{cases} 0 & x < 2 \\ 1 & x \geq 2 \end{cases}.$$

Find the probability density function for y.

2. Box 1 contains 1000 light bulbs of which 10% are defective. Box 2 contains 2000 light bulbs of which 5% are defective.

(a) Suppose a box is given to you at random and you randomly select a lightbulb from the box. If that lightbulb is defective, what is the probability that you chose from Box 1?

(b) Suppose now that a box is given to you at random and you randomly select two lightbulbs from the box. If both lightbulbs are defective, what is the probability that you chose from Box 1?

3. Jim and George are each setting up venture capital portfolios. Suppose that Jim picks $n + 1$ startups to fund and George picks n startups to fund. Suppose that the probability of any startup succeeding is 50% and all of the startups succeed or fail independently. Show that the probability that Jim picks more winners than George is $1/2$.

4. A dart is thrown at a wall. Let (x, y) denote the Cartesian coordinates of the point in the wall pierced by the dart. Suppose x and y are independent Gaussian random variables, each with mean zero and variance σ^2.

(a) Find the probability that the dart will fall within the circle of radius R centered at the point $(0, 0)$.

(b) Find the probability that the dart will hit the first quadrant where x and y are nonnegative.

(c) Find the conditional probability that the dart will fall within the circle of radius R centered at $(0, 0)$ given that the dart hits the first quadrant.

5. Let A and B be positive semidefinite matrices.

(a) Define the matrix $C^{(1)}$ to have entries $C_{ij}^{(1)} = A_{ij}B_{ij}$. Is $C^{(1)}$ positive semidefinite? Prove or give a counterexample.

(b) Define the matrix $C^{(2)}$ to have entries $C_{ij}^{(2)} = \sum_{k=1}^{n} A_{ik}B_{jk}$. Is $C^{(2)}$ positive semidefinite? Prove or give a counterexample.

(c) Define the matrix $C^{(3)}$ to have entries $C_{ij}^{(3)} = A_{ii}B_{ij}^2 A_{jj}$. Is $C^{(3)}$ positive semidefinite? Prove or give a counterexample.

6. Let x, y, and z be zero-mean, unit-variance random variables which satisfy

$$\mathrm{var}(x + y + z) = 0\,.$$

Compute the covariance matrix of the random vector $[x, y, z]$.

7. Let $\{x_1, x_2, \ldots, x_n, \ldots\}$ be a sequence of independent random variables with identical means $\mathbb{E}[x_i] = m$ and variances $\mathrm{var}(x_i) = \sigma^2$. Define the sample mean and sample mean-square of the first N of the x_i's as

$$\text{sample mean:} \qquad m_N = \frac{1}{N}\sum_{i=1}^{N} x_i$$

$$\text{sample mean-square:} \qquad s_N^2 = \frac{1}{N}\sum_{i=1}^{N} x_i^2$$

(a) Find the mean and variance of the sample mean. Show that

$$\lim_{N\to\infty} \mathbb{E}[(m_N - m)^2] = 0$$

and use this result to deduce that

$$\lim_{N\to\infty} \Pr[|m_N - m| \geq \epsilon] = 0$$

for any $\epsilon > 0$.

(b) Suppose the x_i are zero-mean Gaussian random variables. Find the mean and variance of the sample mean-square. Show that

$$\lim_{N\to\infty} \mathbb{E}[(s_N^2 - \sigma^2)^2] = 0$$

(c) Suppose that the x_is are independent, zero-mean, Gaussian random variables. Are m_N and s_N^2 Gaussian random variables? Explain your reasoning.

8. Let A denote the event that the A's win the world series, let B denote the event that the Golden Bears win the Rose Bowl, let R denote the event that the Raiders win the Super Bowl, and let W denote the event that the Warriors win the NBA Championship. For any subset $S \subset \{A, B, R, W\}$, let p_S denote the probability that the teams in the set S all win their respective championship. Show that the matrix

$$\begin{bmatrix} p_{\{AB\}} & p_{\{ABR\}} & p_{\{ABRW\}} & p_{\{ABW\}} \\ p_{\{ABR\}} & p_{\{BR\}} & p_{\{BRW\}} & p_{\{ABRW\}} \\ p_{\{ABRW\}} & p_{\{BRW\}} & p_{\{RW\}} & p_{\{ARW\}} \\ p_{\{ABW\}} & p_{\{ABRW\}} & p_{\{ARW\}} & p_{\{AW\}} \end{bmatrix}$$

is positive semidefinite.