MIMO Compute-and-Forward

Jiening Zhan, Bobak Nazer, and Michael Gastpar
Wireless Foundations Center, EECS Department
University of California, Berkeley
Berkeley, CA 94709, USA
Email: jiening, bobak, gastpar@eecs.berkeley.edu

Uri Erez
Dept. of Electrical Engineering - Systems
Tel-Aviv University
Tel-Aviv 69978, Israel
Email: uri@eng.tau.ac.il

***Abstract*—In many network communication scenarios, a relay in the network may only need to recover and retransmit an equation of the transmitted messages. In previous work, it has been shown that if each transmitter employs the same lattice code, the interference structure of the channel can be exploited to recover an equation much more efficiently than possible with standard multiple-access strategies. Here, we generalize this compute-and-forward framework to the multiple antenna setting. Our results show that it is often beneficial to use extra antennas at the receiver to rotate the channel coefficients towards the nearest integer vector instead of separating out the transmitted signals. We also demonstrate that in contrast to classical strategies, the multiplexing gain of compute-and-forward increases if the transmitters have channel state information. Finally, we apply our scheme to the two way relay network and observe performance gains over traditional strategies.**

I. INTRODUCTION

Consider M transmitters connected to a single receiver through a multiple-access channel (MAC). The MAC capacity region gives the best possible rates for the receiver to recover all of the transmitted messages. However, in many settings, such as network coding [1], it may only be of interest to decode a *function of the transmitted messages*. In these cases, the MAC capacity region is too pessimistic and significant gains are possible through the use of structured codes [2].

In [3], it was shown that for a Gaussian MAC with channel output $y[i] = \sum_{m=1}^{M} h_m x[i] + z[i]$ with power P per user and noise variance N, any linear function of the messages can be recovered at rate:

$$R = \max_{\lambda \in \mathbb{C}} \log \left(\frac{P}{|\lambda|^2 N + P \|\lambda \mathbf{h} - \mathbf{a}\|^2} \right) \tag{1}$$

where $\mathbf{a} = [a_1 \; a_2 \; \ldots \; a_M]^T$ represents the integer-valued coefficients of the desired linear equation. Note that the non-integer part of the channel vector $\mathbf{h}$ appears as a noise term in the rate expression. This is simply because the underlying channel codes of this *compute-and-forward* scheme are drawn from a lattice and only integer combinations of lattice points will be lattice points themselves. This *approximation error* can be somewhat mitigated by scaling the channel output by λ prior to decoding but, in general, this integer penalty will persist.

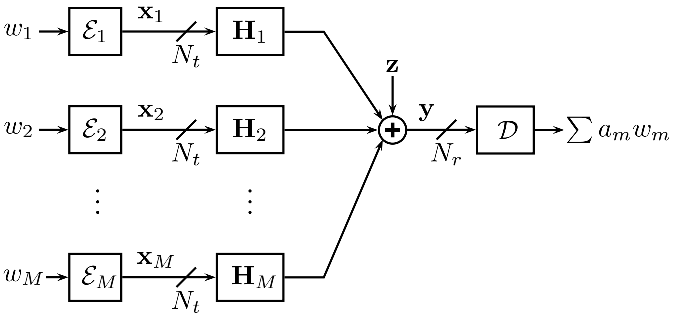

Fig. 1. Decoding an equation over a MIMO multiple-access channel.

In this paper, we show that if the receiver is equipped with more than one antenna, it can steer the channel coefficients towards integer values much more effectively. Hence, multiple antennas can considerably reduce the approximation error and increase the compute-and-forward rate. We first find a closed form solution for the optimal linear receiver for the case when the transmitters have no channel state information. We then demonstrate that the multiplexing gain of our strategy is fundamentally different from standard multiple-access schemes. Specifically, we show that channel state information (CSI) at a transmitter can buy degrees-of-freedom for MIMO compute-and-forward while, at best, transmitter CSI is worth a power gain for the classical MIMO MAC scheme. Finally, we apply our scheme to the multiple antenna two-way relay channel. In certain parameter regimes, our strategy outperforms either compress-and-forward or decode-and-forward. Overall, our results demonstrate that lattice codes are also useful for attaining higher rates than possible with i.i.d. random codes in the multiple-antenna setting.

A. Related Work

The MIMO MAC capacity region is well-studied in the literature (see, for instance [4], [5]). Recently, several groups have independently shown that structured codes can improve upon the rates attainable with the usual i.i.d. random coding techniques for many network communication scenarios [6]–[11]. In particular, for Gaussian channel models, nested lattice codes as developed in [12] can be extremely useful.

J. Zhan, B. Nazer, and M. Gastpar were supported by NSF grants CCR-0347298, CNS-0627024, and CCF-0830428. Additionally, J. Zhan was supported by an NSF Graduate Fellowship. U. Erez was supported by ISF grant 1234/09.

II. DEFINITIONS

A. Problem Statement

Consider a MIMO multiple-access channels with M users, N_t transmit antennas per user, and N_r receive antennas per user (see Figure 1). Let $\mathbf{x}_m[i] \in \mathbb{C}^{N_t \times 1}$ denote the channel input from user m at time i. The channel inputs for each user must satisfy the usual power constraint:

$$\frac{1}{n} \sum_{i=1}^{n} \|\mathbf{x}_m[i]\|^2 \leq \mathsf{SNR} \tag{2}$$

The channel output $\mathbf{y}[i] \in \mathbb{C}^{N_r \times 1}$ at time i is specified by:

$$\mathbf{y}[i] = \sum_{m=1}^{M} \mathbf{H}_m \mathbf{x}_m[i] + \mathbf{z}[i] \tag{3}$$

where $\mathbf{H}_m \in \mathbb{C}^{N_r}$ is the fading matrix seen by user m with entries drawn i.i.d from $\mathcal{CN}(0,1)$ and $\mathbf{z}[i]$ is a length-N_r noise vector with entries also drawn i.i.d. from $\mathcal{CN}(0,1)$. We assume that the receiver always has perfect knowledge of the channel matrix. The transmitters always know their target rate but may not have any knowledge of the channel realizations.

Let $R > 0$ be the desired target rate. Each user has two message vectors $\mathbf{w}_m^R, \mathbf{w}_M^I \in \mathbb{F}_p^k$ where $\mathbb{F}_p$ is a prime-sized finite field and $k = nR(2\log_2 p)^{-1}$. One of these vectors is encoded and transmitted on the real part of the channel input and the other on the imaginary part.

The receiver, given the channel output $\mathbf{y} \in \mathbb{C}^n$, attempts to recover two equations of the transmitted messages. Specifically, the receiver wants to decode:

$$\mathbf{u}^R = \sum_{m=1}^{M} a_m^R \mathbf{w}_m^R - \sum_{m=1}^{M} a_m^I \mathbf{w}_m^I \tag{4}$$

$$\mathbf{u}^I = \sum_{m=1}^{M} a_m^I \mathbf{w}_m^R + \sum_{m=1}^{M} a_m^R \mathbf{w}_m^I \tag{5}$$

where $a_m^R, a_M^I \in \mathbb{F}_p$ and all operations are over $\mathbb{F}_p$. The structure of these two equations is chosen to match the complex addition performed naturally by the channel. Let $a_m = a_m^R + j a_m^I$ be the embedding of the equation coefficients for user m from $\mathbb{F}_p \times \mathbb{F}_p$ into $\mathbb{C}$. We denote the vector of these embedded coefficients by $\mathbf{a}^T = [a_1 \ a_2 \ \cdots \ a_M]$.

Remark 1: For real-valued channels, the receiver would simply recover $\sum a_m^R \mathbf{w}_m$ as the desired equation.

Definition 1: We say that rate R is achievable for decoding equations $\mathbf{u}^R, \mathbf{u}^I$ with coefficients specified by $\mathbf{a}$ if for any $\epsilon > 0$ and n large enough, there exist encoding and decoding functions $\mathcal{E}_1, \ldots, \mathcal{E}_m, \mathcal{D}$ such that the receiver can produce estimates $\hat{\mathbf{u}}^R, \hat{\mathbf{u}}^I$ satisfying

$$\Pr\left(\left\{\hat{\mathbf{u}}^R \neq \mathbf{u}^R\right\} \cap \left\{\hat{\mathbf{u}}^I \neq \mathbf{u}^I\right\}\right) < \epsilon. \tag{6}$$

Given a full rank set of such equations at rate R it is clear that a receiver could solve for the original messages.

Remark 2: One could also consider decoding multiple linear equations of the messages simultaneously as well as multiple receivers, each with different channel matrices, along the lines of [3].

III. ACHIEVABLE RATES

Our results build on the scheme for single-antenna case from [3].

Theorem 1 (Nazer-Gastpar): Given a single-antenna multiple-access channel, $y[i] = \sum_{m=1}^{M} h_m x[i] + z[i]$, with P power per user and variance N noise, the following rate is achievable for decoding the equation with coefficients specified by $\mathbf{a} \in \mathbb{C}^M$

$$R = \log\left(\frac{P}{N + P\|\mathbf{h} - \mathbf{a}\|^2}\right) \tag{7}$$

The proof relies on each encoder using the same nested lattice code. See [13] for more details. Note that there are two noise penalties. The first is due to the usual additive noise term. The second noise term is unique to compute-and-forward and is due to the non-integer part of the channel coefficients. We will use the extra antennas to rotate the channel coefficients towards integers with an eye to reducing this penalty.

Our MIMO compute-and-forward scheme uses the above result combined with appropriate precoding vectors. Assume that all transmitters have full channel knowledge. The length-N_t channel input from encoder m at time i is given by $\mathbf{x}_m[i] = \mathbf{c}_m \tilde{x}[i]$ where $\tilde{x}[i]$ is chosen as in the scalar case above. Prior to decoding, the receiver projects the channel output using the length-N_r vector $\mathbf{b}$ to get $\tilde{y}[i] = \mathbf{b}^T \mathbf{y}[i]$. Combining these steps we get a scalar MAC:

$$\tilde{h}_m = \mathbf{b}^T \mathbf{H}_m \mathbf{c}_m \tag{8}$$

$$\tilde{y}[i] = \sum_{m=1}^{M} \tilde{h}_m \tilde{x}[i] + \tilde{z}[i] \tag{9}$$

where $\tilde{z}[i]$ is i.i.d. circularly symmetric Gaussian noise with variance $\|\mathbf{b}\|^2$. This gives us the following theorem.

Theorem 2: Given a multiple-antenna multiple-access channel, $\mathbf{y}[i] = \sum_{m=1}^{M} \mathbf{H}_m \mathbf{x}[i] + \mathbf{z}[i]$, N_t transmit antennas and N_r receive antennas, the following rate is achievable for decoding the equation with coefficients given by $\mathbf{a}$

$$R(\mathbf{a}) = \max_{\mathbf{b}, \mathbf{c}_m} \log\left(\frac{\mathsf{SNR}}{\|\mathbf{b}\|^2 + \mathsf{SNR}\sum_{m=1}^{M} |\mathbf{b}^T \mathbf{H}_m \mathbf{c}_m - a_m|^2}\right)$$
$$\text{such that } \|\mathbf{c}_m\|^2 \leq 1 \qquad m = 1, 2, \ldots, M \tag{10}$$

Intuitively, these rotation vectors should be used to maneuver the channel coefficients towards the desired equation coefficients. If the receiver has as many antennas as there are users, i.e $N_r = M$, it can set the channel coefficients to be integers and completely null the approximation error.

In general, finding the optimal $\mathbf{b}$ and $\mathbf{c}_m$ for channel matrices $\mathbf{H}_m$ and desired coefficients $\mathbf{a}$ is a non-convex problem. Below, we find the optimal vectors for the special case where the transmitters have no knowledge of the channel realization. We then show how to find the optimal transmit vectors $\mathbf{c}_m$ for a fixed receive vector $\mathbf{b}$.

IV. OBLIVIOUS TRANSMITTERS

In practice, channel knowledge is obtained at the receiver through pilot measurements and at the transmitters through feedback. In certain wireless settings, feedback from the receiver may be ineffective and hence channel knowledge is not feasible at the transmitters. Without channel knowledge, extra transmit antennas are not useful in our lattice scheme since the transmitters cannot steer the fading coefficients closer to integers. In this section, we present the achievable rates for the special case of a single transmit antenna in the absence of transmitter channel knowledge.

Our channel matrices become vectors of the form $\mathbf{h}_1, ..., \mathbf{h}_M$. We define the global channel matrix $\mathbf{G}^* \in \mathbf{C}^{M \times N_r}$ to have rows $\mathbf{h}_1^T, ..., \mathbf{h}_M^T$. Before presenting the rate for our scheme, we first observe that when all the individual signals have to be decoded, the capacity region is given by the set of all rate vectors $(R_1, ..., R_M)$ satisfying

$$\sum_{i \in \mathcal{S}} R_i \leq \log\det\left(\mathbf{I} + \mathsf{SNR}\sum_{i \in \mathcal{S}} \mathbf{h}_i\mathbf{h}_i^*\right) \tag{11}$$

for all $\mathcal{S} \in \{1, ..., M\}$ [4]. This rate is achievable using random coding.

The achievable rate for our lattice scheme simplifies as follows. For an equation with coefficient vector $\mathbf{a}$, successful decoding is possible if each message is encoded at rate less than

$$R(\mathbf{a}) = \max_{\mathbf{b} \in \mathbb{C}^{N_r}} \log\left(\frac{\mathsf{SNR}}{||\mathbf{b}||^2 + \mathsf{SNR}||\mathbf{G}^*\mathbf{b} - \mathbf{a}||^2}\right) \tag{12}$$

We observe that if there are enough receive antennas, i.e $N_r \geq M$, then the error from approximating the channel coefficients by integers can nulled at the cost of incurring additional noise. However, for finite SNR, it is suboptimal to cancel all the interference. Rather, the optimal receiver projection vector is $\mathbf{b_{OPT}} = \left(\mathbf{GG}^* + \frac{1}{\mathsf{SNR}}\mathbf{I}\right)^{-1}\mathbf{Ga}$. This follows from the fact that optimization problem can be rewritten as a quadratic program. Using the optimal projection vector at the decoder, we can derive the following expression for the achievable rate.

Theorem 3: Given a multiple-access channel with N_r receive antennas and no channel state information at the transmitters, the following rate is achievable for decoding the equation with coefficients $\mathbf{a}$:

$$R(\mathbf{a}) = -\log \mathbf{a}^*\mathbf{VDV}^*\mathbf{a} \tag{13}$$

where $\mathbf{V} \in \mathbb{C}^{M \times M}$ is the right eigenmatrix of $\mathbf{G}$ and $\mathbf{D} \in \mathbb{R}^{M \times M}$ is a diagonal matrix with elements,

$$\mathbf{D}_{i,i} = \begin{cases} \frac{1}{\mathsf{SNR}\lambda_i + 1} & i \leq \text{rank}(\mathbf{G}) \\ 1 & i > \text{rank}(\mathbf{G}) \end{cases} \tag{14}$$

where λ_i is the ith eigenvalue of $\mathbf{G}^*\mathbf{G}$.

Proof: Define $f : \mathbb{C}^{N_r} \rightarrow \mathbb{C}$ as follows:

$$f(\mathbf{b}) = \frac{1}{\mathsf{SNR}}\|\mathbf{b}\|^2 + \|\mathbf{G}^*\mathbf{b} - \mathbf{a}\|^2 \tag{15}$$

Let $\mathbf{U\Sigma V}^*$ be the singular value decomposition of $\mathbf{G}$ with $\mathbf{U} \in \mathbb{C}^{Nr \times Nr}, \mathbf{\Sigma} \in \mathbb{C}^{Nr \times M}, \mathbf{V} \in \mathbb{C}^{M \times M}$. Note that in this SVD representation, $\Sigma_{i,i} = \sigma_i$ (the ith singular value of $\mathbf{G}$) and $\Sigma_{i,j} = 0$ for all $i \neq j$. Evaluating f at $\mathbf{b}_{OPT}$ gives

$$\begin{aligned} f(\mathbf{b}_{OPT}) &= \tfrac{1}{\mathsf{SNR}}\mathbf{b}^*\mathbf{b} + \mathbf{b}^*\mathbf{GG}^*\mathbf{b} - \mathbf{b}^*\mathbf{Ga} - \mathbf{a}^*\mathbf{G}^*\mathbf{b} + \mathbf{a}^*\mathbf{a} \\ &= \mathbf{b}^*(\mathbf{GG}^* + \tfrac{1}{\mathsf{SNR}}\mathbf{I})\mathbf{b} - \mathbf{b}^*\mathbf{Ga} - \mathbf{a}^*\mathbf{G}^*\mathbf{b} + \mathbf{a}^*\mathbf{a} \\ &= \mathbf{b}^*\mathbf{Ga} - \mathbf{b}^*\mathbf{Ga} - \mathbf{a}^*\mathbf{G}^*\mathbf{b} + \mathbf{a}^*\mathbf{a} \\ &= -\mathbf{a}^*\mathbf{G}^*\mathbf{b} + \mathbf{a}^*\mathbf{a} \\ &= -\mathbf{a}^*\mathbf{G}^*\left(\mathbf{GG}^* + \frac{1}{\mathsf{SNR}}\mathbf{I}\right)^{-1}\mathbf{Ga} + \mathbf{a}^*\mathbf{a} \\ &= -\mathbf{a}^*\mathbf{V\Sigma}^*\mathbf{U}^*\left(\mathbf{U\Sigma\Sigma}^*\mathbf{U}^* + \frac{1}{\mathsf{SNR}}\mathbf{I}\right)^{-1}\mathbf{U\Sigma V}^*\mathbf{a} \\ &\quad + \mathbf{a}^*\mathbf{Ia} \\ &= \mathbf{a}^*\mathbf{V}\left(\mathbf{I} - \mathbf{\Sigma}^*\left(\mathbf{\Sigma\Sigma}^* + \frac{1}{\mathsf{SNR}}\mathbf{I}\right)^{-1}\mathbf{\Sigma}\right)\mathbf{V}^*\mathbf{a} \\ &= \mathbf{a}^*\mathbf{VDV}^*\mathbf{a} \end{aligned}$$

Equation (13) follows by noting that $R(\mathbf{a}) = -\log f(\mathbf{b}_{OPT})$. ■

The theorem above suggests that our scheme performs well when the coefficients vector $\mathbf{a}$ aligns in the direction of $\mathbf{v}_{max}$ where $\mathbf{v}_{max}$ is the largest eigenvector of the matrix $\mathbf{G}^*\mathbf{G}$. This reinforces the idea that structured codes can outperform random codes when the channel and the desired function are suitably matched. We illustrate this point with the following example.

Example 1: Consider a 2 user, 1 transmit antenna and 2 receive antenna MIMO MAC with global channel matrix

$$\mathbf{G} = \frac{1}{2}\begin{bmatrix} \sqrt{K}+1 & \sqrt{K}-1 \\ \sqrt{K}-1 & \sqrt{K}+1 \end{bmatrix}$$

where $K \geq 1$.

The eigenvalues of $\mathbf{G}^*\mathbf{G}$ are found to be $\lambda_{max} = K$ and $\lambda_{min} = 1$ with corresponding eigenvectors $\mathbf{v}_{max} = \frac{1}{\sqrt{2}}[1, 1]^T$ and $\mathbf{v}_{min} = \frac{1}{\sqrt{2}}[1, -1]^T$. We are interested in recovering the sum of the input signals, i.e $\mathbf{a} = [1, 1]^T$. The achievable rates via MIMO compute-and-forward (MCF) and random coding (RAND) are given by

$$R_{\text{MCF}} = \frac{1}{2}\log\left(1 + \frac{K}{2}\mathsf{SNR}\right) \tag{16}$$

$$R_{\text{RAND}} = \frac{1}{4}\log(1 + \mathsf{SNR}) + \frac{1}{4}\log(1 + K\mathsf{SNR}) \tag{17}$$

At high SNR, the rates scale as

$$R_{\text{MCF}} \sim \frac{1}{2}\log\left(\frac{K}{2}\mathsf{SNR}\right) \qquad R_{\text{RAND}} \sim \frac{1}{2}\log\left(\sqrt{K}\mathsf{SNR}\right) \tag{18}$$

where R_{RAND} follows from (11).

For large values of K, MIMO compute-and-forward outperforms random coding by a substantial amount. Note that in this example, the coefficients $\mathbf{a}$ aligns perfectly in the direction of maximum eigenvector $\mathbf{v}_{max}$.

Until now, we have implicitly assumed that we are interested in a particular equation of the transmitted codewords. However, in networks with intermediate relay nodes, it is often sufficient to know any equation of the inputs, so long as the full set of equations is full rank. In the rest of this section, we analyze the rate of decoding the best integer equation given the channel coefficients. We define R_{BEST} to be the maximum achievable rate over all functions with integer coefficients (except the all zeros vector).

$$R_{\text{BEST}} = \max_{\mathbf{a} \in \mathbb{Z}^M + j\mathbb{Z}^M \setminus \{\mathbf{0}\}} - \log \mathbf{a}^* \mathbf{V} \mathbf{D} \mathbf{V}^* \mathbf{a} \tag{19}$$

The optimization problem presented in (19) is a minimal lattice point search and is known to be NP hard. Fortunately, it can be shown that any $\mathbf{a}$ with $\|\mathbf{a}\| \leq (1+\lambda_{\text{MAX}})\mathsf{SNR}$ results in zero rate, which makes the search feasible for low to moderate values of SNR.

We can also show that R_{BEST} is lower and upper bounded as follows:

$$-\log\left(\frac{1}{M}\sum_{i=1}^{M}\frac{1}{1+\lambda_i \mathsf{SNR}}\right) \leq R_{\text{BEST}} \leq \log(1+\lambda_{\text{MAX}}\mathsf{SNR}).$$

Furthermore, in the special case of $M = 2$ users and $N_r = 2$ receive antennas, R_{BEST} can be also be lower bounded by

$$R_{\text{BEST}} \geq R_{\text{RAND}} + \log\left(\frac{\sqrt{\gamma}}{\min_{(\alpha,\beta)\in\mathcal{A}} \alpha\gamma + \beta(1-\gamma)}\right)$$

where γ and $\mathcal{A}$ are given by

$$\gamma = \frac{\lambda_{\text{MIN}}\mathsf{SNR}+1}{\lambda_{\text{MAX}}\mathsf{SNR}+1} \tag{20}$$

$$\mathcal{A} = \left\{\left(1, \frac{1}{5}\right), \left(2, \frac{1}{10}\right)\right\} \tag{21}$$

for $\gamma \in \left(\frac{1}{16}, \frac{5+\sqrt{6}}{19}\right)$, $R_{\text{BEST}} > R_{\text{RAND}}$. The proof is omitted due to space constraints.

V. TRANSMITTER CHANNEL KNOWLEDGE

When the transmitters have channel state information, precoding can help steer the fading coefficients closer to integers. In this section, we show that channel knowledge at the transmitters can increase the multiplexing gain of MIMO compute-and-forward. Let the multiplexing gain be denoted by $r = \lim_{\mathsf{SNR}\to\infty} \frac{R}{\log \mathsf{SNR}}$ where R is the achievable rate.

Recall that (10) is non-convex in general. However, for a fixed $\mathbf{b}$, the optimization over $\mathbf{c}_1, \ldots \mathbf{c}_m$ is convex. Define $\mathbf{p}_m^* = \mathbf{b}^T \mathbf{H}_m$ for $m = 1, \ldots M$. If $|a_m| \leq \|\mathbf{p}_m\|$, the optimal $\mathbf{c}_m$ for a fixed $\mathbf{b}$ is given by

$$\mathbf{c}_m = \left\{\frac{a_m \mathbf{p}_m}{\|\mathbf{p}_m\|^2} + \mathbf{v} : \mathbf{v} \perp \mathbf{p}_m, \frac{|a_m|^2}{\|\mathbf{p}_m\|^2} + \|\mathbf{v}\|^2 \leq 1\right\} \tag{22}$$

Here, transmitter m can steer its channel coefficient to the desired integer since $\mathbf{b}^T \mathbf{H} \mathbf{c}_m = a_m$. If $|a_m| \geq \|\mathbf{p}_m\|$, the optimal $\mathbf{c}_m$ (for a fixed $\mathbf{b}$) is given by

$$\mathbf{c}_m = \frac{a_m \mathbf{p}_m}{|a_m|\|\mathbf{p}_m\|} \tag{23}$$

Note that this is equivalent to Equation (22) if $|a_m| = \|\mathbf{p}_m\|$. Furthermore, if $|a_m| > \|\mathbf{p}_m\|$, transmitter m does not have enough power to steer its channel coefficient to a_m. If the receiver chooses $\mathbf{b}$ such that $\|\mathbf{H}_m^* \mathbf{b}\| \geq |a_m|$ for all m, then the fading coefficients can be steered to integers by precoding alone. Thus, the error from approximating the channel coefficients as integers can be completely eliminated and the optimal multiplexing gain $r = 1$ is achievable.

Multiplexing gain can also be observed in the case of partial channel state information. Consider a scenario where the transmitters and the receiver have only one antenna and a subset of the transmitters have global channel knowledge while the rest are ignorant of the channel. Let $\Gamma \subset \{1, \ldots, M\}$ denote the set of transmitters with channel knowledge. We require that the receiver recovers an equation of the transmitted messages with all non-zero coefficients.

Using random coding, the best performance is given by the maximum symmetric rate point in the MIMO MAC region. Recall that channel knowledge at the transmitters translates to only a power gain for the MIMO MAC region. Hence, the multiplexing gain remains unchanged.

$$r_{\text{RAND}} = \frac{1}{M} \tag{24}$$

On the contrary, transmit channel knowledge increases the multiplexing gain for our lattice scheme.

$$r_{\text{COMPUTE}} = \min\left\{\frac{1}{M-|\Gamma|}, 1\right\} \tag{25}$$

Proof Sketch: For simplicity, we give a proof sketch for the real valued channel. The complex version follows using the same ideas. Assume that $|h_m| \geq 1$ for all m (we can achieve this by post-scaling the output by $\max_m \frac{1}{|h_m|}$ at the receiver). This ensures that we decode to non-zero coefficients. Assume that the first $|\Gamma|$ users have channel knowledge. Define the function $\gamma : \mathbb{N} \to \mathbb{R}$ as follows

$$\gamma(b) = \min_{a_m \in \mathbb{Z}} b^2 + \mathsf{SNR} \sum_{m=|\Gamma|+1}^{M} (h_m b - a_m)^2 \tag{26}$$

Dirchlet's approximation theorem guarantees that for any length K vector $\mathbf{p} \in \mathbb{R}^K$, there exists a $q \in \{1, 2\ldots, Q\}$ such that $\|q\mathbf{p} \mod \mathbb{Z}^K\| \leq \frac{K}{Q^{\frac{1}{K}}}$ [14]. Assume that $Q^2 \sim \mathsf{SNR}^{\beta}$ and applying Dirchlet's theorem, it can be shown that there exists a positive integer $b \leq Q$ such that $\gamma(b) \sim \mathsf{SNR}^{\beta} + \mathsf{SNR}^{\left(1-\frac{\beta}{M-|\Gamma|-1}\right)}$. Since $|h_m| \geq 1$ for all m and $b > 0$, the minimizing $a_m \neq 0$ for all m. Let $\kappa = \lceil \max_{m=1,\ldots|\Gamma|} \frac{1}{h_m b} \rceil$. It can be shown that $\gamma(\kappa b)$ also scales as $\mathsf{SNR}^{\beta} + \mathsf{SNR}^{\left(1-\frac{\beta}{M-|\Gamma|-1}\right)}$. For $m = 1, \ldots|\Gamma|$, precode by setting

$$c_m = \frac{1}{h_m \kappa b} \tag{27}$$

By the definition of κ, we ensure that $|c_m| \leq 1$. The final result follows by setting $\beta = \frac{M-|\Gamma|-1}{M-|\Gamma|}$ and noticing that $R_{\text{COMPUTE}} = \log \frac{\mathsf{SNR}}{\gamma}$.

VI. Network Applicaton: Two Way Relay

There has been a great deal of recent interest in the two-way relay channel (see, for instance, [8], [11], [15], [16], and the references therein). The (real-valued) network model is as follows. Two encoders want to exchange messages w_1, w_2 at the highest possible symmetric rate with the help of a relay. The encoders generate channel inputs $x_1[i]$ and $x_2[i]$ subject to individual block power constraints $\frac{1}{n}\sum_{i=1}^{n}(x_m[i])^2 \leq \mathsf{SNR}$. The relay observes $\mathbf{y}[i] = \mathbf{h}_1 x_1[i] + \mathbf{h}_2 x_2[i] + \mathbf{z}[i]$ where $\mathbf{h}_1, \mathbf{h}_2, \mathbf{z}[i]$ are drawn i.i.d. according to $\mathcal{N}(\mathbf{0}, \mathbf{I}^{N_r \times N_r})$. The relay has a noiseless broadcast link back to both users with rate R_0. Note that the transmitters have channel state information. The network is illustrated in a block diagram in Figure 2.

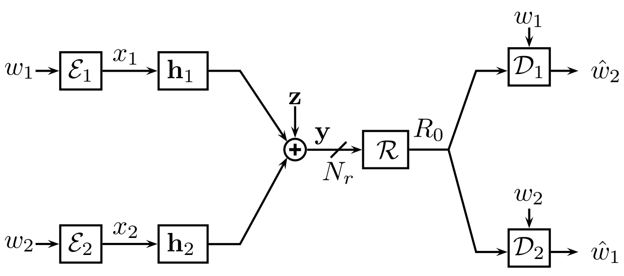

Fig. 2. Two Way Relay

In the single antenna two-way relay, lattice codes allow the relay to recover the sum of the messages at a higher rate than available for decoding the messages individually. When the relay is equipped with multiple antennas, one might expect that they should be used to separate out the individual messages. However, we show that, in a certain regime, it is better to use the antennas to help decode a single equation of the messages.

In Figure 3, we compare the performance (averaged over the channel state) of MIMO compute-and-forward with compress-and-forward and decode-and-forward for $R_0 = 5$ and $N_r = 1, 2$ antennas at the relay. Clearly, adding an antenna at the relay provides a nontrivial increase in throughput. It is also clear that using this antenna to enhance the performance of compute-and-forward results in the highest average rate. This affirms that both transmitter CSI and multiple antennas are beneficial for computing equations in networks.

VII. Conclusions

In this paper, we developed a multiple-antenna extension of the lattice-based compute-and-forward strategy for AWGN networks. We showed that the addition of antennas at the receiver can be used to rotate the channel vector towards integers, thus increasing the compute-and-forward rate. We also showed that channel state information at the transmitters can provide a boost in the multiplexing gain. We applied our framework to a relay network scenario and demonstrated improved performance over traditional strategies.

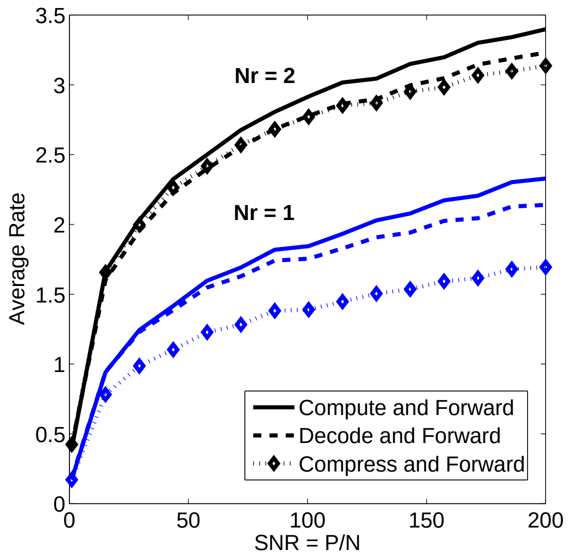

Fig. 3. Average rate for two way relay channel with transmitter CSI and $N_r = 1, 2$ antennas at the relay, Ro = 5

References

[1] R. Ahlswede, N. Cai, S.-Y. R. Li, and R. W. Yeung, “Network information flow,” *IEEE Trans. Info. Theory*, vol. 46, pp. 1204–1216, July 2000.

[2] B. Nazer and M. Gastpar, “Computation over multiple-access channels,” *IEEE Trans. Info. Theory*, vol. 53, pp. 3498–3516, Oct. 2007.

[3] B. Nazer and M. Gastpar, “Compute-and-forward: Harnessing interference with structured codes,” in *ISIT 2008*, Toronto, Canada.

[4] A. Goldsmith, S. A. Jafar, N. Jindal, and S. Vishwanath, “Capacity limits of MIMO channels,” *IEEE JSAC*, vol. 21, pp. 684–702, June 2003.

[5] D. Tse and P. Viswanath, *Fundamentals of Wireless Communication*. Cambridge: Cambridge University Press, 2005.

[6] B. Nazer and M. Gastpar, “The case for structured random codes in network capacity theorems,” *Euro Trans. Telecomm.*, vol.19, no.4, pp.455-474, June 2008. To appear.

[7] T. Philosof, A. Khisti, U. Erez, and R. Zamir, “Lattice strategies for the dirty multiple access channel,” in *ISIT 2007*, Nice, France.

[8] K. Narayanan, M. P. Wilson, and A. Sprintson, “Joint physical layer coding and network coding for bi-directional relaying,” in *Allerton 2007*, Monticello, IL.

[9] D. Krithivasan and S. Pradhan, “Lattices for distributed source coding: Jointly Gaussian sources and reconstruction of a linear function,” *IEEE Trans. Info. Theory*, Submitted July 2007. See http://arxiv.org/abs/0707.3461.

[10] A. Sanderovich, M. Peleg, and S. Shamai, “Scaling laws in decentralized processing of interfered Gaussian channels,” in *IZS 2008*, Zurich, Switzerland.

[11] W. Nam, S.-Y. Chung, and Y. H. Lee, “Capacity bounds for two-way relay channels,” in *IZS 2008*, Zurich, Switzerland.

[12] U. Erez and R. Zamir, “Achieving $\frac{1}{2}\log(1+\mathsf{SNR})$ on the AWGN channel with lattice encoding and decoding,” *IEEE Trans. Info. Theory*, vol. 50, pp. 2293–2314, Oct. 2004.

[13] B. Nazer and M. Gastpar, “Compute-and-forward: Harnessing interference with structured codes,” *To be submitted to IEEE Trans. Info. Theory*.

[14] I. Bernik and M. M. Dodson, “Metric Diophantine Approximation on Manifolds,” Cambridge University Press, 1991.

[15] D. Gunduz, A. Goldsmith, and H. V. Poor, “MIMO two-way relay channel: Diversity-multiplexing tradeoff analysis,” *Asilomar 2008*. See http://arxiv.org/abs/0812.3642.

[16] R. Vaze and R. W. Heath Jr., “On the capacity and diversity-multiplexing tradeoff of the two-way relay channel,” See http://www.arxiv.org/0810.3900, Oct. 2008.