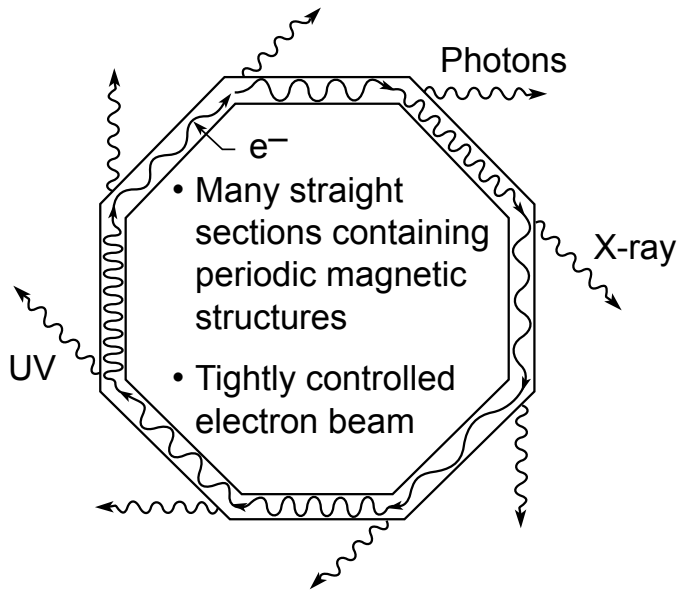
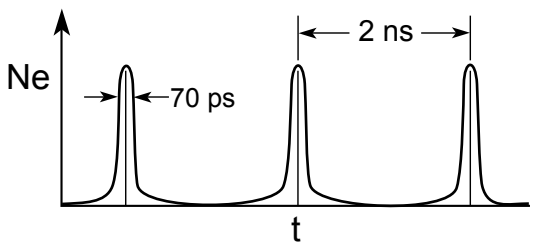
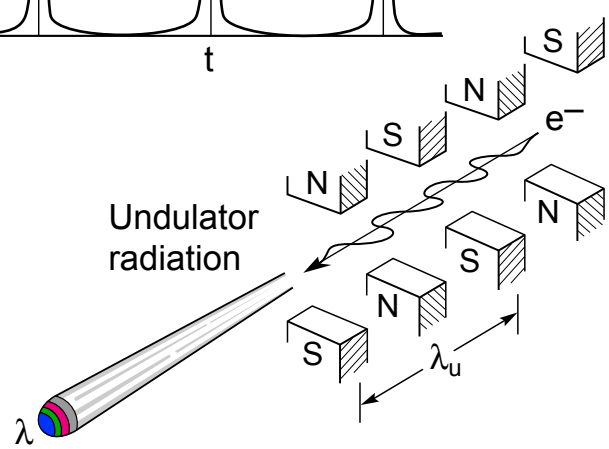
Intro to Synchrotron Radiation, Bending Magnet Radiation







Bending Magnet:

$$\hbar\omega_c = \frac{3e\hbar B\gamma^2}{2m} \qquad (5.7)$$

Wiggler:

$$\hbar\omega_c = \frac{3e\hbar B\gamma^2}{2m} \qquad (5.80)$$

$$n_c = \frac{3K}{4}\left(1 + \frac{K^2}{2}\right) \qquad (5.82)$$

$$P_T = \frac{\pi e K^2 \gamma^2 I N}{3\epsilon_0 \lambda_u} \qquad (5.85)$$

Undulator:

$$\lambda = \frac{\lambda_u}{2\gamma^2}\left(1 + \frac{K^2}{2} + \gamma^2\theta^2\right) \qquad (5.28)$$

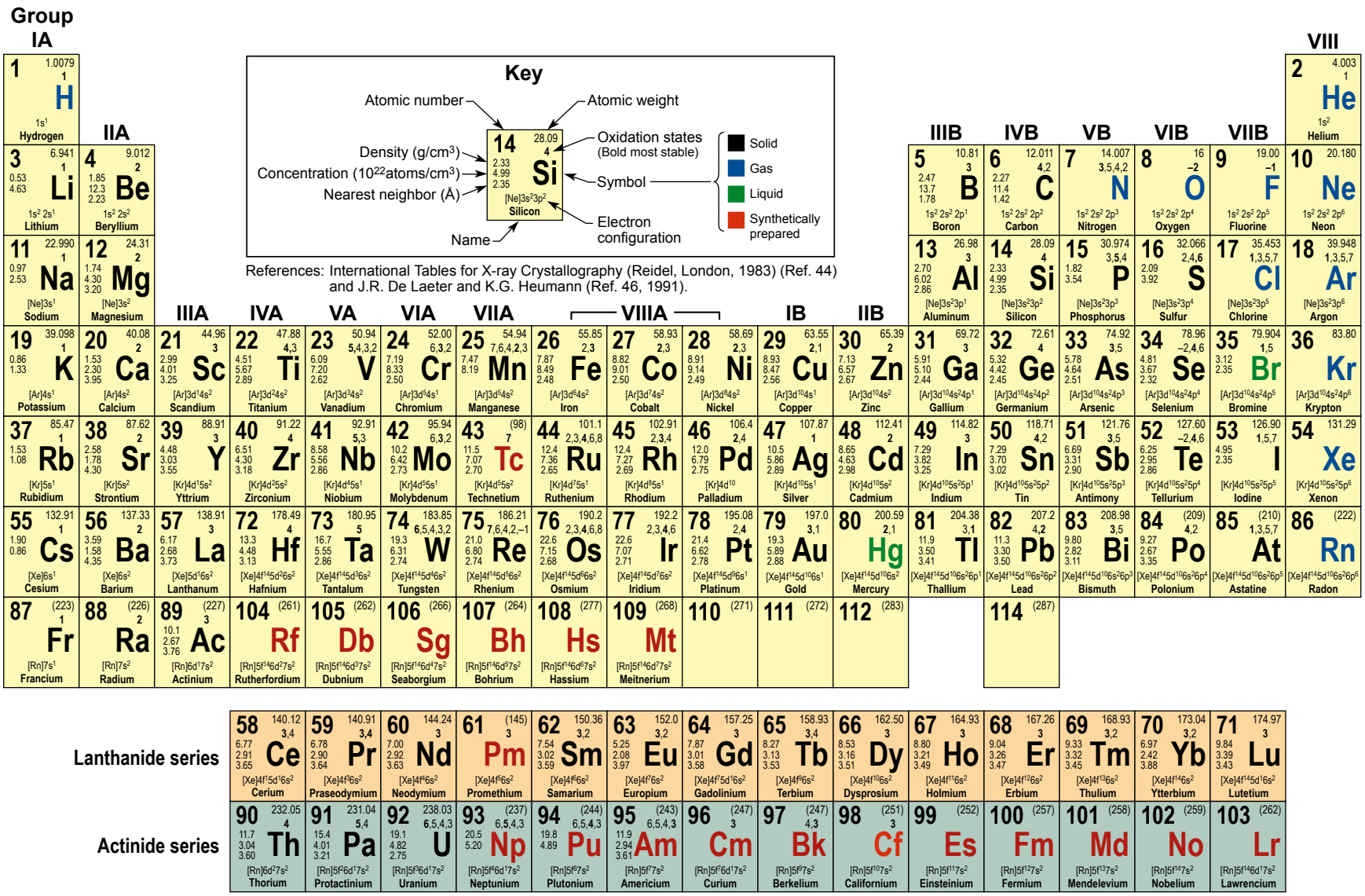
$$K = \frac{eB_0\lambda_u}{2\pi mc} \qquad (5.18)$$

$$\theta_{\text{cen}} = \frac{1}{\gamma^*\sqrt{N}} \qquad (5.15)$$

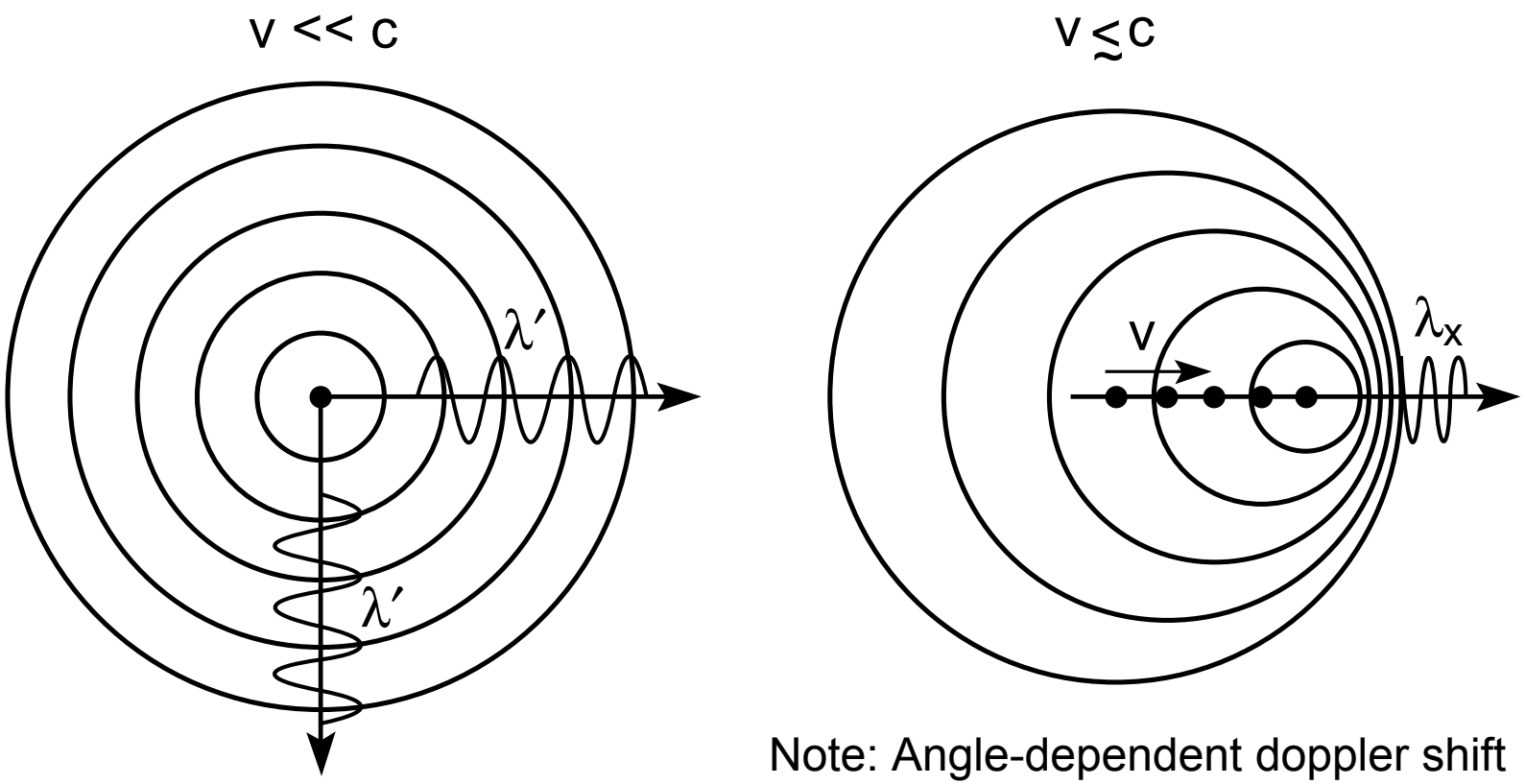
$$\left.\frac{\Delta\lambda}{\lambda}\right|_{\text{cen}} = \frac{1}{N} \qquad (5.14)$$

$$\bar{P}_{\text{cen}} = \frac{\pi e\gamma^2 I}{\epsilon_0\lambda_u}\frac{K^2}{\left(1 + \frac{K^2}{2}\right)^2} f(K) \qquad (5.41)$$

Broadly Tunable Radiation is Needed to Probe the Primary Resonances of the Elements



Synchrotron Radiation from Relativistic Electrons



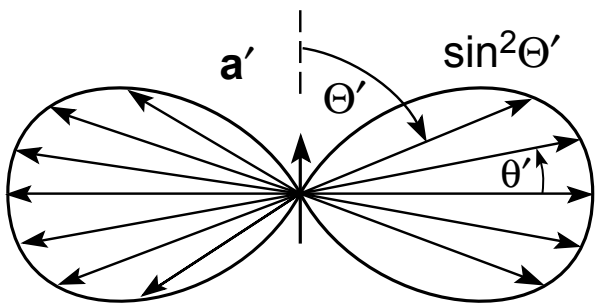
Note: Angle-dependent doppler shift

$$\lambda = \lambda' \left(1 - \frac{v}{c}\cos\theta\right) \qquad \lambda = \lambda' \gamma \left(1 - \frac{v}{c}\cos\theta\right)$$

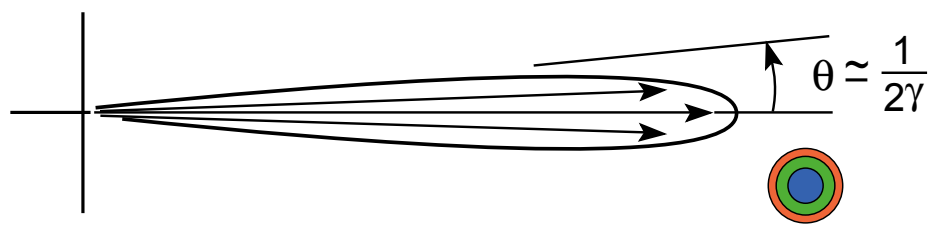
$$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$$

Synchrotron Radiation in a Narrow Forward Cone

Frame moving with electron



Laboratory frame of reference

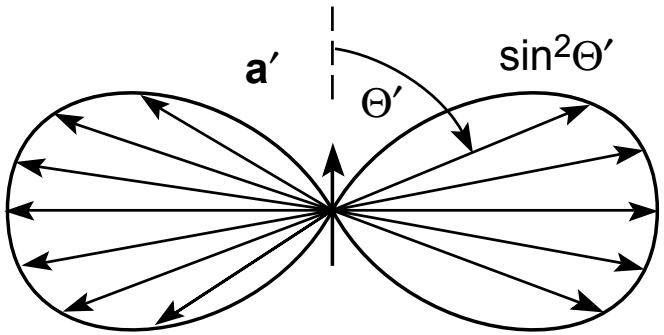


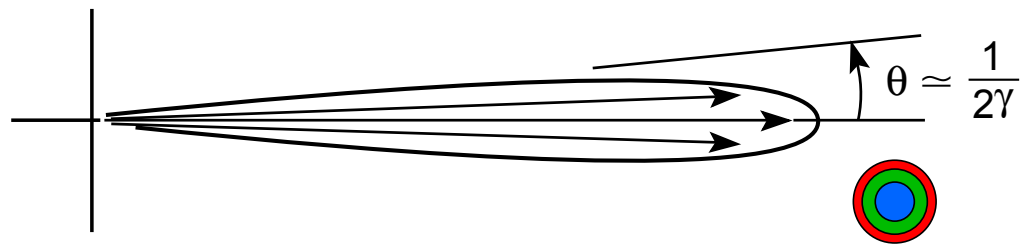
$$\tan\theta = \frac{\sin\theta'}{\gamma(\beta + \cos\theta')} \qquad (5.1)$$

$$\theta \simeq \frac{1}{2\gamma} \qquad (5.2)$$

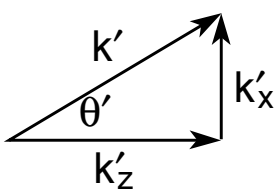
Relativistic Electrons Radiate in a Narrow Forward Cone

Dipole radiation





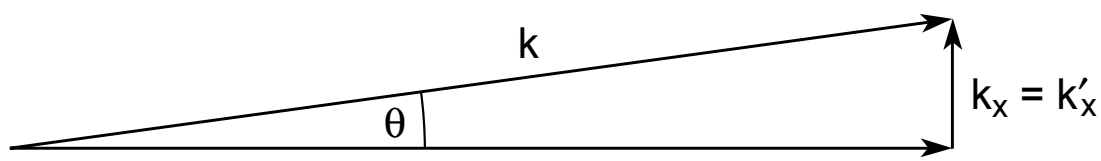
Frame of reference moving with electrons



$k' = 2\pi/\lambda'$

Lorentz transformation

Laboratory frame of reference



$k_z = 2\gamma k'_z$(Relativistic Doppler shift)

$$\theta \simeq \frac{k_x}{k_z} \simeq \frac{k'_x}{2\gamma k'_z} = \frac{\tan\theta'}{2\gamma} \simeq \frac{1}{2\gamma}$$

Some Useful Formulas for Synchrotron Radiation

$$\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} = \frac{1}{\sqrt{1-\beta^2}} \; ; \quad \beta = \frac{v}{c}$$

$$E_e = \gamma mc^2, \quad \boldsymbol{p} = \gamma m\mathbf{v}$$

$$\gamma = \frac{E_e}{mc^2} = 1957\, E_e(\mathrm{GeV})$$

$$\hbar\omega \cdot \lambda = 1239.842 \text{ eV} \cdot \text{nm}$$

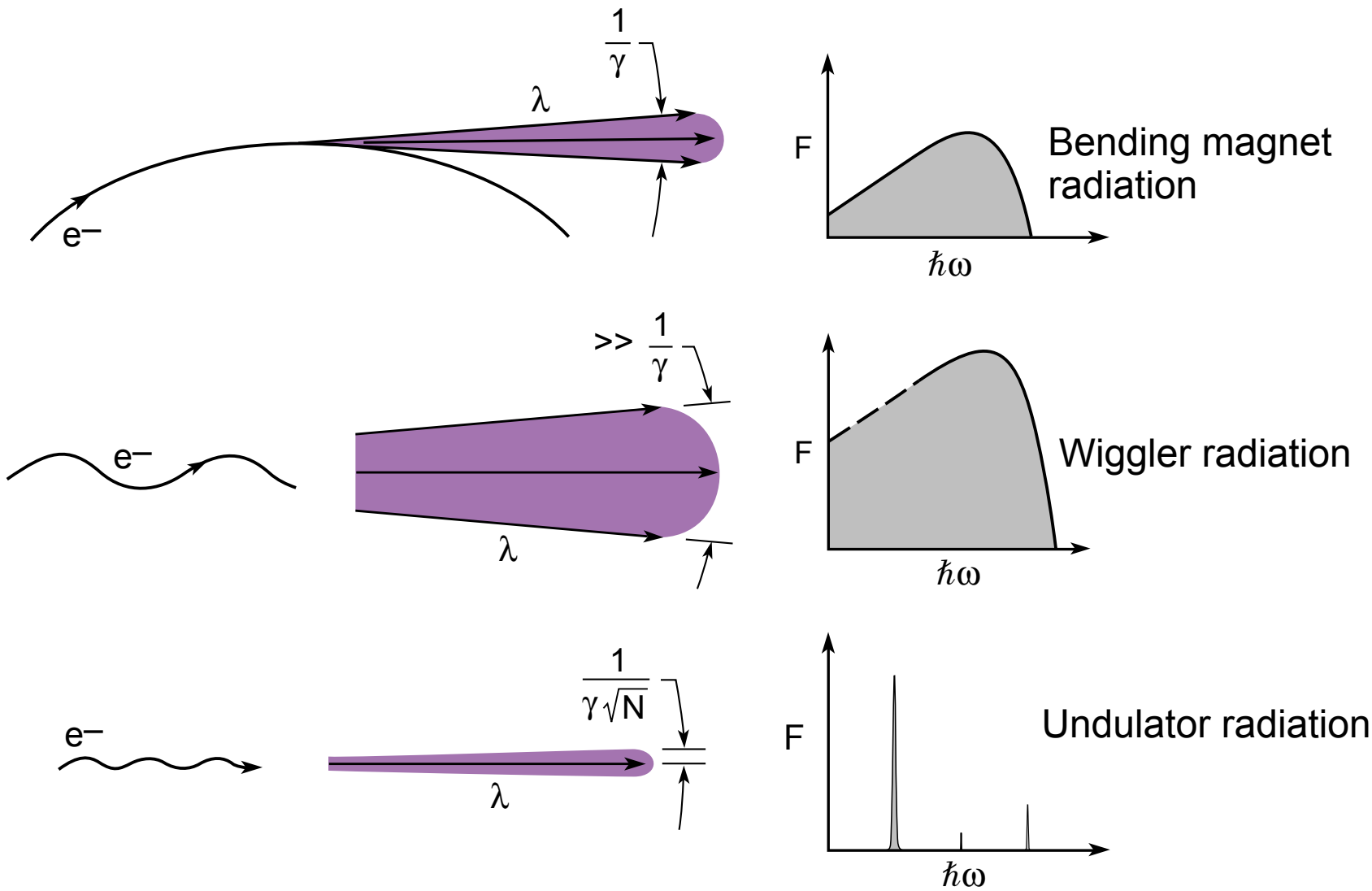
$$1 \text{ watt} \Rightarrow 5.034 \times 10^{15} \lambda[\text{nm}] \, \frac{\text{photons}}{\text{s}}$$

Bending Magnet: $E_c = \dfrac{3e\hbar B\gamma^2}{2m}$, $E_c(\mathrm{keV}) = 0.6650 E_e^2(\mathrm{GeV}) B(\mathrm{T})$

Undulator: $\lambda = \dfrac{\lambda_u}{2\gamma^2}\left(1 + \dfrac{K^2}{2} + \gamma^2\theta^2\right)$; $E(\mathrm{keV}) = \dfrac{0.9496 E_e^2(\mathrm{GeV})}{\lambda_u(\mathrm{cm})\left(1 + \frac{K^2}{2} + \gamma^2\theta^2\right)}$

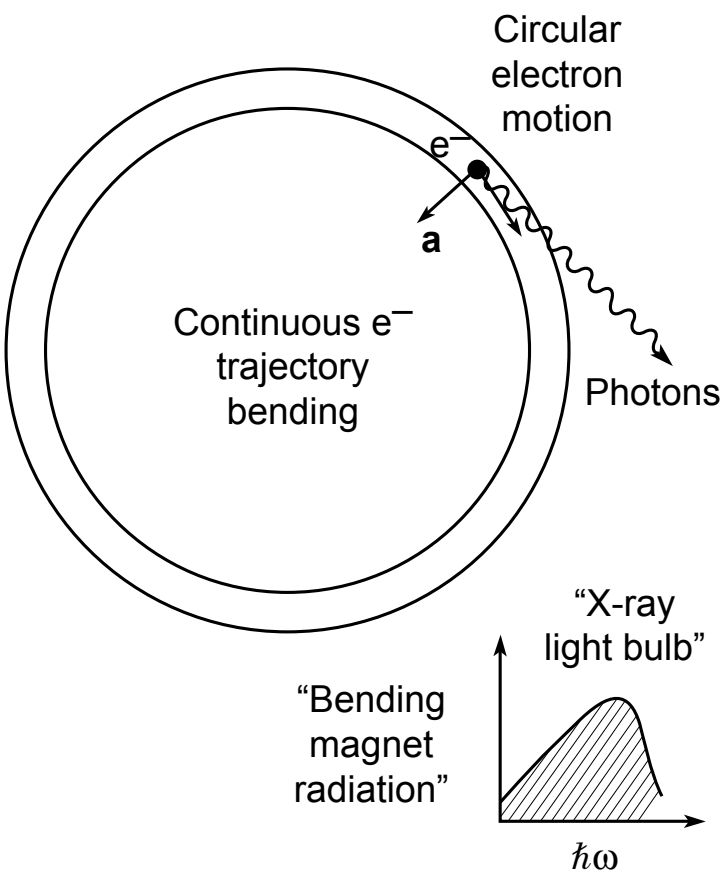
where $K \equiv \dfrac{eB_0\lambda_u}{2\pi mc} = 0.9337 B_0(\mathrm{T})\lambda_u(\mathrm{cm})$

Three Forms of Synchrotron Radiation

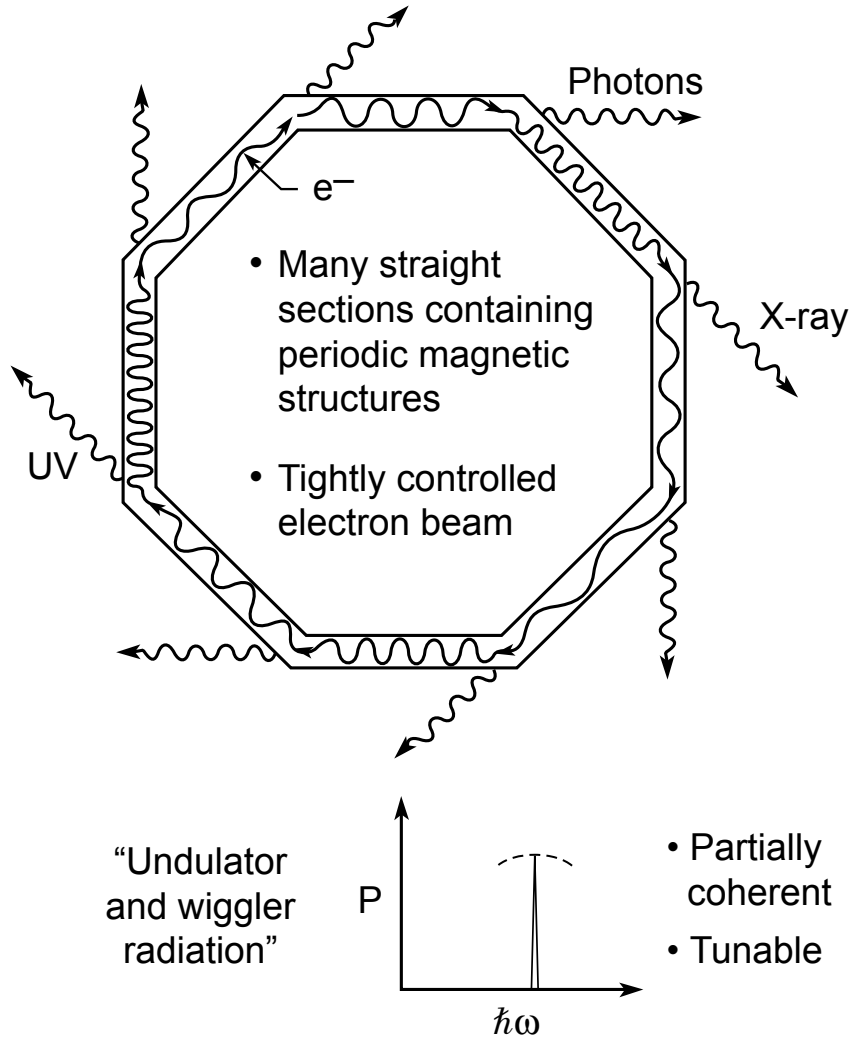


Modern Synchrotron Radiation Facility

Older Synchrotron Radiation Facility



Modern Synchrotron Radiation Facility



The ALS with San Francisco in the Background

1.9 GeV, γ = 3720, 197 m circumference

France's ESRF is Well Situated

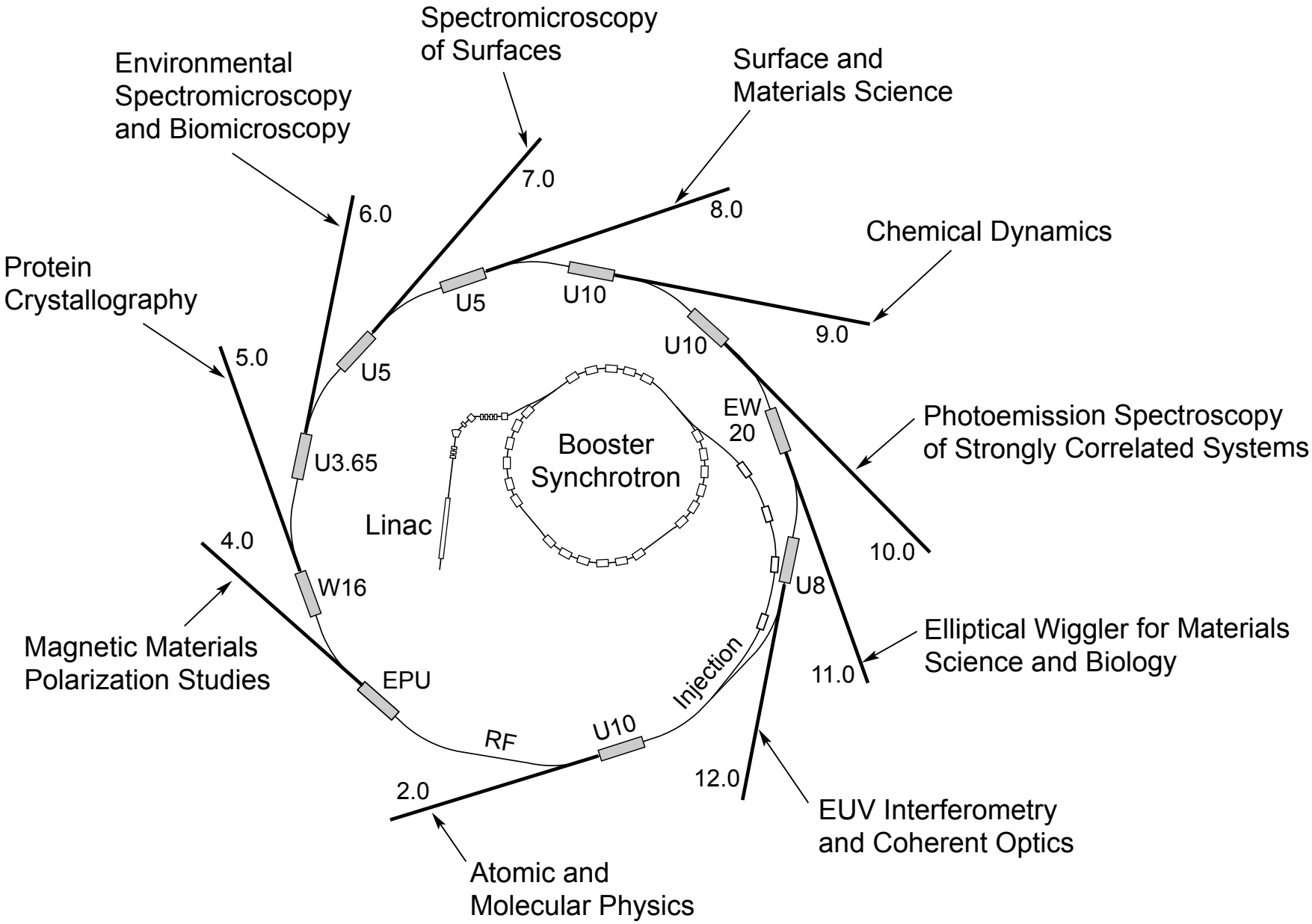
6 GeV; γ = 11, 800; 884m circumference

Bounded by two rivers

SPring-8 in Hyogo Prefecture, Japan

8 GeV; γ = 15,700; 1.44 km circumference

A Single Storage Ring Serves Many Scientific User Groups



Bending Magnet Radius

The Lorentz force for a relativistic electron in a constant magnetic field is

$$\mathbf{F} = \frac{d\mathbf{p}}{dt} = -e\mathbf{v} \times \mathbf{B}$$

where $\mathbf{p} = \gamma m\mathbf{v}$. In a fixed magnetic field the rate of change of electron energy is

$$\frac{dE_e}{dt} = \mathbf{v} \cdot \mathbf{F} = \underbrace{-e\mathbf{v} \cdot (\mathbf{v} \times \mathbf{B})}_{\equiv 0}$$

thus with $E_e = \gamma mc^2$

$$\frac{dE_e}{dt} = \frac{d}{dt}(\gamma mc^2) = 0$$

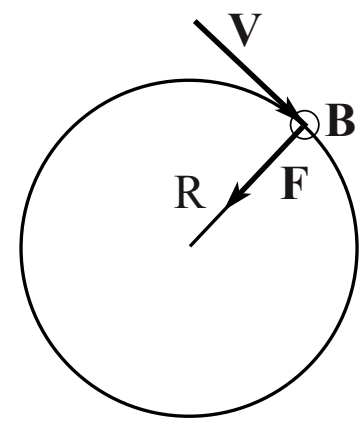
$$\therefore \gamma = \text{constant}$$

and the force equation becomes

$$\frac{d\mathbf{p}}{dt} = \gamma m \frac{d\mathbf{v}}{dt} = -e\mathbf{v} \times \mathbf{B}$$

$$\gamma m \left(-\frac{\mathrm{v}^2}{R}\right) = -e\mathrm{v}B$$

$$\therefore \quad \boxed{R = \frac{\gamma m\mathrm{v}}{eB} \simeq \frac{\gamma mc}{eB}}$$



$$\mathrm{v} = \beta c$$

$$\beta \to 1$$

$$a = \frac{-\mathrm{v}^2}{R}$$

Bending Magnet Radiation

$$2\,\Delta\tau = \tau_e - \tau_r$$

$$2\,\Delta\tau = \frac{\text{arc length}}{\text{v}} - \frac{\text{radiation path}}{c}$$

$$2\,\Delta\tau \simeq \frac{R\cdot 2\theta}{\text{v}} - \frac{2R\sin\theta}{c}$$

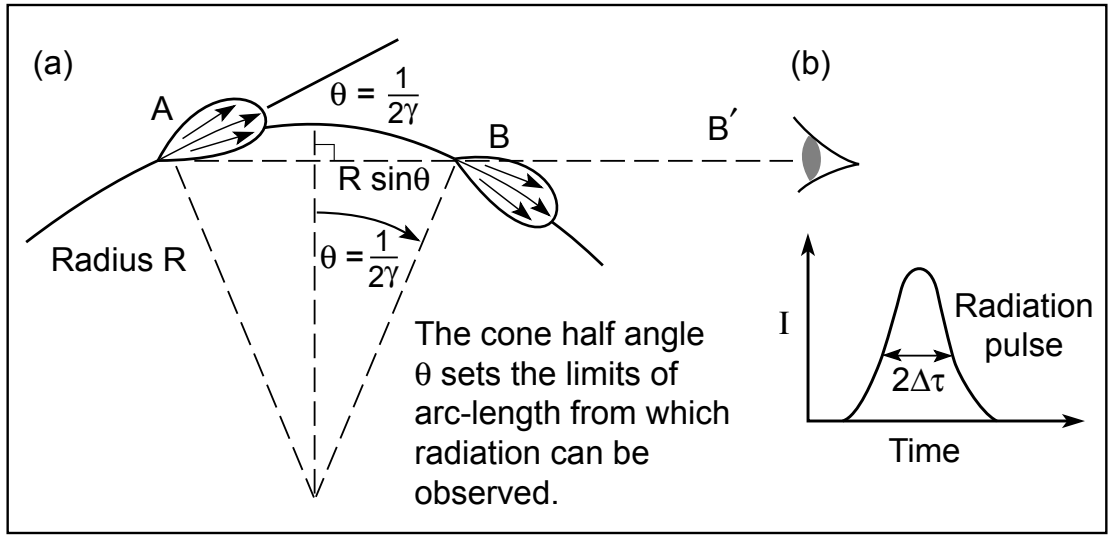
With $\theta \simeq 1/2\gamma$, $\sin\theta \simeq \theta$

$$2\,\Delta\tau \simeq \frac{R}{\gamma\text{v}} - \frac{R}{\gamma c} = \frac{R}{\gamma}\left(\frac{1}{\text{v}} - \frac{1}{c}\right)$$

With $\text{v} = \beta c$

$$2\,\Delta\tau \simeq \frac{R}{\gamma\beta c}(1-\beta) \quad \text{but} \quad (1-\beta) \simeq \frac{1}{2\gamma^2} \quad \text{and} \quad R \simeq \frac{\gamma mc}{eB}$$

$$\therefore\ 2\Delta\tau = \frac{m}{2eB\gamma^2}$$



Bending Magnet Radiation (continued)

From Heisenberg's Uncertainty Principle for rms pulse duration and photon energy

$$\Delta E \cdot \Delta\tau \geq \hbar/2$$

thus

$$\Delta E \geq \frac{\hbar}{2\Delta\tau} \qquad (5.4b)$$

$$\Delta E \geq \frac{\hbar}{m/2eB\gamma^2}$$

Thus the single-sided rms photon energy width (uncertainty) is

$$\Delta E \geq \frac{2e\hbar B\gamma^2}{m} \qquad (5.4c)$$

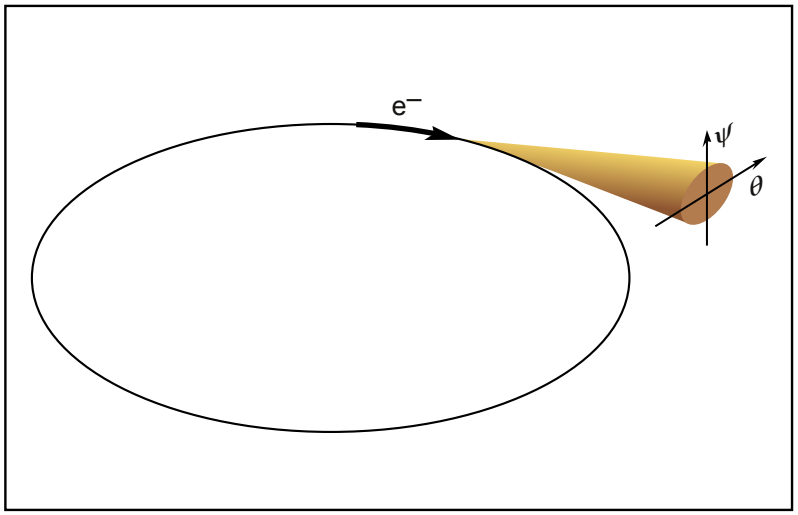
A more detailed description of bending magnet radius finds the critical photon energy

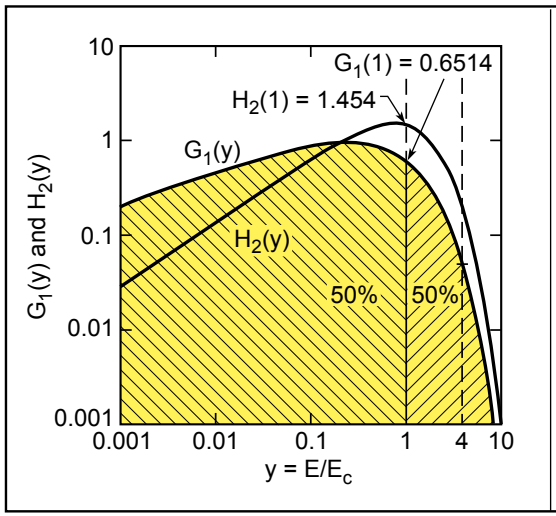
$$\boxed{E_c = \hbar\omega_c = \frac{3e\hbar B\gamma^2}{2m}} \qquad (5.7a)$$

In practical units the critical photon energy is

$$E_c(\text{keV}) = 0.6650 E_e^2(\text{GeV}) B(\text{T}) \qquad (5.7b)$$

Bending Magnet Radiation





y	$G_1(y)$	$H_2(y)$
0.0010	2.131×10^{-1}	2.910×10^{-2}
0.0100	4.450×10^{-1}	1.348×10^{-1}
0.1000	8.182×10^{-1}	6.025×10^{-1}
0.3000	9.177×10^{-1}	1.111×10^{0}
0.5000	8.708×10^{-1}	1.356×10^{0}
0.7000	7.879×10^{-1}	1.458×10^{0}
1.000	6.514×10^{-1}	1.454×10^{0}
3.000	1.286×10^{-1}	5.195×10^{-1}
5.000	2.125×10^{-2}	1.131×10^{-1}
7.000	3.308×10^{-3}	2.107×10^{-2}
10.00	1.922×10^{-4}	1.478×10^{-3}

$$\boxed{E_c = \hbar\omega_c = \frac{3e\hbar B\gamma^2}{2m}} \tag{5.7a}$$

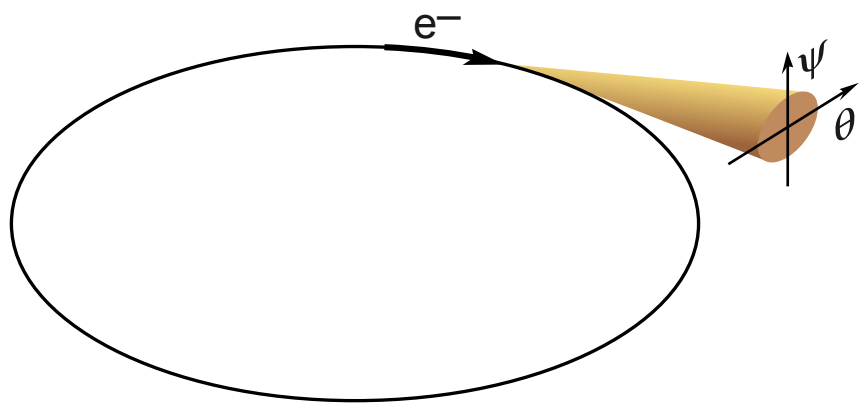
$$E_c(\text{keV}) = 0.6650 E_e^2(\text{GeV}) B(\text{T}) \tag{5.7b}$$

$$\gamma = \frac{E_e}{mc^2} = 1957 E_e(\text{GeV}) \tag{5.5}$$

$$\left.\frac{d^3 F_B}{d\theta\, d\psi\, d\omega/\omega}\right|_{\psi=0} = 1.33 \times 10^{13} E_e^2(\text{GeV}) I(\text{A}) H_2(E/E_c) \frac{\text{photons/s}}{\text{mrad}^2 \cdot (0.1\%\ \text{BW})} \tag{5.6}$$

$$\frac{d^2 F_B}{d\theta\, d\omega/\omega} = 2.46 \times 10^{13} E_e(\text{GeV}) I(\text{A}) G_1(E/E_c) \frac{\text{photons/s}}{\text{mrad} \cdot (0.1\%\text{BW})} \tag{5.8}$$

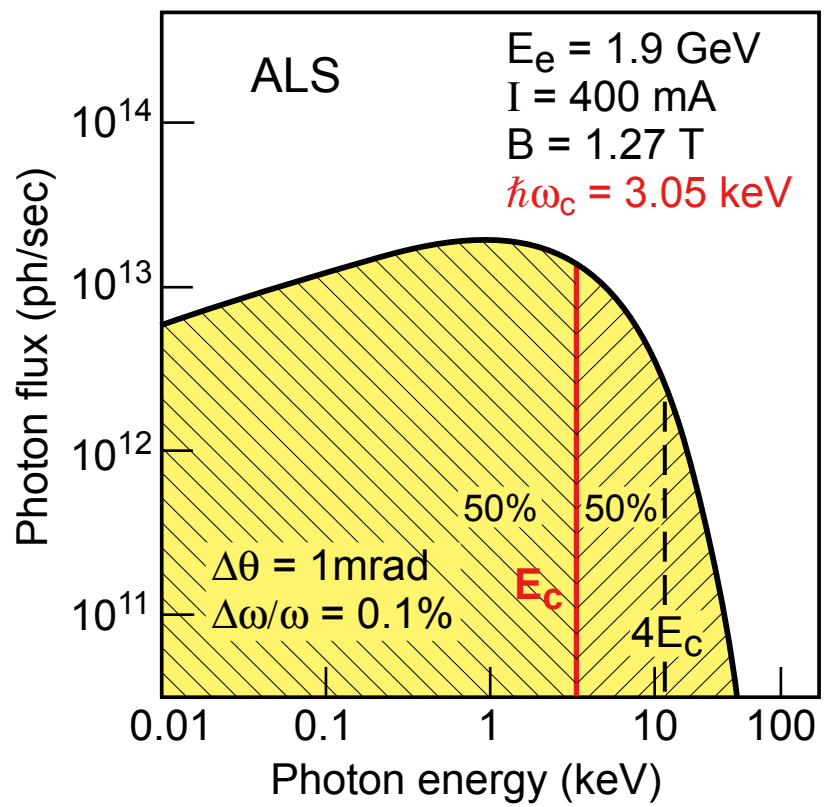
Bending Magnet Radiation Covers a Broad Region of the Spectrum, Including the Primary Absorption Edges of Most Elements



$$E_c = \hbar\omega_c = \frac{3e\hbar B\gamma^2}{2m} \qquad (5.7a)$$

$$E_c(\text{keV}) = 0.6650 E_e^2(\text{GeV}) B(\text{T}) \qquad (5.7b)$$

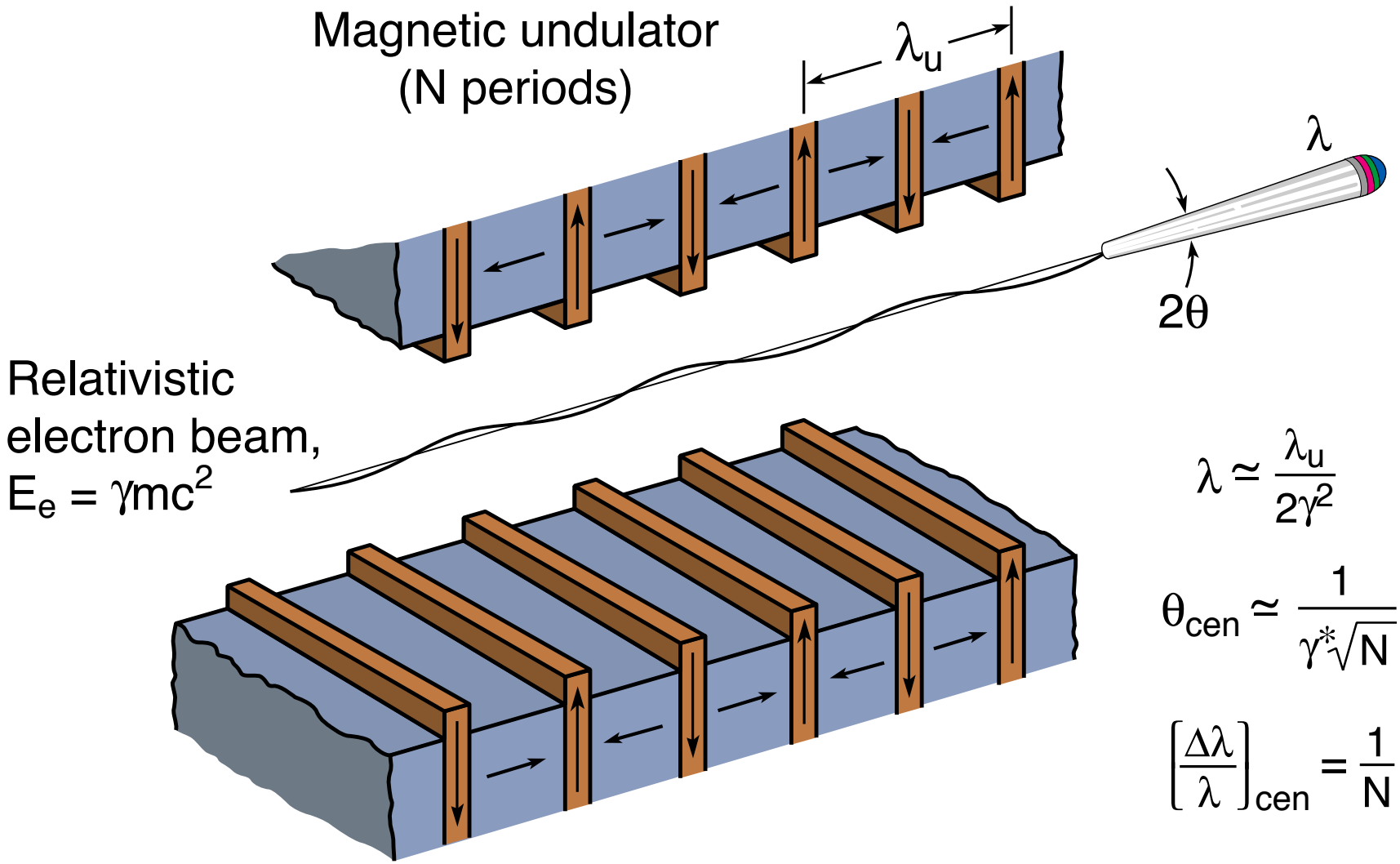
$$\frac{d^2 F_B}{d\theta\, d\omega/\omega} = 2.46 \times 10^{13} E_e(\text{GeV}) I(\text{A}) G_1(E/E_c) \frac{\text{photons/s}}{\text{mrad}\cdot(0.1\%\text{BW})} \qquad (5.8)$$



Advantages:
- covers broad spectral range
- least expensive
- most accessable

Disadvantages:
- limited coverage of hard x-rays
- not as bright as undulator

Narrow Cone Undulator Radiation, Generated by Relativistic Electrons Traversing a Periodic Magnet Structure

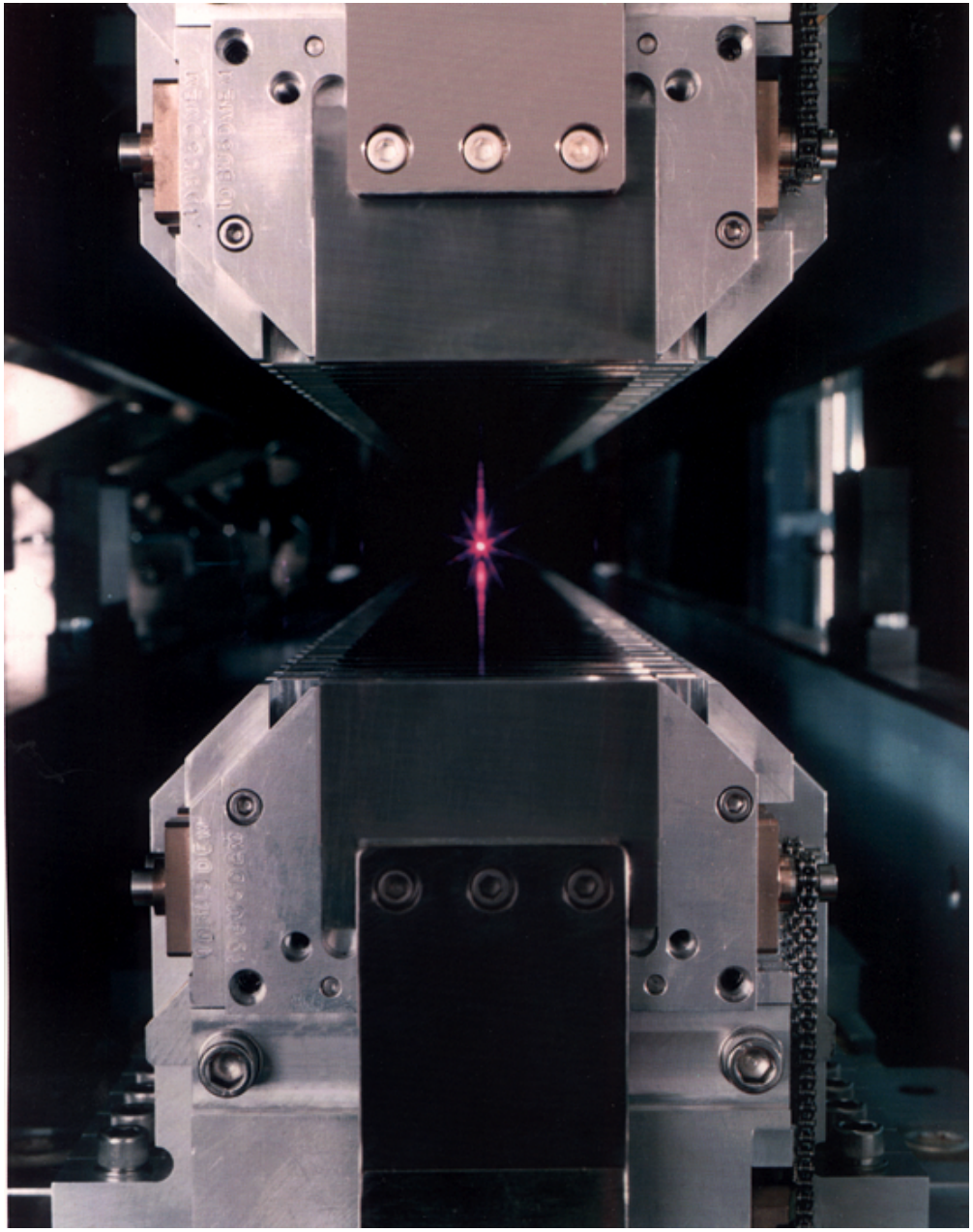


$$\lambda \simeq \frac{\lambda_u}{2\gamma^2}$$

$$\theta_{cen} \simeq \frac{1}{\gamma^* \sqrt{N}}$$

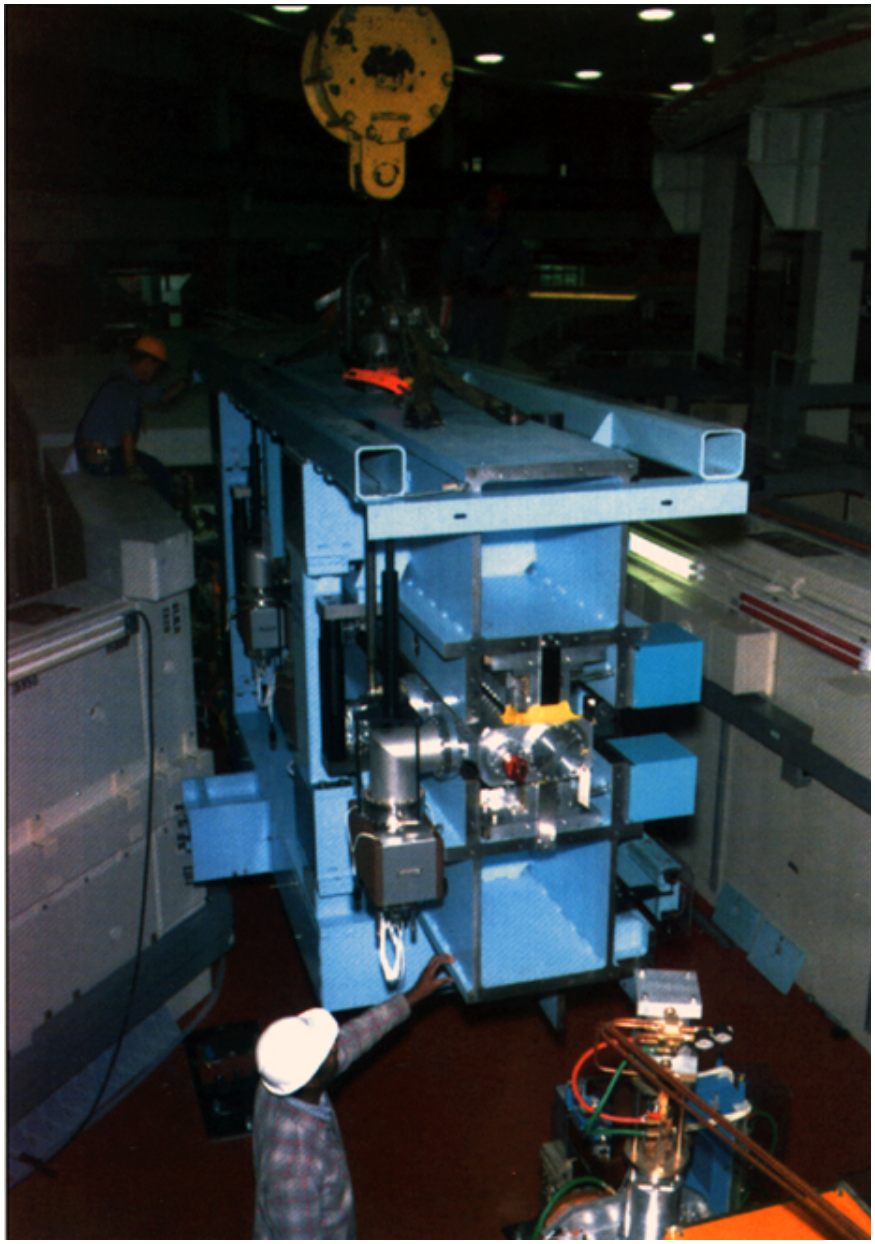
$$\left[\frac{\Delta\lambda}{\lambda}\right]_{cen} = \frac{1}{N}$$

An Undulator Up Close



ALS U5 undulator, beamline 7.0, N = 89, λ_u = 50 mm

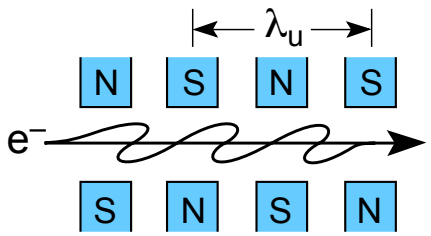
Installing an Undulator at Berkeley's Advanced Light Source



ALS Beamline 9.0 (May 1994), N = 55, λ_u = 80 mm

Undulator Radiation

Laboratory Frame of Reference

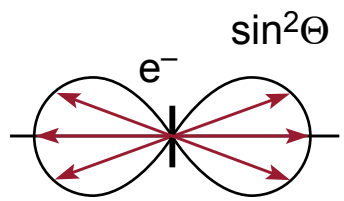


$E = \gamma mc^2$

$$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$$

N = # periods

Frame of Moving e^-



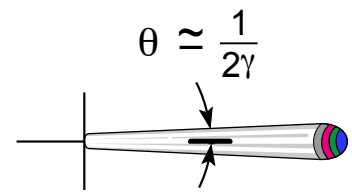
e^- radiates at the Lorentz contracted wavelength:

$$\lambda' = \frac{\lambda_u}{\gamma}$$

Bandwidth:

$$\frac{\lambda'}{\Delta\lambda'} \simeq N$$

Frame of Observer



Doppler shortened wavelength on axis:

$$\lambda = \lambda'\gamma(1 - \beta\cos\theta)$$

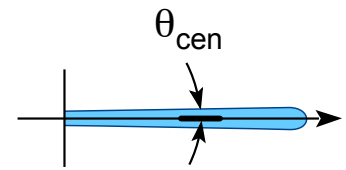
$$\lambda = \frac{\lambda_u}{2\gamma^2}(1 + \gamma^2\theta^2)$$

Accounting for transverse motion due to the periodic magnetic field:

$$\boxed{\lambda = \frac{\lambda_u}{2\gamma^2}\left(1 + \frac{K^2}{2} + \gamma^2\theta^2\right)}$$

where $K = eB_0\lambda_u / 2\pi mc$

Following Monochromator



For $\frac{\Delta\lambda}{\lambda} \simeq \frac{1}{N}$

$$\theta_{cen} \simeq \frac{1}{\gamma\sqrt{N}}$$

typically

$$\theta_{cen} \simeq 40\ \mu\text{rad}$$

Physically, where does the $\lambda = \lambda_u/2\gamma^2$ come from?

The electron "sees" a Lorentz contracted period

$$\lambda' = \frac{\lambda_u}{\gamma} \tag{5.9}$$

and emits radiation in its frame of reference at frequency

$$f' = \frac{c}{\lambda'} = \frac{c\gamma}{\lambda_u}$$

Observed in the laboratory frame of reference, this radiation is Doppler shifted to a frequency

$$f = \frac{f'}{\gamma(1 - \beta\cos\theta)} = \frac{c}{\lambda_u(1 - \beta\cos\theta)} \tag{5.10}$$

On-axis ($\theta = 0$) the observed frequency is

$$f = \frac{c}{\lambda_u(1 - \beta)}$$

Physically, where does the $\lambda = \lambda_u/2\gamma^2$ come from?

$$f = \frac{c}{\lambda_u(1-\beta)}$$

By definition $\gamma = \frac{1}{\sqrt{1-\beta^2}}$; $\gamma^2 = \frac{1}{(1-\beta)(1+\beta)} \simeq \frac{1}{2(1-\beta)}$

thus

$$f = \frac{2\gamma^2 c}{\lambda_u}$$

and the observed wavelength is

$$\lambda = \frac{c}{f} = \frac{\lambda_u}{2\gamma^2} \qquad (5.11)$$

Give examples.

What about the off-axis $\theta \neq 0$ radiation?

For $\theta \neq 0$, take $\cos\theta = 1 - \frac{\theta^2}{2} + \ldots$, then

$$f = \frac{c}{\lambda_u(1 - \beta\cos\theta)} \qquad (5.10)$$

$$f = \frac{c/\lambda_u}{1 - \beta\,(1 - \theta^2/2 + \ldots)} = \frac{c/\lambda_u}{1 - \beta + \beta\theta^2/2 - \ldots} = \frac{c/(1-\beta)\lambda_u}{1 + \beta\theta^2/2(1-\beta)\ldots}$$

$$f = \frac{2\gamma^2 c/\lambda_u}{1 + \gamma^2\theta^2} = \frac{2\gamma^2 c}{\lambda_u(1 + \gamma^2\theta^2)}$$

The observed wavelength is then

$$\lambda = \frac{\lambda_u}{2\gamma^2}(1 + \gamma^2\theta^2) \qquad (5.12)$$

exhibiting a reduced Doppler shift off-axis, i.e., longer wavelengths.
This is a simplified version of the "Undulator Equation".

The Undulator's "Central Radiation Cone"

With electrons executing N oscillations as they traverse the periodic magnet structure, and thus radiating a wavetrain of N cycles, it is of interest to know what angular cone contains radiation of relative spectral bandwidth

$$\frac{\Delta\lambda}{\lambda} = \frac{1}{N} \qquad (5.14)$$

Write the undulator equation twice, once for on-axis radiation ($\theta = 0$) and once for wavelength-shifted radiation off-axis at angle θ:

$$\lambda_0 + \Delta\lambda = \frac{\lambda_u}{2\gamma^2}(1 + \gamma^2\theta^2)$$

$$\lambda_0 = \frac{\lambda_u}{2\gamma^2}$$

divide and simplify to

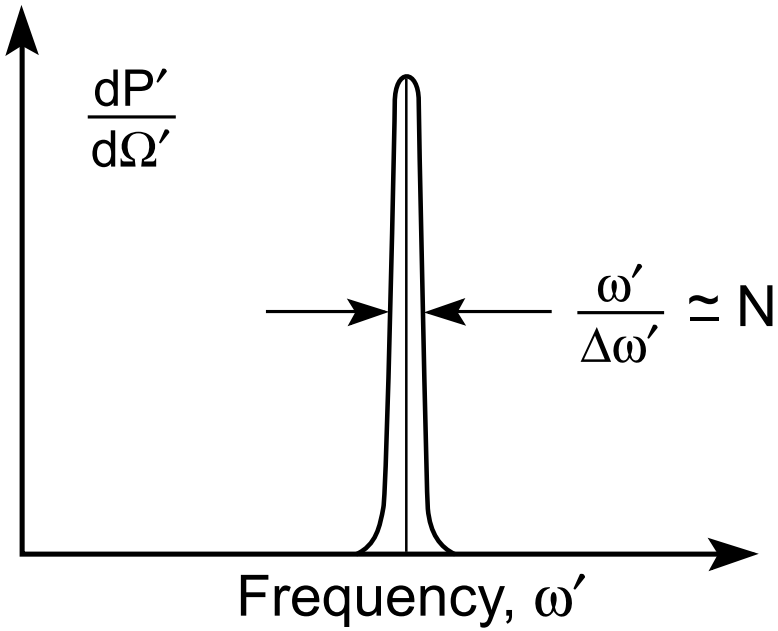
$$\frac{\Delta\lambda}{\lambda} \simeq \gamma^2\theta^2 \qquad (5.13)$$

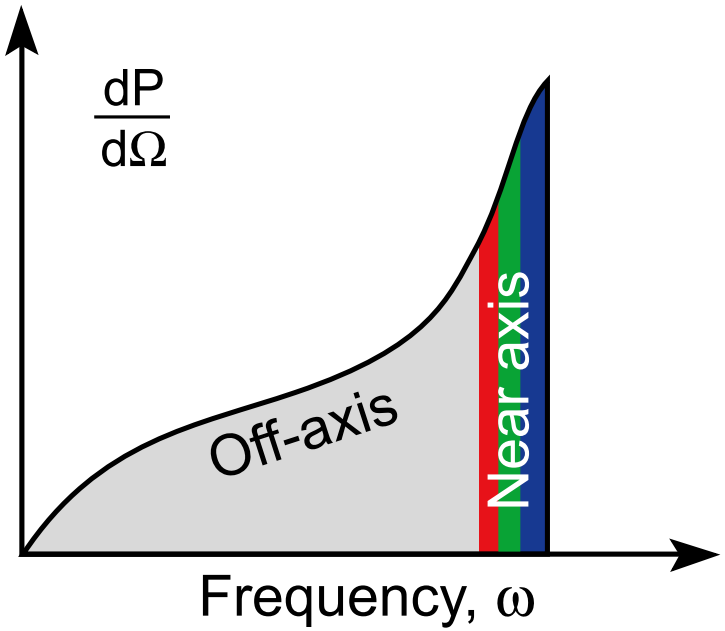
Combining the two equations (5.13 and 5.14)

defines θ_{cen} : $\gamma^2\theta_{cen}^2 \equiv \frac{1}{N}$, which gives $\boxed{\theta_{cen} \simeq \frac{1}{\gamma\sqrt{N}}}$ (5.15)

This is the half-angle of the "central radiation cone", defined as containing radiation of $\Delta\lambda/\lambda = 1/N$.

The Undulator Radiation Spectrum in Two Frames of Reference



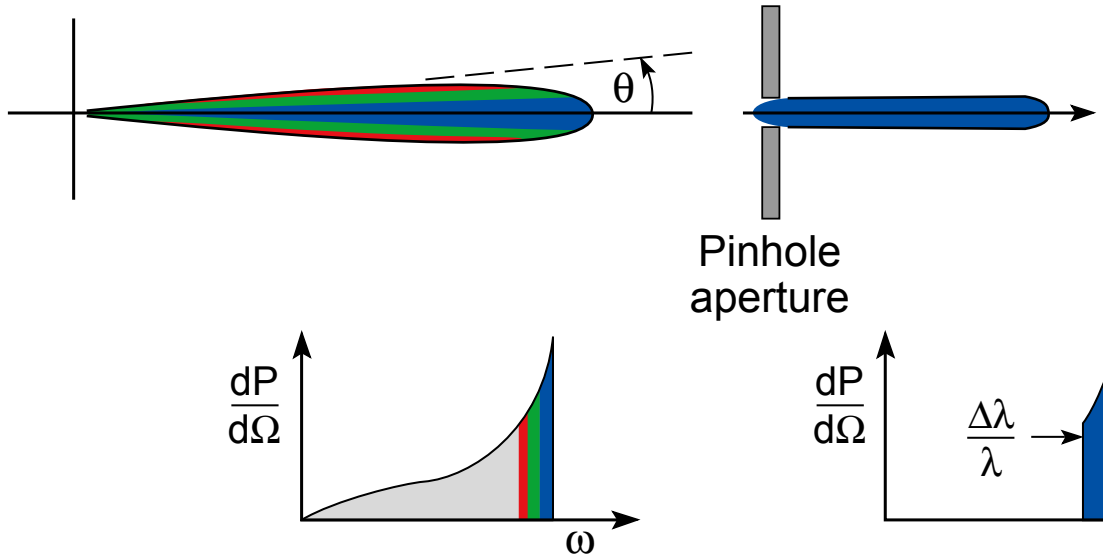


Execution of N electron oscillations produces a transform-limited spectral bandwidth, $\Delta\omega'/\omega' = 1/N$.

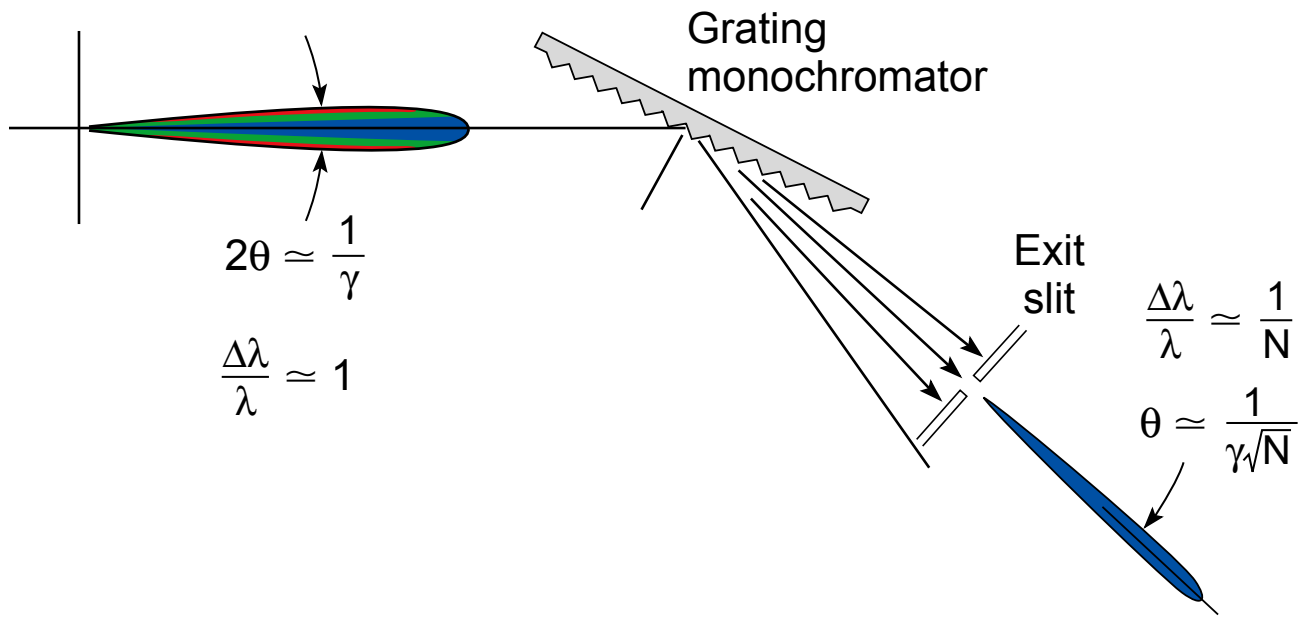
The Doppler frequency shift has a strong angle dependence, leading to lower photon energies off-axis.

The Narrow (1/N) Spectral Bandwidth of Undulator Radiation Can be Recovered in Two Ways

With a pinhole aperture



With a monochromator

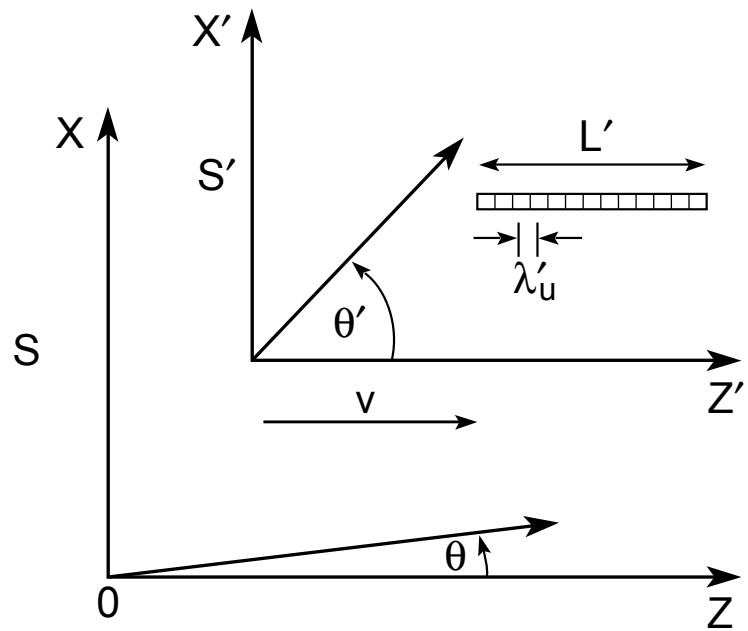


Lorentz Space-Time Transformations (Appendix F)

$$z = \gamma(z' + \beta c t') \quad \text{(F.1a)}$$

$$t = \gamma\left(t' + \frac{\beta z'}{c}\right) \quad \text{(F.1b)}$$

$$y = y' \quad \text{and} \quad x = x' \quad \text{(F.1c)}$$



$$z' = \gamma(z - \beta c t) \quad \text{(F.2a)}$$

$$t' = \gamma\left(t - \frac{\beta z}{c}\right) \quad \text{(F.2a)}$$

$$y' = y \quad \text{and} \quad x' = x \quad \text{(F.2a)}$$

$$\beta \equiv \frac{v}{c} \quad \text{(F.3)}$$

$$\gamma \equiv \frac{1}{\sqrt{1 - \beta^2}} \quad \text{(F.4)}$$

Lorentz Transformations: Frequency, Angles, Length and Time

Doppler frequency shifts

$$\omega = \omega' \gamma (1 + \beta \cos\theta') \qquad \text{(F.8a)}$$

$$\omega' = \omega \gamma (1 - \beta \cos\theta) \qquad \text{(F.8b)}$$

Lorentz contraction of length

$$L' = L/\gamma \qquad \text{(F.12)}$$

Time dilation

$$\Delta t' = \Delta t/\gamma \qquad \text{(F.13)}$$

Angular transformations

$$\cos\theta = \frac{\cos\theta' + \beta}{1 + \beta\cos\theta'} \qquad \text{(F.9a)}$$

$$\cos\theta' = \frac{\cos\theta - \beta}{1 - \beta\cos\theta} \qquad \text{(F.9b)}$$

$$\sin\theta = \frac{\sin\theta'}{\gamma(1 + \beta\cos\theta')} \qquad \text{(F.10b)}$$

$$\sin\theta' = \frac{\sin\theta}{\gamma(1 - \beta\cos\theta)} \qquad \text{(F.10a)}$$

$$\tan\theta = \frac{\sin\theta'}{\gamma(\cos\theta' + \beta)} \qquad \text{(F.11a)}$$

$$\tan\theta' = \frac{\sin\theta}{\gamma(\cos\theta - \beta)} \qquad \text{(F.11b)}$$