**Jordan canonical form** 

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Let $A$ be any complex $n \times n$ matrix. The expression $\det(sI - A)$ is a polynomial in $s$ of degree $n$. It is called the *characteristic polynomial* of $A$. It is a *monic* polynomial, i.e. the coefficient of the highest degree term ($s^n$) is 1.

By the fundamental theorem of algebra, every polynomial with complex coefficients factors into degree one factors with complex coefficients. If the polynomial is monic, so are the factors, i.e each factor is of the form $(s - \lambda)$ for some complex number $\lambda$. These complex numbers are the *roots* of the polynomial. The roots of the characteristic polynomial of a matrix are called its *eigenvalues.*

In general, a polynomial can have repeated roots. Suppose $\det(sI - A)$ has $d$ distinct roots $\lambda_1, \lambda_2, \ldots, \lambda_d$ having respective multiplicities $a_1, a_2, \ldots, a_d$, i.e.

$$\det(sI - A) = \prod_{i=1}^{d} (s - \lambda_i)^{a_i} \ .$$

$a_i$ is called the *algebraic multiplicity* of the eigenvalue $\lambda_i$. We have $\sum_{i=1}^{d} a_i = n$.

The *eigenspace* of $A$ corresponding to the eigenvalue $\lambda_i$ is the null space of the matrix $\lambda_i I - A$, denoted $\mathcal{N}(\lambda_i I - A)$. It is nontrivial, i.e. it has dimension at least 1, as can be checked by realizing that $\lambda_i I - A$ is a *singular* complex matrix, i.e. has determinant equal to zero, which in turn is equivalent to $\lambda_i$ being a root of the characteristic polynomial of $A$, i.e. being an eigenvalue of $A$. The dimension of the null space of $\lambda_i I - A$, denoted $g_i$, is called the *geometric multiplicity* of the eigenvalue $\lambda_i$.

This handout will mention several facts without proof. Some of these facts may be new to you. If so, you may want to look up their proofs. You will find good treatments of this in most books on linear algebra (e.g. Strang) or abstract algebra (e.g. Hungerford or Jacobson). Another way to get intuition is to play around with several examples. You can do this quite easily by using MATLAB to do the calculations for you.

In general, for each eigenvalue $\lambda_i$, we have $g_i \leq a_i$, i.e. the dimension of the eigenspace of the eigenvalue at most equals its multiplicity as a root of the characteristic polynomial, and the inequality can be strict. One has

$$\mathcal{N}(\lambda_i I - A) \subseteq \mathcal{N}((\lambda_i I - A)^2) \subseteq \ldots$$

Strict inclusion will hold till a certain stage, call it $m_i$, after which all the subspaces in this sequence will be equal, i.e. $m_i$ is defined by

$$\mathcal{N}((\lambda_i I - A)^{m_i - 1}) \overset{\subset}{\neq} \mathcal{N}((\lambda_i I - A)^{m_i}) = \mathcal{N}((\lambda_i I - A)^{m_i + 1}) = \mathcal{N}((\lambda_i I - A)^{m_i + 2}) = \ldots$$

The dimension of $\mathcal{N}((\lambda_i I - A)^{m_i})$ will equal $a_i$, the algebraic multiplicity of $\lambda_i$. $\mathcal{N}((\lambda_i I - A)^{m_i})$ is called the *generalized eigenspace* of the eigenvalue $\lambda_i$. Note that the generalized eigenspace of an eigenvalue equals its eigenspace precisely when the geometric multiplicity of the eigenvalue equals its algebraic multiplicity. In particular, this is always true if the algebraic multiplicity of the eigenvalue is 1, i.e., if it is a simple root of the characteristic polynomial. However, this condition is not necessary. For instance, each eigenvalue of a symmetric matrix has its geometric multiplicity equal to its algebraic multiplicity (i.e. the dimension of its eigenspace equals its multiplicity as a root of the characteristic polynomial) irrespective of what its algebraic multiplicity is.

Every matrix $A$ can be put in *Jordan canonical form* by a similarity transformation (change of basis). In fact, one can choose a basis $b_{i1}, \ldots, b_{ia_i}$ for each generalized eigenspace $\mathcal{N}((\lambda_i I - A)^{m_i})$, $1 \le i \le d$, such that if $U$ denotes the matrix

$$U = [b_{11} \ldots b_{1a_1} b_{21} \ldots b_{2a_2} \ldots b_{d1} \ldots b_{da_d}]$$

then one has $U^{-1}AU = J$, where $J$ has the block diagonal form

$$J = \mathrm{diag}(J_{11}, \ldots, J_{1g_1}, J_{21}, \ldots, J_{2g_2}, \ldots, J_{d1}, \ldots, J_{dg_d}) \ .$$

Here each $J_{ij}$ for $1 \le j \le g_i$ is an $m_{ij} \times m_{ij}$ matrix of the form

$$J_{ij} = \begin{bmatrix} \lambda_i & 1 & 0 & \ldots & 0 \\ 0 & \lambda_i & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & 1 \\ 0 & 0 & 0 & \ldots & \lambda_i \end{bmatrix} .$$

Note that, for each $1 \le i \le d$,

$$a_i = \sum_{j=1}^{g_i} m_{ij} \ .$$

For each $1 \le i \le d$, one has $\max_j m_{ij} = m_i$, where $m_i$ was defined above. The dimension of the eigenspace $\mathcal{N}(\lambda_i I - A)$ equals $\sum_{j=1}^{g_i} 1(m_{ij} \ge 1) = \sum_{j=1}^{g_i} 1 = g_i$, the dimension of $\mathcal{N}((\lambda_i I - A)^2)$ equals $\sum_{j=1}^{g_i} 1(m_{ij} \ge 1) + \sum_{j=1}^{g_i} 1(m_{ij} \ge 2)$ and so on, with the dimension of $\mathcal{N}((\lambda_i I - A)^{m_i})$ equaling

$$\sum_{j=1}^{g_i} 1(m_{ij} \ge 1) + \ldots + \sum_{j=1}^{g_i} 1(m_{ij} \ge m_i) = \sum_{j=1}^{g_i} m_{ij} = a_i \ .$$

Further, the columns $b_{i1}, b_{i(1+m_{i1})}, \ldots, b_{i(1+m_{i1}+\ldots m_{i(g_i-1)})}$ (of which there are exactly $g_i$) form a basis for the eigenspace of $\lambda_i$.

Note that if we write

$$U^{-1} = \left[c_{11}^T \ldots c_{1a_1}^T c_{21}^T \ldots c_{2a_2}^T \ldots c_{d1}^T \ldots c_{da_d}^T\right]^T$$

then the rows $c_{i1}, \ldots, c_{ia_i}$ are a basis for the left generalized eigenspace corresponding to the eigenvalue $\lambda_i$, and $c_{im_{i1}}, c_{i(m_{i1}+m_{i2})}, \ldots, c_{i(m_{i1}+\ldots m_{ig_i})}$ are a basis for the left eigenspace corresponding to the eigenvalue $\lambda_i$.

Since $A = UJU^{-1}$ we can now calculate the powers of $A$ quite easily : $A^n = UJ^nU^{-1}$ and $J^n = \mathrm{diag}(J^n_{11}, \ldots, J^n_{1g_1}, J^n_{21}, \ldots, J^n_{2g_2}, \ldots, J^n_{d1}, \ldots, J^n_{dg_d})$ . Here

$$J^n_{ij} = \begin{bmatrix} \lambda_i^n & \binom{n}{1}\lambda_i^{n-1} & \binom{n}{2}\lambda_i^{n-2} & \ldots & \binom{n}{m_{ij}-1}\lambda_i^{n-m_{ij}+1} \\ 0 & \lambda_i^n & \binom{n}{1}\lambda_i^{n-1} & \ldots & \binom{n}{m_{ij}-2}\lambda_i^{n-m_{ij}+2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & \lambda_i^n \end{bmatrix} .$$

Check this !!