Homework 4: Due in class on Thursday 4/12

1. An individual has initial capital S_0. At the beginning of time period k, $k \geq 0$, the individual has capital S_k. This must be allocated, consuming C_k, investing I_k at a sure rate that will return rI_k by the beginning of the next period, and investing J_k in a risky venture that will return $Z_k J_k$ by the end of the next period, where Z_k is a nonnegative random variable having some distribution $P(Z_k \in dz) = F(dz)$, and the $(Z_k, k \geq 0)$ are i.i.d.. We must have $S_k = C_k + I_k + J_k$, and we have $S_{k+1} = (S_k - C_k) + rI_k + Z_k J_k$, according to the preceding description. The utility of consumption is described by a function $R(c), c \geq 0$. The objective is to maximize $E[\sum_{k=0}^{\infty} \alpha^k R(C_k)]$, where $0 < \alpha < 1$ is a given discount factor.

   (a) Set the problem up as a discounted dynamic programming problem and write down the corresponding Bellman equation.

   (b) If $R(c) = c^{\beta}$, where $0 < \beta < 1$, show that the optimal policy allocates a fixed fraction of one's current fortune to each of the three alternatives. Also show in this case that $J^*(s) = KR(s)$ for some fixed constant K, where $J^*(s)$ denotes the optimal overall discounted utility when the initial capital is s.

   (c) Suppose that $R(c)$ is a concave function of c. Suppose $E[Z_k] < r$. How much of the current capital will an optimal stationary Markov strategy allocate to the risky asset?

2. Let μ_1 and μ_2 define stationary Markov policies in a finite state finite control space discounted dynamic programming problem with one step costs $g(i,u)$ and state transition probabilities $p_{ij}(u)$. Thus μ_1 and μ_2 are functions from the state space $\mathcal{X}$ to the set of controls $\mathcal{U}$.

   (a) Let μ_3 denote a stationary Markov policy that chooses actions to minimize

   $$g(i,u) + \alpha \sum_j p_{ij}(u) \min\{J^*_{\mu_1}(j), J^*_{\mu_2}(j)\} ,$$

   where J^*_{μ} denotes the optimal overall discounted cost when the stationary Markov control strategy μ is in effect. Show that

   $$J^*_{\mu_3} \leq \min\{J^*_{\mu_1}, J^*_{\mu_2}\} ,$$

   (the inequality is meant to hold state by state, as usual).

   (b) Let μ_4 be defined by

   $$\mu_4(i) = \begin{cases} \mu_1(i) & \text{if } J^*_{\mu_1}(i) \leq J^*_{\mu_2}(i) \\ \mu_2(i) & \text{if } J^*_{\mu_2}(i) < J^*_{\mu_1}(i) \end{cases} , \quad .$$

   Show that

   $$J^*_{\mu_4} \leq \min\{J^*_{\mu_1}, J^*_{\mu_2}\} .$$

3. Problem 7.7 on pg. 447 of Vol. 1 of the text. (Read Example 7.3.2. on pp. 420 -421 of Vol. 1 of the text.)
4. Problem 1.9 on pp. 78 -79 of Vol. 2 of the text.
5. Problem 1.15 on pg. 82 of Vol. 2 of the text.