EE223: Stochastic Systems: Estimation and Control. **SP'07**

Lecture 15 — March 6

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15.1 Today's topics

We continue our discussion in the partially observed finite horizon linear quadratic setting. Recall that we are dealing with a newly defined state at time k, which is the conditional law of the old state.

15.2 Review

We define a new state at time k, which is the conditional law of the old state X_k given $I_k^\Psi = (Y_0^\Psi, \cdots, Y_k^\Psi, U_0^\Psi, \cdots, U_{k-1}^\Psi)$. Then the cost-to-go at time N is

$$J_N(\lambda) = E_\lambda[X^T Q_N X] = \int_{\Re^n} (x^T Q_N x)\lambda(dx) \tag{15.1}$$

where λ is the conditional law at time N and X here is considered as a dummy variable having the distribution λ.

In the previous lecture we evaluated $J_{N-1}(\lambda)$ as follows:

- Take X in $\Re^n$ with distribution λ;
- Let $\tilde{X} = A_{N-1}X + B_{N-1}u + w_{N-1}$ (for fixed u);
- Let $Y = C_N\tilde{X} + v_N$.

Then we have:

$$J_{N-1}(\lambda) = \min_u \left[E_\lambda[X^T Q_{N-1} X] + u^T R_{N-1} u + E[J_N(T_{N-1,N}(\lambda, u, Y))]\right], \tag{15.2}$$

where $T_{N-1,N}(\lambda, u, y)$ is the conditional law of the state at time N given that we used control u and that we observed $Y = y$. Note that the expectation in the third term inside the minimum is evaluated over Y, since the observation itself is random. Further, the distribution of this observation Y itself depends on λ.

In the previous lecture, we saw that:

$$J_{N-1}(\lambda) = E_\lambda[X^T K_{N-1} X] + E_{\lambda^0}[X^T \Gamma_{N-1} X] + E[w_{N-1}^T K_N w_{N-1}], \tag{15.3}$$

and that the associated optimal control is:

$$u_{N-1} = L_{N-1} m_{N-1}, \tag{15.4}$$

where $m_{N-1} = E_\lambda[X]$. Recall that in this equation λ^0 denotes the centered version of λ. Today, we will compute $J_{N-2}(\lambda)$. For convenience, we designate the respective terms in the optimal cost-to-go at time $N-1$ as ①,②, ③in order.

To find $J_{N-2}(\lambda)$,

- Start with X in $\Re^n$ with distribution λ;
- Let $\tilde{X} = A_{N-2}X + B_{N-2}u + w_{N-2}$ (for fixed u);
- Let $Y = C_{N-1}\tilde{X} + v_{N-1}$.

Then we have:

$$J_{N-2}(\lambda) = \min_u \left[E_\lambda[X^T Q_{N-2} X] + u^T R_{N-2} u + E[J_{N-1}(T_{N-2,N-1}(\lambda, u, Y))] \right]. \tag{15.5}$$

We now observe that:

$$E_\lambda[X^T Q_{N-2} X] = m^T Q_{N-2} m + E_{\lambda^0}[X^T Q_{N-2} X] \ , \tag{15.6}$$

where $m = E_\lambda[X]$. Further,

$$J_{N-1}(T_{N-2,N-1}(\lambda, u, Y)) \tag{15.7}$$

is the sum of three terms, ①, ②, and ③. Term ③is just an additive constant term and does not affect the calculation of the optimal control. Term ①is

$$\begin{aligned} & E[E_{T_{N-2,N-1}(\lambda,u,Y)}[X^T K_{N-1} X]] && (15.8)\\ = \ & E[\int x^T K_{N-1} x \mathbb{P}(\tilde{X} \in dx | Y)] && (15.9)\\ = \ & \int x^T K_{N-1} x \mathbb{P}(\tilde{X} \in dx) && (15.10)\\ = \ & E[\tilde{X} K_{N-1} \tilde{X}] && (15.11)\\ = \ & (A_{N-2}m + B_{N-2}u)^T K_{N-1} (A_{N-2}m + B_{N-2}u) && (15.12)\\ & + E_{\lambda^0}[(A_{N-2}X)^T K_{N-1} (A_{N-2}X)] + E[w_{N-2}^T K_{N-1} w_{N-2}] \ . && (15.13) \end{aligned}$$

This calculation is similar to one done in the previous lecture.

Term ②is

$$E[E_{T^0_{N-2,N-1}(\lambda,u,Y)}[X^T \Gamma_{N-1} X]] \ , \tag{15.14}$$

where $T^0_{N-2,N-1}(\lambda, u, Y)$ denotes the centered version of distribution $T_{N-2,N-1}(\lambda, u, Y)$.

Note that

$$T^0(\lambda, u, y) = T^0(\lambda^0, 0, y - C_{N-1}(A_{N-2}m + B_{N-2}u)) \tag{15.15}$$

and the law of the observation Y is the translate by $C_{N-1}(A_{N-2}m+B_{N-2}u)$ of the law of the observation if the initial distribution were λ^0 and the control were 0. Hence this contribution does not depend on u and depends on λ only through λ^0. Let's call it $C_{N-2,N-1}(\lambda^0)$.

By putting together these intermediate results, we have

$$\begin{aligned} J_{N-2}(\lambda) &= E_{\lambda^0}[X^T Q_{N-2} X] \\ &+ E_{\lambda^0}[X^T A_{N-2}^T K_{N-1} A_{N-2} X] \\ &+ C_{N-2,N-1}(\lambda^0) \\ &+ \sum_{l=N-2}^{N-1} w_l^T K_{l+1} w_l \\ &+ \min_u [m^T Q_{N-2} m + u^T R_{N-2} u + (A_{N-2}m + B_{N-2}u)^T K_{N-1}(A_{N-2}m + B_{N-2}u)]. \end{aligned} \tag{15.16}$$

As far as the last term concerned, we see, as in the fully observed case, that the minimum occurs at $u_{N-2} = L_{N-2}m$ and yields $m^T K_{N-2} m$. In addition, by noting the relation

$$Q_{N-2} + A_{N-2}^T K_{N-1} A_{N-2} = K_{N-2} + \Gamma_{N-2} \ , \tag{15.17}$$

we have

$$J_{N-2}(\lambda) = E_\lambda[X^T K_{N-2} X] + E_{\lambda^0}[X^T \Gamma_{N-2} X] + C_{N-2,N-1}(\lambda^0) + \sum_{l=N-2}^{N-1} w_l^T K_{l+1} w_l \ ,$$

which we may write as

$$J_{N-2}(\lambda) = E_\lambda[X^T K_{N-2} X] + C_{N-2,N-2}(\lambda^0) + C_{N-2,N-1}(\lambda^0) + \sum_{l=N-2}^{N-1} w_l^T K_{l+1} w_l \ ,$$

where $C_{N-2,N-2}(\lambda^0)$ is defined to be $E_{\lambda^0}[X^T \Gamma_{N-2} X]$. Note that, defining $C_{N-1,N-1}(\lambda^0)$ to be $E_{\lambda^0}[X^T \Gamma_{N-1} X]$, our earlier formula for $J_{N-1}(\lambda)$ could have been written as:

$$J_{N-1}(\lambda) = E_\lambda[X^T K_{N-1} X] + C_{N-1,N-1}(\lambda^0) + \sum_{l=N-1}^{N-1} w_l^T K_{l+1} w_l \ .$$

This suggests that we should have in general that

$$J_k(\lambda) \ = \ E_\lambda[X^T K_k X] + \sum_{l=k}^{N-1} C_{k,l}(\lambda^0) + \sum_{l=k}^{N-1} w_l^T K_{l+1} w_l \tag{15.18}$$

where

$$
\begin{aligned}
C_{k,k}(\lambda^0) &= E_{\lambda^0}[X^T \Gamma_k X] && (15.19)\\
\text{and } C_{k,l}(\lambda^0) & \quad \text{for } l \geq k+1 \text{ comes from } C_{k+1,l}(\cdot) \text{ via} && (15.20)\\
C_{k,l}(\lambda) &= E[C_{k+1,l}(T^0_{k,k+1}(\lambda, u, Y))], && (15.21)
\end{aligned}
$$

which depends only on λ^0 but does not depend on u. This can be verified as above by noting that

$$T^0_{k,k+1}(\lambda, u, y) = T^0_{k,k+1}(\lambda^0, 0, y - C_{k+1}(A_k m + B_k u)) \ ,$$

where $m = E_\lambda[X]$ and by noting that the distribution of $C_{k+1}(A_k X + B_k u)$ when X has the distribution λ is the translate by $C_{k+1}(A_k m + B_k u)$ of the distribution of $C_{k+1}(A_k X)$ when X has the distribution λ^0.

There remains the problem of computing the conditional expectation of the state given the information available to the controller, at each time. This can be done under assumptions of joint Gaussianity on the noise variables and the initial condition (in addition to the usual independence assumptions).

15.3 Vector Gaussianity

A Gaussian random vector $Z \in \Re^d$ is one whose joint characteristic function has the form:

$$\Phi_Z(\eta) = e^{i\eta^T m - \eta^T K \eta}, i = \sqrt{-1} \tag{15.22}$$

for some $m \in \Re^d$ and positive semidefinite matrix K. This definition works even in the absence of a density for Z. Here, we recall the definition:

$$\Phi_Z(\eta) \equiv E[e^{i\eta^T Z}] \tag{15.23}$$

Note that $m = E[Z]$ and $K = E[(Z-m)(Z-m)^T]$.

If K is positive definite, the coordinates of Z have joint density

$$f_Z(z) = \frac{1}{(2\pi)^{d/2}(detK)^{1/2}} e^{-\frac{1}{2}(z-m)^T K^{-1}(z-m)} \tag{15.24}$$

$Z_1, Z_2, \cdots, Z_d$ are called jointly Gaussian if $[Z_1, Z_2, \cdots, Z_d]^T$ is vector Gaussian.

Remark 15.1. *Suppose $Z_1, Z_2, \ldots, Z_L$ are jointly Gaussian with means $m_1, m_2, \ldots, m_L$ respectively.*

If

$$E\left[\begin{bmatrix} Z_1 - m_1 \\ Z_2 - m_2 \\ \vdots \\ Z_L - m_L \end{bmatrix} \begin{bmatrix} (Z_1-m_1)^T & (Z_2-m_2)^T & \ldots & (Z_L-m_L)^T \end{bmatrix}\right] = \begin{bmatrix} K_{11} & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & K_{LL} \end{bmatrix}$$

then, $Z_1, Z_2, \ldots, Z_L$ are mutually independent. That is, uncorrelatedness implies mutual independence for jointly Gaussian random vectors.

If Z_1 and Z_2 are jointly Gaussian random vectors with respective means m_1 and m_2, then

$$E[Z_1|Z_2] = m_1 + A(Z_2 - m_2) \text{ for some matrix } A. \tag{15.25}$$

To see this, note that

$$E[(Z_1 - (m_1 + A(Z_2 - m_2)))(Z_2 - m_2)^T] = K_{12} + AK_{22}\ , \tag{15.26}$$

where $K_{12} = E[(Z_1 - m_1)(Z_2 - m_2)^T]$. If this can be made zero for some A, then $Z_1 - (m_1 + A(Z_2 - m_2))$ would be independent of $(Z_2 - m_2)$. i.e.,

$$E[(Z_1 - (m_1 + A(Z_2 - m_2)))f(Z_2)] = 0 \text{ for all functions } f(\cdot)\ , \tag{15.27}$$

which would prove the claim.

If K_{22} is invertible, set $A = -K_{12}K_{22}^{-1}$. Otherwise, find B having the rank of K_{22} and satisfying $B(Z_2 - m_2) = Z_3$ with $K_{33} = E[Z_3Z_3^T]$ invertible (we can always do this). Then we have

$$\begin{aligned} K_{13} &= K_{12}B^T, &(15.28)\\ K_{33} &= BK_{22}B^T\ , &(15.29) \end{aligned}$$

where $K_{13} = E[(Z_1 - m_1)Z_3^T]$. Set $A = -K_{13}K_{33}^{-1}B$.