EE 223: Stochastic Estimation and Control	**Spring 2007**

Lecture 14 - March 1

Lecturer: Venkat Anantharam *Scribe: Lian Yu*

Notation from the last lecture:

$$\begin{aligned} K_N &= Q_N \\ K_k &= A_k^T K_{k+1} A_k - \Gamma_k + Q_k \end{aligned}$$

where $\Gamma_k = A_k^T K_{k+1} B_k (B_k^T K_{k+1} B_k + R_k)^{-1} B_k^T K_{k+1}$.

Also, $L_k = -(R_k + B_k^T K_{k+1} B_k)^{-1} B_k^T K_{k+1} A_k$.

1 Fully Observed LQ Problem

Recall the fully observed LQ problem. The linear system evolves as:

$$x_{k+1} = A_k x_k + B_k u_k + w_k \qquad k = 0, 1, \ldots, N-1,$$

and the objective (informally) is to minimize the quadratic cost:

$$\min E \left[\sum_{k=0}^{N-1} (X_k^T Q_k X_k + U_k^T R_k U_k) + X_N^T Q_N X_N \right]$$

where Q_k is a positive semidefinite matrix for $k = 0, \ldots, N$, and R_k is a positive definite matrix for $k = 0, \ldots, N-1$. The minimization is over all causal strategies where the controller has access to the states. The optimal cost-to-go is given by:

$$J_k(x) = x^T K_k x + \sum_{l=k}^{N-1} w_l^T K_{l+1} w_l$$

and the optimal control at time k is $u_k = L_k x_k$.

2 Partially Observed LQ Problem

We have the same dynamics:

$$x_{k+1} = A_k x_k + B_k u_k + w_k \qquad k = 0, 1, \ldots, N-1.$$

The controller has causal access to observations given by:

$$y_k = C_k x_k + v_k \qquad k = 0, 1, \ldots, N.$$

The objective (informally) is:

$$\min E\left[\sum_{k=0}^{N-1}(X_k^T Q_k X_k + U_k^T R_k U_k) + X_N Q_N^T X_N\right]$$

where the minimization is now over a different set of strategies than in the fully observed case. We know from the general theory that we should be writing a DP recursion backwards in time for functions $J_k(\lambda)$, where λ ranges over probabilities distributions on $\mathbb{R}^n$, starting with $J_N(\lambda)$ given by:

$$\begin{aligned} J_N(\lambda) &= \text{expected final cost if the conditional law of } X_N \text{ given} \\ &\quad\ (Y_0, \ldots, Y_N, U_0, \ldots, U_{N-1}) \text{ is } \lambda \\ &= E_\lambda[X^T Q_N X] \\ &= \int_{\mathbb{R}^n} x^T Q_N x \lambda(dx). \end{aligned}$$

Let $m = E_\lambda[X] = \int_{\mathbb{R}^n} x\lambda(dx)$. Note that:

$$\begin{aligned} E_{\lambda^0}[X^T Q_N X] &= \int_{\mathbb{R}^n} x^T Q_n x \lambda^0(dx) \\ &= \int_{\mathbb{R}^n} (x-m)^T Q_N (x-m)\lambda(dx) \\ &= \left[\int_{\mathbb{R}^n} x^T Q_N x \lambda(dx)\right] - m^T Q_N m \\ &= E_\lambda[X^T Q_N X] - (E_\lambda[X])^T Q_N (E_\lambda[X]) \end{aligned}$$

where λ^0 is the centered probability distribution corresponding to λ, i.e. the translate of λ that results in a distribution with mean the zero vector in $\mathbb{R}^n$. Thus we may write:

$$J_N(\lambda) = E_{\lambda^0}[X^T Q_N X] + (E_\lambda[X])^T Q_N (E_\lambda[X]) \ .$$

Now let us try to compute $J_{N-1}(\lambda)$ (for λ a probability distribution on $\mathbb{R}^n$) using the DP recursion. Think of $X \in \mathbb{R}^n$ drawn with distribution λ and think of applying a control $u \in \mathbb{R}^n$; the next state is then $A_{N-1}X + B_{N-1}u + w_{N-1}$. We observe $Y = C_N(A_{N-1}X + B_{N-1}u + w_{N-1}) + v_N$ and then compute the conditional law at time N, i.e. $T_{N-1,N}(\lambda, u, Y)$. Note that in this expression the distribution of Y depends on λ. The DP equation for period $N-1$ is:

$$J_{N-1}(\lambda) = \min_u \left\{E_\lambda[X^T Q_{N-1} X] + u^T R_{N-1} u + E[J_N(T_{N-1.N}(\lambda, u, Y))]\right\} \ ,$$

where the expectation in the third term in the minimization is over the random variable Y. Define $\tilde{X} := A_{N-1}X + B_{N-1}u + w_{N-1}$. Note that

$$T_{N-1,N}(\lambda, u, y)(dx) = P(\tilde{X} \in dx \mid Y = y) \ .$$

Thus we have:

$$J_N(T_{N-1,N}(\lambda, u, Y)) = \int x^T Q_N x P(\tilde{X} \in dx | Y),$$

so we have:

$$\begin{aligned} E[J_N(T_{N-1,N}(\lambda, u, Y))] &= \int x^T Q_N x P(\tilde{X} \in dx) \\ &= E[\tilde{X}^T Q_N \tilde{X}]. \end{aligned}$$

But

$$\begin{aligned} E[\tilde{X}^T Q_{N-1} \tilde{X}] &= E[w_{N-1}^T Q_N w_{N-1}] + E_{\lambda^0}[(A_{N-1}X)^T Q_N (A_{N-1}X)] \\ &\quad + (A_{N-1}m + B_{N-1}u)^T Q_N (A_{N-1}m + B_{N-1}u) \ , \end{aligned}$$

where $m = E_\lambda[X]$ and λ^0 is the centered distribution corresponding to λ. Also,

$$E_\lambda[X^T Q_{N-1} X] = m^T Q_{N-1} m + E_{\lambda^0}[X^T Q_{N-1} X].$$

Substituting these into the right hand side of the expression for $J_{N-1}(\lambda)$ we get:

$$J_{N-1}(\lambda) = m^T Q_{N-1} m + \min_u \left\{ (A_{N-1}m + B_{N-1}u)^T Q_N (A_{N-1}m + B_{N-1}u) + u^T R_{N-1} u \right\}$$

$$+ E_{\lambda^0}[X^T (Q_{N-1} + A_{N-1}^T Q_N A_{N-1}) X] + E[w_{N-1}^T Q_N w_{N-1}] \ .$$

We see that, as in the fully observed case, the minimum occurs when $u = L_{N-1}m$ and also that

$$\begin{aligned} J_{N-1}(\lambda) &= m^T K_{N-1} m + E[w_{N-1}^T Q_N w_{N-1}] + E_{\lambda^0}[X^T Q_{N-1} X] + E_{\lambda^0}[(A_{N-1}X)^T Q_N (A_{N-1}X)] \\ &= E_\lambda[X^T K_{N-1} X] + E[w_{N-1}^T Q_N w_{N-1}] + E_{\lambda^0}[X^T (Q_{N-1} + A_{N-1}^T Q_N A_{N-1} - K_{N-1}) X]. \end{aligned}$$

But $Q_{N-1} - K_{N-1} = \Gamma_{N-1} - A_{N-1}^T Q_N A_{N-1}$, so the third term is

$$E_{\lambda^0}[X^T \Gamma_{N-1} X].$$

We therefore have:

$$J_{N-1}(\lambda) = E_\lambda[X^T K_{N-1} X] + E_{\lambda^0}[X^T \Gamma_{N-1} X] + E[w_{N-1}^T Q_N w_{N-1}] \ .$$

Note the analogy with the fully observed case, with the appearance of a new term, $E_{\lambda^0}[X^T \Gamma_{N-1} X]$.