EE 223: Stochastic Estimation and Control **Spring 2007**

Lecture 17 – March 15

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17.1 Ricatti Equation

$$P_{k+1} = A^T(P_k - P_k B(B^T P_k B + R)^{-1} B^T P_k)A + Q$$

If $Q = C^T C$, (A, B) is controllable and (A, C) is observable then for every initial condition P_0, $\lim_{k\to\infty} P_k = P$ where P is the unique fixed point of:

$$P = A^T(P - PB(B^T PB + R)^{-1} B^T P)A + Q$$

If $L = -(B^T PB + R)^{-1} B^T PA$ then $A + BL$ is stable.

Time invariant LQG problem where the asymptotic version of the Kalman filter is used for state conditional mean estimation (suboptimal, but time invariant) and the asymptotic version of the linear control is used (also suboptimal).

Earlier:

$$X_{k+1} = A_k X_k + B_k u_k + w_k$$
$$y_k = C_k X_k + v_k$$

Kalman filter:

$$\hat{X}_{(k+1)|k} = A_k \hat{X}_{k|k}$$
$$\hat{X}_{k|k} = \hat{X}_{k|(k-1)} + \Sigma_{k|(k-1)} C_k^T (C_k \Sigma_{k|(k-1)} C_k^T + N_k)^{-1} (y_k - C_k \hat{X}_{k|(k-1)})$$

Where:

$$\hat{X}_{k|(k-1)} = E[X_k \mid y_0, ..., y_{k-1}]$$
$$\hat{X}_{k|k} = E[X_k \mid y_0, ..., y_k]$$
$$\Sigma_{k|(k-1)} = E[(X_k - \hat{X}_{k|(k-1)})(X_k - \hat{X}_{k|(k-1)})^T]$$

$$\Sigma_{k|k} = E[(X_k - \hat{X}_{k|k})(X_k - \hat{X}_{k|k})^T]$$

In the time invariant case: $A_k = A$, $B_k = B$, $C_k = C$ but the Kalman filter is still time varying.

$$\Sigma_{(k+1)|k} = A_k(\Sigma_{k|(k-1)} - \Sigma_{k|(k-1)}C_k^T(C_k\Sigma_{k|(k-1)}C_k^T + N_k)^{-1}C_k\Sigma_{k|(k-1)})A_k^T + M_k$$

$(A_k, B_k, C_k, M_k, N_k)$ do not depend on *k* in the time invariant case.

Appealing to the general Ricatti equation convergence theorem we see that if $M = DD^T$, (A, C) is observable and (A, D) is controllable then for every initial condition $\Sigma_{0|-1}$ $\lim_{k\to\infty} \Sigma_{k|(k-1)} = \Sigma$ and is the unique fixed point of:

$$\Sigma = A(\Sigma - \Sigma C^T(C\Sigma C^T + N)^{-1}C\Sigma)A^T + M$$

And further:

$$A - A\Sigma C^T(C\Sigma C^T + N)^{-1}C$$

is stable.

17.2 Approximate Estimators

(no control)

$$\hat{X}^{@}_{(k+1)|k} = A\hat{X}^{@}_{k|(k-1)} + A\Sigma C^T(C\Sigma C^T + N)^{-1}(Y_k - C\hat{X}^{@}_{k|(k-1)})$$

Starting with $\hat{X}^{@}_{0|-1} = E[X_0]$

<u>Thm</u>: Define $e_k = X_k - \hat{X}^{@}_{k|(k-1)}$ (one step prediction error for the approximate filter). Then:

$$e_{k+1} = (A - A\Sigma C^T(C\Sigma C^T + N)^{-1}C)e_k + w_k - A\Sigma C^T(C\Sigma C^T + N)^{-1}v_k$$

$A - A\Sigma C^T(C\Sigma C^T + N)^{-1}C$ is stable so this estimator has asymptotically bounded error covariance.

A calculation shows that the limit covariance matrix of the one step prediction error of the approximate filter satisfies the algebraic Riccati equation (third displayed equation on pg. 496 of the text). Thus the approximate filter has the same asymptotic prediction error covariance as the true filter.

Now imagine using asymptotic control applied to the approximate estimator:

$$X_{k+1} = AX_k + BL\hat{X}^{@}_{k|k} + w_k$$

$$\hat{X}^{@}_{(k+1)|(k} = A\hat{X}^{@}_{k|k} + BL\hat{X}^{@}_{k|k} + \Sigma C^T (C\Sigma C^T + N)^{-1} (CAX_k + CBL\hat{X}^{@}_{k|k} + Cw_k + v_k - C(A\hat{X}^{@}_{k|k} + BL\hat{X}^{@}_{k|k}))$$

If we ignore the w_k and v_k terms and subtract the second equation from the first we get:

$$X - \hat{X}^{@}_{(k+1)|(k+1)} = (A - \Sigma C^T (C\Sigma C^T + N)^{-1} CA)(X_k - \hat{X}^{@}_{k|k})$$

Where the matrix $(A - \Sigma C^T (C\Sigma C^T + N)^{-1} CA)$ is stable. It follows that:

$\begin{pmatrix} X_{k+1} \\ \hat{X}^{@}_{(k+1)|(k+1)} \end{pmatrix}$ is a $2n$-dimensional linear system with noise $\begin{pmatrix} w_k \\ v_k \end{pmatrix}$

So (in the absence of noise) $X_{k+1} - \hat{X}^{@}_{(k+1)|(k+1)} \to 0$ so the top equation looks like:

$$X_{k+1} = (A + BL)X_k - BL(X_k - \hat{X}^{@}_{k|k}) \text{ + “no noise”}$$

Which implies $X_k \to 0$ so the overall system is a stable $2n$-dimensional linear system driven by noise.

Since the asymptotic covariance of the prediction error of the approximate filter is the same as that of the true filter, the asymptotic cost per unit time of the appproximately optimal control strategy that uses the asymptotic gain applied to the estimates of the approximate filter can be seen to equal the asymptotic cost per unit time of the optimal control strategy using the optimal filter.

17.3 Infinite Horizon Dynamic Programming

Consider stationary dynamical system and try to address the steady state aspects directly. One way to do this is to use discounted dynamic programming.

Assume there is some $0 < \alpha < 1$ and cost paid at time *k* is α^k times that paid at time 0. In the case of the fully observed DP,

$$X_{k+1} = f(X_k, u_k, w_k) \text{ where } f \text{ is time invariant and } w_k \sim iid$$

The objective (informally) is to minimize:

$$E[\sum_{k=0}^{\infty} \alpha^k g(X_k, u_k, w_k)] \text{ where } g \text{ is time invariant.}$$

Formally, define a strategy $\Psi = (\upsilon_0, \upsilon_1, ...)$ where $U_k(X_0, ..., X_k)$ is the control applied at time *k*.

The approach to solving this is by using the finite horizon results and letting $N \to \infty$ (*N* is the horizon.)

Given any $J : x \to \Re$

Consider the minimization of:

$$E[\sum_{k=0}^{N-1} \alpha^k g(X_k^{\Psi}, u_k^{\Psi}, w_k) + \alpha^N J(X_N^{\Psi})]$$

Over all $\Psi = (\upsilon_0, ... \upsilon_{N-1})$. We know how to solve this. We could define:

$$J_k^{(N)}(x) = \min_u E[\alpha^k g(x, u, w_k) + J_{k+1}^{(n)}(f(x, u, w_k))]$$

$$J_N^{(N)}(x) = \alpha^N J(x)$$

<u>Idea</u>: consider $\frac{J_k^{(N)}(x)}{\alpha^k}$

This suggests examination of the mapping:

$$(TJ)(x) = \min_u E[g(x, u, w) + \alpha J(f(x, u, w))]$$