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HURST INDEX OF FUNCTIONS OF LONG RANGE DEPENDENT MARKOV CHAINS

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Abstract

A positive recurrent, aperiodic Markov chain is said to be long range dependent (LRD) when the indicator function of a particular state is LRD. This happens if and only if the return time distribution for that state has infinite variance. We investigate the question of whether other instantaneous functions of the Markov chain also inherit this property. We provide conditions under which the function has the same degree of long range dependence as the chain itself. We illustrate our results through three examples in diverse fields: queuing networks, source compression, and finance.

Keywords: Markov chain; long range dependence; Hurst index

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1. Introduction

A stationary random process (X_n) with $E[X_n^2] < \infty$ is said to be long range dependent (LRD) if

$$\limsup_{n\to\infty} \sum_{r=1}^{n} \mathrm{cov}(X_0, X_r) = \infty.$$

The degree of long range dependence is measured by the Hurst index H ($\frac{1}{2} \le H \le 1$).

$$H := inf \left\{ h : \limsup_{n\to\infty} \frac{\sum_{r=1}^{n} \mathrm{cov}(X_0, X_r)}{n^{2h-1}} < \infty \right\}.$$

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Equivalently, we can write

$$H := inf\left\{h : \limsup_{n\to\infty} \frac{\mathrm{var}(\sum_{i=1}^{n} X_i)}{n^{2h}} < \infty\right\}.$$

Take (M_n), a positive-recurrent, aperiodic, discrete time, countable state Markov chain with state space $\mathbb{N}$, where $\mathbb{N}$ denotes the set of natural numbers. The chain is in stationarity with stationary distribution π. The indicator function $1(M_n = i)$ of state i of this chain is LRD if and only if indicator functions of every state is LRD [4]. When this is true, (M_n) is said to be an LRD Markov chain. Moreover, the Hurst index of these functions is also a class property [4]. The common Hurst index H is said to be the Hurst index of the chain.

In [4] it is proved that a Markov chain is LRD if and only if the return time distribution of any state has infinite variance. It is also argued that finite weighted sums of indicator functions on this chain also inherit this property. It is natural to conjecture that this might be true for all functions of the chain. However, this conjecture is easily disproved, most easily by considering a constant function (also see the two counter examples in [4]). It is then of considerable interest to find which functions of an LRD Markov chain are also LRD.

Let $\rho_n = \rho(M_n)$ be an L_2 function of M_n. In this paper, we provide conditions under which one can infer the long range dependence of (ρ_n) from that of (M_n).

Our main result, given in section 4, provides a technical condition under which the rate of growth of $\sum_{r=1}^{n} \mathrm{cov}(X_0, X_r)$ is identical for $X_n = \rho_n$ and $X_n = 1(M_n = i)$. We set up the proof with a collection of lemmas presented in section 3. For convenience, most of the notation is collected together in section 2.

There are many interesting scenarios where the results of this paper might be useful. In the second half of the paper, we collect three such examples. Section 5 discusses a simple queuing network of two parallel queues. One queue is driven by an LRD process, whereas the other one is driven by a short range dependent process. We model the inputs and queue lengths by countable state Markov chains, and show that under longest queue first scheduling both queues are LRD.

A particularly novel example is given in section 6, where we re-prove a recent result in the source coding of LRD sequences [13]. We show that the code length process of any lossless encoder which is compressing an LRD renewal process must dominate an

LRD process with the same Hurst index as the source process.

The last example is about long range dependence in financial series. We discuss how our model can explain the LRD behavior observed in some instantaneous functions of the absolute returns of some asset.

2. Notation and setup

(M_n) is a positive-recurrent, discrete time, countable state Markov chain with state space $\mathbb{N}$ and stationary distribution π_i, $i \in \mathbb{N}$. Most of the notation we use is borrowed from [5].

$\rho : \mathbb{N} \to \mathbb{R}$ is such that $\sum_{i\in\mathbb{N}} \rho(i)^2\pi_i < \infty$.

$\rho_n := \rho(M_n)$.

$\mu := \sum_i \rho(i)\pi_i$, is the mean of ρ.

$p_{ij}^{(n)} := P(M_n = j|M_0 = i), n \geq 0$.

${}_kp_{ij}^{(n)} := P(M_n = j; M_l \neq k, 0 < l < n|M_0 = i), n > 0$.

${}_kp_{ij}^{*} := \sum_{n=1}^{\infty} {}_kp_{ij}^{(n)}$.

${}_{\mathcal{H}}p_{ij}^{(n)} := P(M_n = j; M_l \notin \mathcal{H}, 0 < l < n|M_0 = i), n > 0$.

${}_{\mathcal{H}}p_{ij}^{*} := \sum_{n=1}^{\infty} {}_{\mathcal{H}}p_{ij}^{(n)}$.

$f_{ij}^{(n)} := {}_jp_{ij}^{(n)}, n > 0$.

$Q_{ij}^{(n)} := \sum_{r=1}^{n}(p_{ij}^{(r)} - \pi_j), n > 0$.

$R_{ij}^{(n)} := \sum_{r=1}^{n} Q_{ij}^{(r)}, n > 0$.

$T_j := \inf_t\{t > 0 : M_t = j\}$.

$m_{ij} := E_i[T_j]$.

$H := inf \left\{h : \limsup_{n\to\infty} \frac{\mathrm{var}(\sum_{i=1}^{n} \mathbf{1}(M_i=1))}{n^{2h}} < \infty\right\}$, the Hurst index of (M_n).

$H_\rho := inf \left\{h : \limsup_{n\to\infty} \frac{\mathrm{var}(\sum_{i=1}^{n} \rho_i)}{n^{2h}} < \infty\right\}$, the Hurst index of (ρ_n).

3. Lemmas

We will rely on several lemmas, most of which are already known.

Lemma 3.1. *Chung [5], chapter 11, Corollary 1 For* $p \geq 0$*,*

$$E_1T_1^p = \infty \iff E_iT_i^p = \infty, \qquad \forall i \in \mathbb{N}$$

Lemma 3.2. *Let (a_n) be an arbitrary sequence and $b_n \to \infty$. c is a finite real number. If*

$$\frac{a_n}{b_n} \to c,$$

then

$$\frac{\sum_{r=1}^{n} a_r}{\sum_{r=1}^{n} b_r} \to c.$$

Proof. Pick N large enough such that $|\frac{a_n}{b_n} - c| < \epsilon, \quad \forall n \geq N$. Pick M large enough such that $\left|\frac{\sum_{r=N}^{M} b_r}{\sum_{r=1}^{M} b_r}\right| > 1 - \epsilon$ and $\left|\frac{\sum_{r=1}^{N} a_r}{\sum_{r=N}^{M} b_r}\right| < \epsilon$ (always possible since $b_n \to \infty$). Then

$$\left|\frac{\sum_{r=1}^{M} a_r}{\sum_{r=1}^{M} b_n} - c\right| < \epsilon + \frac{\epsilon}{1-\epsilon}(1 + |c|).$$

Lemma 3.3. *For an LRD Markov chain,*

$$\lim_{n\to\infty} Q_{ij}^{(n)} = \infty, \tag{1}$$

$$\lim_{n\to\infty} \frac{R_{ij}^{(n)}}{n} = \infty, \tag{2}$$

$$\lim_{n\to\infty} \frac{Q_{ij}^{(n)}/\pi_j}{Q_{11}^{(n)}/\pi_1} = 1. \tag{3}$$

Proof. (3) is eq. 8 in [4]. (1) follows from eqs. 8 and 5 of [4]. (2) follows from (1).

We will assume henceforth that n is large enough s.t. $Q_{11}^{(n)}, R_{11}^{(n)} > 1$.

Lemma 3.4. *(i)*

$$cov(\rho_0, \rho_r) = \sum_{i,j} \pi_i p_{ij}^{(r)} (\rho(i) - \mu)(\rho(j) - \mu).$$

(ii)

$$\sum_{r=1}^{n} cov(\rho_0, \rho_r) = \sum_{i,j} \rho(i)\rho(j)\pi_i Q_{ij}^{(n)}.$$

(iii)

$$var(\rho_0 + \ldots + \rho_n) - (n+1)var(\rho_0) = 2\sum_{i,j} \rho(i)\rho(j)\pi_i R_{ij}^{(n)}.$$

Proof. (i) is a simple expansion. (ii) is derived from (i), and (iii) can be found in [4], section 3.

Lemma 3.5. *eq. (1) in Chung [5], theorem 9.1*

$$p_{ij}^{(r)} = {}_1p_{ij}^{(r)} + \sum_{m=1}^{r-1} {}_1p_{i1}^{(m)} p_{1j}^{(r-m)}, \quad r \geq 1. \tag{4}$$

Lemma 3.6. *(Carpio & Daley [4], 2.12)*

$$\begin{aligned} Q_{11}^{(n)} &\sim (\pi_1)^2 \sum_{u=1}^{\infty} \min(u,n) \sum_{s=u+1}^{\infty} f_{11}^{(s)} \\ &= (\pi_1)^2 \sum_{u=1}^{\infty} \sum_{r=1}^{\min(u,n)} \sum_{s=u+1}^{\infty} f_{11}^{(s)} \\ &= (\pi_1)^2 \sum_{r=1}^{n} \sum_{u=r}^{\infty} \sum_{s=u+1}^{\infty} f_{11}^{(s)}. \end{aligned}$$

Lemma 3.7.

$$\lim_{n\to\infty} \frac{1}{Q_{11}^{(n)}/\pi_1} \sum_{i,j} \pi_i \sum_{r=1}^{n} {}_1p_{ij}^{(r)} = 1$$

Proof.

$$\begin{aligned} \sum_{i,j} \pi_i \sum_{r=1}^{n} {}_1p_{ij}^{(r)} &= \sum_{r=1}^{n} \sum_{i,j} \pi_i \, {}_1p_{ij}^{(r)} \\ &=^a \sum_{r=1}^{n} \sum_{i} \pi_i \sum_{u=r}^{\infty} f_{i1}^{(r)} \\ &=^b \sum_{r=1}^{n} \sum_{u=r}^{\infty} \frac{1}{m_{11}} \sum_{s=u}^{\infty} f_{11}^{(s)} \\ &= \frac{1}{m_{11}} \sum_{r=1}^{n} \sum_{u=r}^{\infty} f_{11}^{u} + \sum_{r=1}^{n} \sum_{u=r}^{\infty} \frac{1}{m_{11}} \sum_{s=u+1}^{\infty} f_{11}^{(s)} \\ &= \frac{1}{m_{11}} \sum_{r=1}^{n} P_1(T_1 \geq r) + \sum_{r=1}^{n} \sum_{u=r}^{\infty} \frac{1}{m_{11}} \sum_{s=u+1}^{\infty} f_{11}^{(s)} \\ &\sim \frac{1}{\pi_1^2 m_{11}} Q_{11}^{(n)} = \frac{Q_{11}^{(n)}}{\pi_1} \end{aligned}$$

since $\sum_{r=1}^{n} P_1(T_1 \geq r) \leq m_{11}$ and by lemma (3.6). Here (a) uses $\sum_j {}_1p_{ij}^{(r)} = \sum_r^{\infty} f_{i1}^{(r)}$, which are equivalent ways of expressing the probability of going from i to any other state without going to 1 in r steps. This expression also appears chapter 9 of [5] (proof of thm. 6). (b) uses the fact $P_\pi(T_1 = r) = \frac{P_1(T_1 \geq r)}{m_{11}}$, where T_1 is the first return time to 1 at stationarity.

Lemma 3.8. *Let $M > 0$ be a finite number,*

$$\lim_{n\to\infty} \frac{1}{Q_{11}^{(n)}/\pi_1} \sum_{\{i<M\}\cup\{j<M\}} \pi_i \sum_{r=1}^{n} {}_1p_{ij}^{(r)} = 0.$$

Proof. Pick m s.t. ${}_1p_{1i}^{(m)} > 0$, then

$${}_1p_{1i}^{(m)}\, {}_1p_{ij}^{*} \le {}_1p_{1j}^{*} = \pi_j/\pi_1.$$

Thus, there exists a finite constant C_M s.t. ${}_1p_{ij}^{*} < C_M \pi_j$ for all $i < M$. Conclude

$$\sum_{i<M,j} \pi_i \sum_{r=1}^{n} {}_1p_{ij}^{(r)} \le C_M \sum_{i<M,j} \pi_i\pi_j \le C_M.$$

Similarly, there exists a finite constant D_M s.t. ${}_1p_{ij}^{*} \le 1 + {}_1p_{jj}^{*} \le D_M$ for all $j < M$.

$$\sum_{i,j<M} \pi_i \sum_{r=1}^{n} {}_1p_{ij}^{(r)} \le D_M \sum_{i,j<M} \pi_i \le M D_M.$$

Using (1) we conclude the proof.

Lemma 3.9. *([4], pg 1051)*

$$\left| \frac{Q_{1j}^{(n)}/\pi_j}{Q_{11}^{(n)}/\pi_1} \right| \le 1.$$

Lemma 3.10.

$$\left| \sum_{r=1}^{n} \sum_{i,j} \pi_i |\rho(i)\rho(j)| {}_1p_{ij}^{(r)} - \sum_{r=1}^{n} \sum_{i,j} \pi_i |\rho(i)\rho(j)| {}_{\mathcal{H}}p_{ij}^{(r)} \right| \le (|\mathcal{H}|+1) C_{\mathcal{H}} \sum_{i,j} \pi_i\pi_j |\rho(i)\rho(j)|,$$

where $\mathcal{H}$ is any non-empty set with a finite number of states and $C_{\mathcal{H}}$ is a constant that depends only on $\mathcal{H}$.

Proof. Let $\mathcal{H}' = \mathcal{H} \cup \{k\}$, $k \notin \mathcal{H}$. We will argue by induction. We write

$$\sum_{r=1}^{n} {}_{\mathcal{H}}p_{ij}^{(r)} - {}_{\mathcal{H}'}p_{ij}^{(r)} = \sum_{r=1}^{n} P(M_r = j; M_l \notin \mathcal{H}, 1 \le l < r; M_l = k, \text{for some } 1 \le l < r | M_0 = i)$$

$$= \sum_{r=1}^{n} \sum_{m=1}^{r-1} {}_{\mathcal{H}'}p_{ik}^{(m)}\, {}_{\mathcal{H}}p_{kj}^{(r-m)}$$

$$= \sum_{m=1}^{n-1} {}_{\mathcal{H}'}p_{ik}^{(m)} \sum_{r=m+1}^{n} {}_{\mathcal{H}}p_{kj}^{(r-m)}$$

$$\le \underbrace{\left(\sum_{m=1}^{\infty} {}_{\mathcal{H}'}p_{ik}^{(m)} \right)}_{C_1} \underbrace{\left(\sum_{r=1}^{\infty} {}_{\mathcal{H}}p_{kj}^{(r)} \right)}_{{}_{\mathcal{H}}p_{kj}^{*}}.$$

C_1 is bounded above by 1 since

$$\sum_{m=1}^{\infty} {}_{\mathcal{H}'}p_{ik}^{(m)} \leq \sum_{m=1}^{\infty} {}_{k}p_{ik}^{(m)} = 1.$$

Let $h \in \mathcal{H}$. m is s.t. ${}_{h}p_{hk}^{(m)} > 0$.

$${}_{h}p_{hk}^{(m)} {}_{\mathcal{H}}p_{kj}^{*} \leq {}_{h}p_{hj}^{*} = \pi_j/\pi_h$$

Thus ${}_{\mathcal{H}}p_{kj}^{*} \leq \pi_j/({}_{h}p_{hk}^{(m)}\pi_h) = C_{\mathcal{H}'}$.

$$\sum_{r=1}^{n}\sum_{i,j} \pi_i|\rho(i)\rho(j)|{}_{\mathcal{H}}p_{ij}^{(r)} - \sum_{r=1}^{n}\sum_{i,j} \pi_i|\rho(i)\rho(j)|{}_{\mathcal{H}'}p_{ij}^{(r)} \leq C_{\mathcal{H}'}\sum_{i,j}\pi_i\pi_j|\rho(i)\rho(j)|.$$

Therefore adding or subtracting a state from the set $\mathcal{H}$ (as long as the resulting set is non-empty) only affects the sum in question by a bounded amount. As a result, replacing $\mathcal{H}$ by $\{1\}$ can change the sum by at most $(1+|\mathcal{H}|)C_{\mathcal{H}}\sum_{i,j}\pi_i\pi_j|\rho(i)\rho(j)|$. (Add state 1 if it is not already in set $\mathcal{H}$. Then subtract all other states until only state 1 is left.)

4. Main results

Theorem 4.1. *Let*

(condition 1)

$$\lim_{n\to\infty}\frac{1}{Q_{11}^{(n)}/\pi_1}\sum_{r=1}^{n}\sum_{i,j}\pi_i(\rho(i)-c)(\rho(j)-c){}_{\mathcal{H}}p_{ij}^{(r)} = 0$$

for some constant c, and non-empty, finite set $\mathcal{H}$.

(condition 2)

$$\lim_{L\to\infty}\limsup_{n\to\infty}\frac{1}{Q_{11}^{(n)}/\pi_1}\sum_{r=1}^{n}\sum_{i,j}\pi_i|\rho(i)\rho(j)|1(|\rho(i)|>L,|\rho(j)|>L){}_{\mathcal{H}}p_{ij}^{(r)} = 0$$

Then,

$$\lim_{n\to\infty}\frac{var(\sum_{r=1}^{n}\rho_i)}{R_{11}^{(n)}/\pi_1} = (\mu-c)^2.$$

Moreover, if $c \neq \mu$, then $H_\rho = H$.

Proof. By (1) and lemma (3.10) the conditions are equivalent to

(condition 1)

$$\lim_{n\to\infty} \frac{1}{Q_{11}^{(n)}/\pi_1} \sum_{r=1}^{n} \sum_{i,j} \pi_i(\rho(i)-c)(\rho(j)-c)_1 p_{ij}^{(r)} = 0$$

for some constant c, and

(condition 2)

$$\lim_{L\to\infty} \limsup_{n\to\infty} \frac{1}{Q_{11}^{(n)}/\pi_1} \sum_{r=1}^{n} \sum_{i,j} \pi_i |\rho(i)\rho(j)| 1(|\rho(i)|, |\rho(j)| > L)_1 p_{ij}^{(r)} = 0.$$

Define

$$\overline{\rho}^M(i) = \begin{cases} \rho(i) & , i \leq M \\ c & , i > M \end{cases} .$$

$\overline{\mu}^M = E[\bar{\rho}_n^M]$, $\underline{\rho}^M(i) = \rho(i) - \bar{\rho}^M(i)$, and $\underline{\mu}^M = E[\underline{\rho}_n^M]$. We adopt the shorthand notation:

$$\phi_n = \frac{(\rho_0 + \ldots + \rho_n) - (n+1)\mu}{\sqrt{2R_{11}^{(n)}/\pi_1}},$$

$$\overline{\phi}_n^M = \frac{(\bar{\rho}_0^M + \ldots + \bar{\rho}_n^M) - (n+1)\overline{\mu}^M}{\sqrt{2R_{11}^{(n)}/\pi_1}},$$

$$\underline{\phi}_n^M = \phi_n - \overline{\phi}_n^M.$$

We will be referring to the reverse triangle inequality for random variables:

$$\left| \sqrt{\mathrm{var}(\phi_n)} - \sqrt{\mathrm{var}(\overline{\phi}_n^M)} \right| \leq \sqrt{\mathrm{var}(\underline{\phi}_n^M)} \tag{5}$$

Using lemma 3.5, write 3.4(i) as

$$\sum_{r=1}^{n} \mathrm{cov}(\rho_0,\rho_r) = \sum_{i,j} \pi_i(\rho(i)-\mu)(\rho(j)-\mu) \sum_{r=1}^{n} {}_1p_{ij}^{(r)} + \tag{6}$$
$$\sum_{i,j} \pi_i \sum_{r=1}^{n} \sum_{m=1}^{r-1} {}_1p_{i1}^{(m)} p_{1j}^{(r-m)} (\rho(i)-\mu)(\rho(j)-\mu).$$

The second term can be rewritten

$$\sum_{i,j} \pi_i \sum_{r=1}^{n} \sum_{m=1}^{r-1} {}_1p_{i1}^{(m)} p_{1j}^{(r-m)} (\rho(i)-\mu)(\rho(j)-\mu) =$$
$$\sum_{i,j} \pi_i \sum_{m=1}^{n-1} {}_1p_{i1}^{(m)} \sum_{r=m+1}^{n} p_{1j}^{(r-m)} (\rho(i)-\mu)(\rho(j)-\mu)$$

$$= \sum_{m=1}^{n-1} \left(\sum_{r=m+1}^{n} \sum_{i,j} {}_1p_{i1}^{(m)} \pi_i (p_{1j}^{(r-m)} - \pi_j)(\rho(i) - \mu)(\rho(j) - \mu) + \underbrace{\sum_{r=m+1}^{n} \sum_{i,j} {}_1p_{i1}^{(m)} \pi_i \pi_j (\rho(i) - \mu)(\rho(j) - \mu)}_{0} \right)$$

$$= \sum_{m=1}^{n-1} \sum_{i,j} \pi_i {}_1p_{i1}^{(m)} Q_{1j}^{(n-m)} (\rho(i) - \mu)(\rho(j) - \mu).$$

Dividing by $Q_{11}^{(n)}/\pi_1$ we get

$$= \sum_{m=1}^{n-1} \sum_{i,j} \pi_i {}_1p_{i1}^{(m)} \pi_j \frac{Q_{1j}^{(n-m)}/\pi_j}{Q_{11}^{(n)}/\pi_1} (\rho(i) - \mu)(\rho(j) - \mu).$$

By lemma 3.9 we have

$$\sum_j \pi_j \left| \frac{Q_{1j}^{(n-m)}/\pi_j}{Q_{11}^{(n)}/\pi_1} \right| |(\rho(j) - \mu)| < \infty.$$

We also know $\sum_i \pi_i \sum_{m=1}^{n-1} {}_1p_{i1}^{(m)} (\rho(i) - \mu) \to 0$. Therefore

$$\lim_{n\to\infty} \sum_{m=1}^{n-1} \sum_{i,j} \pi_i {}_1p_{i1}^{(m)} \pi_j \frac{Q_{1j}^{(n-m)}/\pi_j}{Q_{11}^{(n)}/\pi_1} (\rho(i) - \mu)(\rho(j) - \mu) = 0.$$

(Dominated convergence) The result has the interpretation that the sum of the covariances between ρ_0 and ρ_n on the event that the chain visits state 1 at least once before time n, is negligible compared to $Q_{11}^{(n)}$.

We want to use these results to conclude var$(\underline{\phi}_n^M) \to 0$. For this we write eq. 6 for $\underline{\rho}^M$, $c = 0$. The first term in eq. 6 reads after a little manipulation

$$\sum_{i,j} \pi_i [\underline{\rho}^M(i) \underline{\rho}^M(j) - \underline{\mu}^M (\underline{\rho}^M(i) + \underline{\rho}^M(j)) + (\underline{\mu}^M)^2] \sum_{r=1}^{n} {}_1p_{ij}^{(r)}. \tag{7}$$

Now assume ρ is bounded. After dividing by $Q_{11}^{(n)}/\pi_1$, the second and third terms are $O(\underline{\mu}^M)$ as $\underline{\mu}^M \to 0$ by lemma 3.7. Since $\underline{\mu}^M \to 0$ with M, these terms go to 0 as $M \to \infty$ uniformly in n.

For the first term in (7), write condition 1 as follows for comparison:

$$\lim_{n\to\infty} \frac{1}{Q_{11}^{(n)}/\pi_1} \Bigg(\sum_{r=1}^{n} \sum_{i\le M, j\le M} \pi_i(\rho(i)-c)(\rho(j)-c)\,{}_1p_{ij}^{(r)}$$

$$+ \sum_{r=1}^{n} \sum_{i\le M, j> M} \pi_i(\rho(i)-c)(\rho(j)-c)\,{}_1p_{ij}^{(r)}$$

$$+ \sum_{r=1}^{n} \sum_{i> M, j\le M} \pi_i(\rho(i)-c)(\rho(j)-c)\,{}_1p_{ij}^{(r)}$$

$$+ \sum_{r=1}^{n} \sum_{i> M, j> M} \pi_i(\rho(i)-c)(\rho(j)-c)\,{}_1p_{ij}^{(r)} \Bigg) = 0.$$

The first three sums have limit 0 because ρ are assumed to be bounded, and by lemma 3.8. The last sum is identical to the first term in (7). Therefore dividing eq. 6 by $Q_{11}^{(n)}/\pi_1$ and applying lemma 3.2 while observing lemma 3.4 (ii) and (iii), we conclude that $\lim_{M\to\infty}\lim_{n\to\infty}\mathrm{var}(\underline{\phi}_n^M) = 0$, and by eq. (5), also that $\lim_{M\to\infty}\lim_{n\to\infty}\mathrm{var}(\overline{\phi}_n^M) = \lim_{n\to\infty}\mathrm{var}(\phi_n)$.

To calculate $\mathrm{var}(\overline{\phi}_n^M)$, rewrite eq. 6 for $\overline{\rho}^M$:

$$\sum_{i,j} \pi_i[(\overline{\rho}^M(i)-c)(\overline{\rho}^M(j)-c) - (\overline{\mu}^M - c)(\overline{\rho}^M(i)+\overline{\rho}^M(j)-2c) + (\overline{\mu}^M-c)^2] \sum_{r=1}^{n} {}_1p_{ij}^{(r)}.$$

The first two sums will go to zero when dividing by $Q_{11}^{(n)}/\pi_1$, by the boundedness of ρ and lemma 3.8 because of truncation. The last term will read

$$(\overline{\mu}^M - c)^2 \frac{1}{Q_{11}^{(n)}/\pi_1} \sum_{i,j} \pi_i \sum_{r=1}^{n} {}_1p_{ij}^{(r)} \to (\overline{\mu}^M - c)^2, n\to\infty$$

by lemma 3.7. By lemma 3.4 (ii) and (iii), and lemma 3.2 this concludes the proof when (ρ_n) is bounded.

When (ρ_n) is not bounded, we truncate by value, i.e. $\tilde{\rho}^L(i) = \rho(i)1(\rho(i)\le L)$, $\tilde{\mu}^L = E[\tilde{\rho}_n^L]$, $\underset{\sim}{\rho}^L(i) = \rho(i) - \tilde{\rho}^L(i)$, and $\underset{\sim}{\mu}^L = E[\underset{\sim}{\rho}_n^L]$. Also define

$$\tilde{\phi}_n^L = \frac{(\tilde{\rho}_0^L + \ldots + \tilde{\rho}_n^L) - (n+1)\tilde{\mu}^L}{\sqrt{2R_{11}^{(n)}/\pi_1}},$$

$$\underset{\sim}{\phi}_n^L = \phi_n - \tilde{\phi}_n^L.$$

We can express $\sum_{r=1}^{n}\mathrm{cov}(\underset{\sim}{\rho}_0^L, \underset{\sim}{\rho}_r^L)$ as in eq. 6, and argue as there that the second term has limit 0 as $n\to\infty$ when divided by $Q_{11}^{(n)}/\pi_1$. The first term also has limit 0 due to the assumed condition 2. We appeal again to lemma 3.4 (ii) and (iii), and lemma 3.2 to argue that $\lim_{L\to\infty}\lim_{n\to\infty}\operatorname{var}(\underset{\sim}{\phi}_n^L) = 0$. By eq. (5), we also get $\lim_{L\to\infty}\lim_{n\to\infty}\operatorname{var}(\tilde{\phi}_n^L) = \lim_{n\to\infty}\operatorname{var}(\phi_n)$. We conclude

$$\lim_{n\to\infty}\frac{\operatorname{var}(\sum_{r=1}^{n}\rho_i)}{R_{11}^{(n)}/\pi_1} = \lim_{L\to\infty}\lim_{n\to\infty}\operatorname{var}(\tilde{\phi}_n^L) = \lim_{L\to\infty}(\tilde{\mu}-c)^2 = (\mu-c)^2.$$

The claim about the Hurst indices can be argued as follows. Consider the expression in lemma 3.4 (ii) for $\rho_n = 1(M_n = 1)$. Dividing by $Q_{11}^{(n)}/\pi_1$, we see that the right hand side has limit $\pi_1^2 > 0$. From the above argument it follows that $\left(\sum_{r=1}^{n}\operatorname{cov}(1(M_0=1),1(M_r=1))\right)/\left(\sum_{r=1}^{n}\operatorname{cov}(\rho_0,\rho_r)\right)$ has a finite, non-zero limit if $\mu \neq c$. It is easily seen from the definition of H that ρ has the same Hurst index as the indicator function $1(M_n = 1)$.

Some remarks about the conditions are in order.

1. They fail to hold if $\lim_i \rho(i)$ exists and is not c.
2. They will hold whenever $\lim_i(\rho(i)-c) = 0$. Specifically when $(\rho(i)-c) = 0$ for i greater than some value.

   Both of these can be seen as direct consequences of lemma 3.7.
3. They are implied by the considerably stronger condition

$$\frac{1}{Q_{11}^{(n)}/\pi_1}\sum_{r=1}^{n}\sum_{i,j}\pi_i|\rho(i)-c||\rho(j)-c|_1 p_{ij}^{(r)} \to 0.$$

The following theorem extends the usefulness of the preceding theorem considerably. It describes the case, when the state space of the Markov chain is divided into a finite number of subsets, with communication between the sets happening almost only through state 1.

Theorem 4.2. *Let* $\{\mathcal{A}_k\}$, $1 \le k \le K$, *be a finite partition of the state space* $\mathbb{N}$. *(condition 1) Let* $\mathcal{H}$ *be a non-empty finite set, and*

$$\lim_{n\to\infty}\frac{1}{Q_{11}^{(n)}/\pi_1}\sum_{r=1}^{n}\sum_{i\in\mathcal{A}_k,j\in\mathcal{A}_l}\pi_i|\rho(i)-\mu||\rho(j)-\mu|_{\mathcal{H}}p_{ij}^{(r)} = 0, \quad \forall k \neq l.$$

Also suppose $\pi_{\mathcal{A}_k}^{\infty} := \lim_{n\to\infty}\frac{\sum_{i,j\in\mathcal{A}_k}\pi_i\sum_{r=1}^{n}{}_1p_{ij}^{(r)}}{\sum_{i,j}\pi_i\sum_{r=1}^{n}{}_1p_{ij}^{(r)}}$ *exists* $\forall k$. *Let there exist constants* $c_k, 1 \le k \le K$, *such that*

(condition 2)

$$\lim_{n\to\infty} \frac{1}{Q_{11}^{(n)}/\pi_1} \sum_{r=1}^{n} \sum_{i,j\in\mathcal{A}_k} \pi_i(\rho(i)-c_k)(\rho(j)-c_k)_{\mathcal{H}} p_{ij}^{(r)} = 0 \quad \forall k,$$

and

(condition 3)

$$\lim_{L\to\infty} \limsup_{n\to\infty} \frac{1}{Q_{11}^{(n)}/\pi_1} \sum_{r=1}^{n} \sum_{i,j\in\mathcal{A}_k} \pi_i|\rho(i)\rho(j)|1(|\rho(i)|>L,|\rho(j)|>L)_{\mathcal{H}} p_{ij}^{(r)} = 0 \quad \forall k.$$

Then,

$$\lim_{n\to\infty} \frac{var(\sum_{r=1}^{n}\rho_i)}{R_{11}^{(n)}/\pi_1} = \sum_{k=1}^{K} \pi_{\mathcal{A}_k}^{\infty}(\mu-c_k)^2.$$

Moreover, if $\pi_{\mathcal{A}_k}^{\infty}(c_k-\mu)\neq 0$ *for some* k*, then* $H_\rho = H$*.*

Remark. If $c_k = c_l$ *for a pair of subsets* $\mathcal{A}_k, \mathcal{A}_l$*, then condition* 1 *is not needed for this particular pair.*

Proof. By (1) and lemma (3.10) the conditions are equivalent to

(condition 1)

$$\lim_{n\to\infty} \frac{1}{Q_{11}^{(n)}/\pi_1} \sum_{r=1}^{n} \sum_{i\in\mathcal{A}_k,j\in\mathcal{A}_l} \pi_i|\rho(i)-\mu||\rho(j)-\mu|_1 p_{ij}^{(r)} = 0, \quad \forall k\neq l,$$

(condition 2)

$$\lim_{n\to\infty} \frac{1}{Q_{11}^{(n)}/\pi_1} \sum_{r=1}^{n} \sum_{i,j\in\mathcal{A}_k} \pi_i(\rho(i)-c_k)(\rho(j)-c_k)_1 p_{ij}^{(r)} = 0, \quad \forall k,$$

(condition 3)

$$\lim_{L\to\infty} \limsup_{n\to\infty} \frac{1}{Q_{11}^{(n)}/\pi_1} \sum_{r=1}^{n} \sum_{i,j\in\mathcal{A}_k} \pi_i|\rho(i)\rho(j)|1(|\rho(i)|,|\rho(j)|>L)_1 p_{ij}^{(r)} = 0, \quad \forall k.$$

We truncate as follows

$$\overline{\rho}^M(i) = \begin{cases} \rho(i) & ,i<M \\ c_k & ,i\geq M, i\in\mathcal{A}_k \end{cases}.$$

$\underline{\rho}^M$, $\overline{\mu}^M$, $\underline{\mu}^M$, $\overline{\phi}^M$, and $\underline{\phi}^M$ are defined as before.

The first sum in eq. 6 can be decomposed as

$$\begin{aligned}
&\sum_{i,j} \pi_i(\rho(i)-\mu)(\rho(j)-\mu)\sum_{r=1}^{n} {}_1p_{ij}^{(r)} = \\
&\quad \sum_{k=1}^{K}\sum_{i,j\in\mathcal{A}_k} \pi_i(\rho(i)-\mu)(\rho(j)-\mu)\sum_{r=1}^{n} {}_1p_{ij}^{(r)} \\
&\quad + \sum_{k,l\in\{1,\ldots,K\},k\neq l}\ \sum_{i\in\mathcal{A}_k,j\in\mathcal{A}_l} \pi_i(\rho(i)-\mu)(\rho(j)-\mu)\sum_{r=1}^{n} {}_1p_{ij}^{(r)}. \qquad (8)
\end{aligned}$$

The first condition ensures that the cross terms on the right are insignificant. Therefore we can work with each subset separately. We will argue as in the proof of theorem 4.1 to show $\mathrm{var}(\underline{\phi}_n^M) \to 0$. The analogue of eq. 7 for each of the remaining sums reads

$$\sum_{i,j\in\mathcal{A}_k} \pi_i[\underline{\rho}^M(i)\underline{\rho}^M(j) - \underline{\mu}^M(\underline{\rho}^M(i)+\underline{\rho}^M(j)) + (\underline{\mu}^M)^2]\sum_{r=1}^{n} {}_1p_{ij}^{(r)}.$$

Assume ρ is bounded. After dividing by $Q_{11}^{(n)}/\pi_1$, the second and third terms are $O(\underline{\mu}^M)$ as $\underline{\mu}^M \to 0$ by lemma 3.7. Since $\underline{\mu}^M \to 0$ as $M\to\infty$, these terms tend to 0 as $M\to\infty$ uniformly in n.

For the first term, we argue exactly as in the proof of theorem 4.1 that condition 1, together with lemma 3.8 implies that this term, when divided by $Q_{11}^{(n)}/\pi_1$ goes to 0 as $n\to\infty$. Applying lemma 3.2 while observing lemma 3.4 (ii) and (iii), we conclude that $\lim_{M\to\infty}\lim_{n\to\infty}\mathrm{var}(\underline{\phi}_n^M) = 0$, and by eq. (5), also that $\lim_{M\to\infty}\lim_{n\to\infty}\mathrm{var}(\overline{\phi}_n^M) = \lim_{n\to\infty}\mathrm{var}(\phi_n)$.

To calculate $\mathrm{var}(\overline{\phi}_n^M)$, rewrite eq. 6 for $\overline{\rho}^M$. We again omit the cross sums:

$$\sum_{i,j\in\mathcal{A}_k} \pi_i[(\overline{\rho}^M(i)-c_k)(\overline{\rho}^M(j)-c_k)-(\overline{\mu}^M-c_k)(\overline{\rho}^M(i)+\overline{\rho}^M(j)-2c_k)+(\overline{\mu}^M-c_k)^2]\sum_{r=1}^{n} {}_1p_{ij}^{(r)}.$$

The first two sums will go to zero due to truncation, boundedness of ρ, and by lemma 3.8, when dividing by $Q_{11}^{(n)}/\pi_1$. The last term will read

$$(\overline{\mu}^M-c_k)^2\frac{1}{Q_{11}/\pi_1}\sum_{i,j\in\mathcal{A}_k}\pi_i\sum_{r=1}^{n} {}_1p_{ij}^{(r)} \to \pi_{\mathcal{A}_k}^{\infty}(\overline{\mu}^M-c_k)^2$$

by lemma 3.7 and the definition of $\pi_{\mathcal{A}_k}^{\infty}$. This concludes the proof when (ρ_n) is bounded.

When (ρ_n) is not bounded, we truncate by value, i.e. $\tilde{\rho}^L(i) = \rho(i)1(\rho(i)\le L)$,

$\tilde{\mu}^L = E[\tilde{\rho}^L_n]$, $\underset{\sim}{\rho}^L(i) = \rho(i) - \tilde{\rho}^L(i)$, and $\underset{\sim}{\mu}^L = E[\underset{\sim}{\rho}^L_n]$. Also define

$$\tilde{\phi}^L_n = \frac{(\tilde{\rho}^L_0 + \ldots + \tilde{\rho}^L_n) - (n+1)\tilde{\mu}^L}{\sqrt{2R^{(n)}_{11}/\pi_1}},$$

$$\underset{\sim}{\phi}^L_n = \phi_n - \tilde{\phi}^L_n.$$

We also parition $\underset{\sim}{\rho}^L_n$ as $\sum_{k=1}^K \underset{\sim}{\rho}^L_n 1(\underset{\sim}{\rho}^L_n \in \mathcal{A}_k)$. Define

$${}_k\underset{\sim}{\phi}^L_n = \frac{\underset{\sim}{\rho}^L_0 1(\underset{\sim}{\rho}^L_0 \in \mathcal{A}_k) + \ldots + \underset{\sim}{\rho}^L_n 1(\underset{\sim}{\rho}^L_n \in \mathcal{A}_k) - (n+1)E(\underset{\sim}{\rho}^L_0 1(\underset{\sim}{\rho}^L_0 \in \mathcal{A}_k))}{\sqrt{2R^{(n)}_{11}/\pi_1}}.$$

We can express $\sum_{r=1}^n \text{cov}(\underset{\sim}{\rho}^L_0 1(\underset{\sim}{\rho}^L_0 \in \mathcal{A}_k), \underset{\sim}{\rho}^L_r 1(\underset{\sim}{\rho}^L_r \in \mathcal{A}_k))$ by writing eq. 6 for $\underset{\sim}{\rho}^L(i)1(\underset{\sim}{\rho}^L(i) \in \mathcal{A}_k)$, and argue as there that the second term has limit 0 as $n \to \infty$ when divided by $Q^{(n)}_{11}/\pi_1$. The first term also has limit 0 due to the assumed condition 3. We appeal again to lemma 3.4 (ii) and (iii), and lemma 3.2 to argue that $\lim_{L\to\infty} \lim_{n\to\infty} \text{var}({}_k\underset{\sim}{\phi}^L_n) = 0$. Applying eq. (5), we conclude that

$$\lim_{L\to\infty} \lim_{n\to\infty} \text{var}(\underset{\sim}{\phi}^L_n) = \lim_{L\to\infty} \lim_{n\to\infty} \text{var}\left(\sum_{k=1}^K {}_k\underset{\sim}{\phi}^L_n\right) = 0.$$

One more application of eq. (5) gives $\lim_{L\to\infty} \lim_{n\to\infty} \text{var}(\tilde{\phi}^L_n) = \lim_{n\to\infty} \text{var}(\phi_n)$. We conclude

$$\lim_{n\to\infty} \frac{\text{var}(\sum_{r=1}^n \rho_i)}{R^{(n)}_{11}/\pi_1} = \lim_{M\to\infty} \lim_{n\to\infty} \text{var}(\overline{\phi}^M_n) =^a \lim_{M\to\infty} \sum_{k=1}^K \pi^\infty_{\mathcal{A}_k} (\overline{\mu}^M - c_k)^2 = \sum_{k=1}^K \pi^\infty_{\mathcal{A}_k} (\mu - c_k)^2,$$

where (a) follows from the bounded version of the theorem proved above.

To prove the remark, consider $\mathcal{A}_k \cup \mathcal{A}_l$ as one subset. We can safely ignore the cross terms in eq. 8, without needing to use condition 1 for the pair $\mathcal{A}_k, \mathcal{A}_l$. We do not use condition 1 in the remaining part of the proof.

All that remains is to note

$$(\overline{\mu}^M - c_k)^2 \frac{1}{Q_{11}/\pi_1} \sum_{i,j \in \mathcal{A}_k \cup \mathcal{A}_l} \pi_i \sum_{r=1}^n {}_1p^{(r)}_{ij} \to \pi^\infty_{\mathcal{A}_k \cup \mathcal{A}_l} (\overline{\mu}^M - c_k)^2$$

where $\pi^\infty_{\mathcal{A}_k \cup \mathcal{A}_l} = \pi^\infty_{\mathcal{A}_k} + \pi^\infty_{\mathcal{A}_l}$.

Now we illustrate the use of these tools with some examples. The first one uses theorem 4.1 directly, while the last two examples use theorem 4.2.

5. Example 1: Longest queue first with mixed heavy and light tailed inputs

This example replicates the conclusion in [12] that long range dependence might spread under LQF scheduling in a parallel queue setting, using a general technique based on the theorems of the preceding section.

There is a single server of rate $R \in \mathbb{N}$ with 2 parallel queues. The queues are fed by independent random processes, each modeled by a discrete time, countable state Markov chain. As an example, we investigate the scenario where X_1 is i.i.d. with heavy tailed ($\text{var}(X_1) = \infty$) arrival distribution on $\mathbb{N}$. $X_2 \in \mathbb{N}$ is either an i.i.d. process with light tailed ($\text{var}(X_2) < \infty$) arrivals or X_2 can be a finite state $\mathbb{N}$-valued Markov chain in stationarity. We assume $E[X_1(0)] + E[X_2(0)] < R$.

Let $Q_1(n), Q_2(n)$ be the stationary queue lengths. We assume that the queue is work conserving, and moreover the scheduling decision at time n (number of packets to be served from each queue at time slot n) is a function of $(Q_1(n), Q_2(n))$, the queue sizes at time n. Given such a scheduling strategy, it is easily verified that $(X_1(n), X_2(n), Q_1(n), Q_2(n))$ is a countable state Markov chain.

Lemma 5.1. $(X_1(n), X_2(n), Q_1(n), Q_2(n))$ *is positive recurrent.*

Proof. $E[X_1(0)] + E[X_2(0)] < R$ implies that the queue process $(Q_1(n), Q_2(n))$ is positive recurrent. Pick $M_1 > 0$ and define the set $S_1 = \{Q_1(n) + Q_2(n) < M_1\}$. The return times to this set have finite mean (say ν). Also define $S_2 = \{X_1(n) + X_2(n) < M_2\}$ (or in the case X_2 is a finite state chain, $S_2 = \{X_1(n) < M_2\}$) where M_2 is large enough such that S_2 is nonempty. $S_1 \cap S_2$ is a nonempty compact set. We claim the return times to this set have a finite mean. Since $1_n(S_2)$ is i.i.d, there is a positive probability (say at least p) of visiting S_2 each time there is a visit to S_1 (independent of previous visits). It is easily seen that the mean return time to $S_1 \cap S_2$ is at most ν/p.

We will look at long range dependence through the Hurst indices of the busy-idle processes of the queues. Let (X_1, Q_1') be the Markov chain if all the capacity were to be allocated to queue 1. Denote by $1(Q_1'(n) = 0)$, the busy-idle process of this queue. We know that the busy periods of Q_1' have infinite variance (see e.g. [3] theorem 8.10.3).

Therefore both the Markov chain (X_1, Q_1') and the function $1(Q_1'(n) = 0)$ are LRD (see the introduction). (X_2, Q_2'), similarly defined, is a short range dependent chain.

Lemma 5.2. *$(X_1(n), X_2(n), Q_1(n), Q_2(n))$ is LRD.*

Proof. Consider the chain $(X_1(n), Q_1'(n), X_2(n), Q_2'(n))$. This chain is LRD because it is a combination of two independent chains (X_1, Q_1') and (X_2, Q_2'), one of which we assume to be LRD. Let t_1 be the return time to a nonempty compact set $S_1 = \{X_1(n), Q_1(n), X_2(n), Q_2(n) < M\}$. Similarly t_2 is the return time to the set $S_2 = \{X_1(n), Q_1'(n), X_2(n), Q_2'(n) < M\}$. Since $Q_1'(n) \leq Q_1(n)$ and $Q_2'(n) \leq Q_2(n)$, t_1 stochastically dominates t_2, and therefore $(X_1(n), X_2(n), Q_1(n), Q_2(n))$ is also LRD.

The question we want to ask then is whether $1(Q_2(n) = 0)$, the busy-idle process of the second queue (fed by short range dependent traffic), is also long range dependent.

$\rho_n := 1(Q_2(n) = 0)$ is an L_2 function of the chain $(X_1(n), X_2(n), Q_1(n), Q_2(n))$. Take $c = 0$ in theorem 4.1. $\mathcal{H} = \{X_1(n), X_2(n), Q_1(n), Q_2(n) \leq R\}$. Condition 2 holds trivially for bounded functions. Thus we are left with having to check the condition

$$\lim_{n\to\infty} \frac{1}{Q_{11}^{(n)}/\pi_1} \sum_{i,j:Q_{2,j}=0,Q_{2,i}=0} \pi_i \sum_{r=1}^{n} {}_{\mathcal{H}}p_{ij}^{(r)} = 0.$$

To see why this is true, note that $\sum_{i,j:Q_{2,j}=0 Q_{2,i}=0} \pi_i \sum_{r=1}^{\infty} {}_{\mathcal{H}}p_{ij}^{(r)}$ is bounded above by 1 plus the stationary time spent in the states $\{Q_2 = 0\}$ before the chain visits $\mathcal{H}$. Note that the length of an idle period for Q_2 has finite expectation. Also note, if an idle period begins at time $n+1$, this implies due to the LQF policy that $Q_1(n) \leq R$, $Q_2(n) \leq R$, $X_1(n) \leq R$, and $X_2(n) \leq R$. Thus between successive idle periods of Q_2, the chain must visit $\mathcal{H}$. The stationary expected time spent in $\{Q_2 = 0\}$ without visiting $\mathcal{H}$ is therefore finite. Since $Q_{11}^{(n)} \to \infty$ (1), the above limit holds. Using theorem 4.1, we conclude that $1(Q_2(n) = 0)$ has the same Hurst index as the chain $(X_1(n), X_2(n), Q_1(n), Q_2(n))$.

The advantage of this approach is that in general the input processes need not be i.i.d. Dependencies can easily be modeled, as long as the sources can be represented as countable state Markov.

6. Example 2: Compressing a long range dependent renewal process

This section provides an alternative proof for the result in [13].

Let (X_n) be a discrete, stationary, ergodic renewal process. We begin by introducing the function

$$\rho_n(X_{-\infty}^n) = -\log P(X_n | X_{-\infty}^{n-1}),$$

which is of central importance to coding theory. The behavior of (ρ_n) restricts the minimum code length of lossless compression algorithms by the following lemma, [1], which is also proved in [10].

Lemma 6.1. (Barron's Lemma.) *Given* $\{c(n), n \geq 1\}$*, positive constants with* $\sum_n 2^{-c(n)} < \infty$*, we have*

$$L_n(X_1^n) \geq -\log P(X_1^n | X_{-\infty}^0) - c(n), \text{ eventually, a.s. } . \tag{9}$$

Here $L_n(X_1^n)$ is the code length for the first n symbols of the source for some lossless coding algorithm that produces bit strings. $c(n)$ can be made logarithmic in n.

By the ergodic theorem, the limit of $\frac{1}{n}\sum_{i=1}^n \rho_i$ as $n \to \infty$ exists a.s. and equals $\eta := E[-\log P(X_1 | X_{-\infty}^0)]$, i.e. the entropy rate of (X_n). This implies the following well known first order converse source coding theorem for such sources.

Theorem 6.1.

$$\liminf_n \frac{1}{n} L_n(X_1^n) \geq \eta, \text{ a.s. } .$$

Lemma 6.1 is strong enough to permit second order refinements to theorem 6.1 once we know more about the process (ρ_n). For example, in [10], it is shown that for certain short range dependent classes of sources (e.g. finite state Markov chains), and appropriate coding schemes (e.g. Lempel-Ziv coding), $(L_n - n\eta)$ satisfies a central limit theorem.

Here, we will prove a second order converse source coding theorem, stating that the bit length process (L_n) will eventually dominate a long range dependent process the growth of whose variance is identical to that of (X_n), so that, in particular, it has the same Hurst exponent as (X_n). The proof relies on our general theorem 4.2. This result provides partial theoretical justification to existing empirical work in the field of variable bit-rate (VBR) video traffic ([2, 8, 14, 7] to cite a few). A conclusion resulting from this work is that long range dependence is omnipresent in VBR video traffic, and persists across a wide variety of codecs. Combined with these observations, the result backs the intuition that for many information sources long range dependence persists under compression.

Theorem 6.2. *Let (X_n) be an aperiodic, long range dependent, stationary, ergodic renewal process. Then, there exists a long range dependent random process (γ_n) such that*

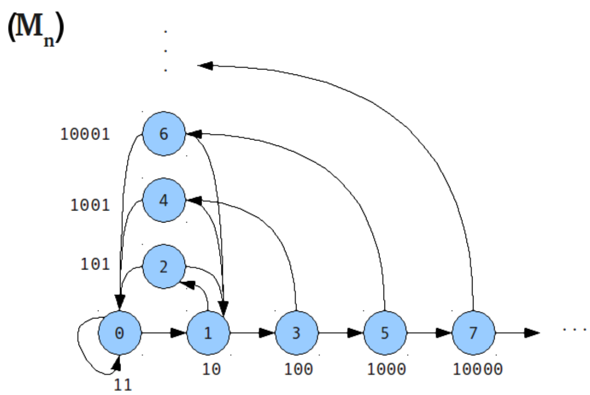
$$L_n(X_1^n) \geq \gamma_n, \text{ eventually, a.s.}$$

for all uniquely decodable source codes. Moreover, (γ_n) has the same Hurst index as (X_n).

Proof. This immediately follows from Barron's lemma once we show (ρ_n) are LRD with the same Hurst index as (X_n). This will follow from theorem 4.2 if we can set up (ρ_n) as a function of a Markov chain.

We construct the following Markov chain (M_n) from the renewal process (X_n) (fig. 1):

- $M_n \in \{0, 1, 2, 3, \ldots\}$.
- $\{M_n = 0\} = \{X_{n-1}^n = 11\}$.
- For $k \in \{1, 2, \ldots\}$
  - $\{M_n = 2k - 1\} =$
    $\{X_n = 0$ and k zeros since last arrival $\}$,
  - $\{M_n = 2k\} =$
    $\{X_n = 1$ and k zeros since last arrival in $X_n\}$.



X_n	1	1	0	0	1	0	0	0	0	1	1	1	0	0	0	1	0	0	0	0	0
M_n	-	0	1	3	4	1	3	5	7	8	0	0	1	3	5	6	1	3	5	7	9

FIGURE 1: Construction of the Markov chain, with an example sequence showing the correspondence with X_n

We establish some notation:

$$\begin{aligned}
(X_n), &\quad \text{stationary renewal process,} \\
&\quad \text{interval-arrival lengths having the law of } T+1; \\
f_T(k) &:= P(T = k); \\
F_T(k) &:= P(T \leq k); \\
\rho_n(X_{-\infty}^n) &:= -\log P(X_n | X_{-\infty}^{n-1}); \\
\eta &:= E[\log P(X_1 | X_{-\infty}^0)].
\end{aligned}$$

One can easily check $\rho_n = \rho(M_n)$, with

- $\rho(0) = -\log f_T(0)$,
- $\rho(2k-1) = -\log P(T > k-1 | T \geq k-1)$,
- $\rho(2k) = -\log P(T = k | T \geq k)$.

We verify:

Lemma 6.2. *ρ_n is an L_2 function of M_n.*

Proof. Let π_i be the stationary distribution of (M_n). Note that $\pi_i > 0 \implies \rho_i < \infty$.

We want to prove

$$\sum \rho(i)^2 \pi_i < \infty.$$

Note that $\pi_{2k+1} = \pi_{2k-1} P(T > k | T \geq k)$, and $\pi_{2k} = \pi_{2k-1} P(T = k | T \geq k)$ for $k = 1, 2, \ldots$. This gives

$$\begin{aligned}
\sum \rho(i)^2 \pi_i &= \pi_0 \rho(0)^2 + \pi_1 \rho(1)^2 \\
&\quad + \sum_{k=1}^{\infty} \pi_{2k-1} P(T = k | T \geq k) \log^2 P(T = k | T \geq k) \\
&\quad + \sum_{k=1}^{\infty} \pi_{2k-1} P(T > k | T \geq k) \log^2 P(T > k | T \geq k), \\
\pi_0 \rho(0)^2 &= (\sum_{k=1}^{\infty} \pi_{2k}) f_T(0) \log^2 f_T(0), \\
\pi_1 \rho(1)^2 &= (\sum_{k=1}^{\infty} \pi_{2k}) (1 - f_T(0)) \log^2 (1 - f_T(0)).
\end{aligned}$$

Since the $p \log^2 p$ terms are bounded above by 1, $\sum \rho_i^2 \pi_i \leq 4$.

Now, to apply theorem 4.2 we partition into 3 sets as follows: $\mathcal{A}_1 = \{i > 0 : i \text{ even}\}$, $\mathcal{A}_2 = \{0\} \cup \{i \text{ odd} : \rho(i) \leq -\log(1 - \epsilon_i)\}$, and $\mathcal{A}_3 = \{i \text{ odd} : \rho(i) > -\log(1 - \epsilon_i)\}$. Here we will will chose $\epsilon_i \downarrow 0$ later. Take $c_1 = c_2 = c_3 = 0$ and $\mathcal{H} = 1$ in that theorem. By the remark to the theorem, we don't need condition 1. We will check conditions 2 and 3 of theorem 4.2 for each of the sets.

When $i, j \in \mathcal{A}_1$ notice ${}_1p_{ij}^{(r)} = 0$, so both conditions hold automatically. For $i, j \in \mathcal{A}_2$, condition 2 holds due to remark no. 2 because the limit of $\rho(i)$ as $i \to \infty$ is zero, and condition 3 holds because ρ is bounded on this set. Thus we focus on $i, j \in \mathcal{A}_3$. Define $\rho(i) =: -\log(1 - \tilde{\epsilon}_i)$. Let subsequence $\{i_k\} = \mathcal{A}_3$. We have $\tilde{\epsilon}_{i_k} \geq \epsilon_{i_k}$.

$\pi_{i_k} \le \pi_1 \prod_{l=1}^{k}(1-\tilde{\epsilon}_{i_k})$, and $\sum_1^\infty {}_1p_{i_k i_j}^{(r)} = \pi_{i_j}/\pi_{i_k}$. We have

$$\begin{aligned}
&\sum_i \rho(i)\pi_i \sum_j \rho(j) \sum_{r=1}^{n} {}_1p_{ij}^{(r)} \\
&\le \sum_k \prod_{l=1}^{k}(1-\tilde{\epsilon}_{i_l})(-\log(1-\tilde{\epsilon}_{i_k})) \sum_{j>k} -\log(1-\tilde{\epsilon}_{i_j}) \prod_{l=k+1}^{j} (1-\tilde{\epsilon}_{i_l}) \\
&= \sum_j \sum_{k<j} (1-\tilde{\epsilon}_{i_k}) \log(1-\tilde{\epsilon}_{i_k})(1-\tilde{\epsilon}_{i_j}) \log(1-\tilde{\epsilon}_{i_j}) \prod_{l=1, l\neq k,j}^{j} (1-\tilde{\epsilon}_{i_l}) \\
&< \sum_j j \prod_{l=3}^{j} (1-\tilde{\epsilon}_{i_l}).
\end{aligned}$$

We can easily choose $\epsilon_i \downarrow 0$ such that this is finite. Dividing by $Q_{11}^{(n)}$, both conditions in theorem 4.2 will be satisfied.

7. Example 3: Long range dependence in financial time series

Let $(P_n, -\infty < n < \infty)$ be the price of some financial asset, and $X_n = \log P_n$. It is an established assumption that the log returns, $r_n = X_n - X_{n-1}$ is well modeled by a martingale difference process. Such a model accounts for the fact that the log returns exhibit little correlation. Nevertheless, it is also a widely observed fact that some instantaneous functions of the log returns, such as $|r_n|^d$, exhibit long memory. (see e.g. [6])

The popular approach to modeling this behavior has been to explicitly write the dependence of the absolute log returns into the statistical description of the model. The result is the various long-memory autoregressive conditional heteroskedasticity (ARCH) process models of financial time series. ([9] for an example)

We want to show in this example that, given a martingale difference sequence (r_n) that can be represented as a function of a long range dependent Markov chain, the outcome that $|r_n|^d$ will exhibit long range dependence should not be considered suprising.

We want to illustrate this with a very simple example based on Mandelbrot's model for wheat prices ([11]). We should note that this simple model is for purposes of illustration only, and does not account for all known properties of financial time series. For instance, it has been observed in many situations that (r_n) has a finite variance, despite having a polynomially decaying marginal distribution. The (r_n) in this example has infinite variance. Nevertheless, the proof scheme used here to establish the long range dependence of $|r_n|^d$ should be applicable much more generally.

Let (W_n) be a stationary random process which models the weather. (W_n) can take on 3 values: good, bad, and neutral $\{g, b, n\}$. The length of a good period, T, (number of consecutive good days) has the same distribution as the length of a bad, or a neutral period. Let $P(T \geq t) = t^{-\alpha}$. T has finite mean but infinite variance (i.e. $1 < \alpha \leq 2$). A good or bad period is followed necessarily by a neutral period. A neutral period is followed by a good or bad period with equal probabilities.

Let $\hat{X}_n$ be the fundamental (log) price of the asset (which can be thought of as summarizing exogenous variables that affect the real price). $\hat{X}_n$ varies as follows: increases by 1 for every good day, decreases by 1 for every bad day, and stays the same for every neutral day. The market calculates the real (log) price by projecting the expected future fundamental price: $X_n = \lim_{t\to\infty} E[\hat{X}_{n+t}|\hat{X}_{-\infty}^n]$.

By construction, (r_n) itself is a martingale difference sequence. We will now show that $\rho_n = |r_n|^d$ is LRD with Hurst index $\frac{1}{2}(3-\alpha)$. ($0 < d < \alpha/2$ for $\mathrm{var}(\rho_0)$ to be finite.)

It can be verified that (also see the calculations in Mandelbrot's original paper [11]) X_n changes as follows: jumps by $E[T]$ on the first good day. Jumps by $-E[T]$ on the first bad day. Increases by $E[T|T \geq t] - E[T|T \geq t-1]$ on the t^{th} good day ($t \geq 2$). Decreases by $E[T|T \geq t] - E[T|T \geq t-1]$ on the t^{th} bad day. The first neutral following t good days decreases X_n by $E[T|T \geq t] - t$. The first neutral following t bad days increases X_n by $E[T|T \geq t] - t$.

Let $J_n = 1$(there is a transition at time n). Let $T_n := \inf_t\{t \geq 0 : W_{n-t-1} \neq W_{n-t-2}\}$ be the number of days since the last transition (0 on the first day following).

Then $M_n = (W_n, J_n, T_n)$ is a countable state, long range dependent Markov chain, with Hurst index $\frac{1}{2}(3-\alpha)$. Moreover, $\rho_n = |r_n|^d$ is a function of M_n:

- $\rho(\{g,b\}, 0, t) = (E[T|T \geq t+2] - E[T|T \geq t+1])^d$
- $\rho(\{n\}, 0, \cdot) = 0$
- $\rho(\{g,b\}, 1, \cdot) = (E[T])^d$
- $\rho(\{n\}, 1, t) = (E[T|T \geq t+1] - (t+1))^d$

Lemma 7.1.

$$E[T|T \geq t+2] - E[T|T \geq t+1] \to \frac{\alpha}{\alpha - 1}, \qquad t \to \infty.$$

Proof.

$$P(T \geq s|T \geq t) = \frac{s^{-\alpha}}{t^{-\alpha}}, \qquad s \geq t$$

$$E[T|T \geq t+1] - E[T|T \geq t] = \sum_{s=t+1}^{\infty} P(T \geq s|T \geq t+1) - P(T \geq s|T \geq t)$$

$$E[T|T \geq t+1] - E[T|T \geq t] = \left((t+1)^{\alpha} - t^{\alpha}\right) \sum_{s=t+1}^{\infty} s^{-\alpha} \to \frac{\alpha}{\alpha - 1}$$

since $\left((t+1)^{\alpha} - t^{\alpha}\right)/t^{\alpha-1} \to \alpha$ and $\frac{1}{\alpha-1}(t+2)^{-\alpha+1} = \int_{t+2}^{\infty} s^{-\alpha} ds < \sum_{s=t+1}^{\infty} s^{-\alpha} < \int_{t+1}^{\infty} s^{-\alpha} ds = \frac{1}{\alpha-1}(t+1)^{-\alpha+1}$.

Lemma 7.2.

$$E[T|T \geq t] - t \leq \frac{t}{\alpha - 1}.$$

Proof.

$$E[T|T \geq t] - t = \sum_{s=t}^{\infty} \frac{s^{-\alpha}}{t^{-\alpha}} \leq \int_{t}^{\infty} s^{-\alpha} ds = \frac{t}{\alpha - 1}.$$

We will utilize theorem 4.2 with $\mathcal{A}_1 = (\{g, b\}, 0, \cdot), \mathcal{A}_2 = (\{n\}, 0, \cdot), \mathcal{A}_3 = (\{g, b\}, 1, \cdot), \mathcal{A}_4 = (\{n\}, 1, \cdot)$. $c_1 = c_4 = \left(\frac{\alpha}{\alpha-1}\right)^d, c_2 = c_3 = 0$. $\mathcal{H} = (\cdot, \cdot, 0)$. We have

$$\begin{aligned} \operatorname{var}(\rho_0) \leq E\rho_0^2 &= \sum_i \pi_i \rho(i)^2 \\ &= \sum_{i \notin \mathcal{A}_4} \pi_i \rho(i)^2 + \sum_{i \in \mathcal{A}_4} \pi_i \rho(i)^2 \qquad \leq C + \sum_{t=1}^{\infty} \frac{1}{2} P(T = t) (\frac{t}{\alpha - 1})^{2d} < \infty \end{aligned}$$

by lemma 7.2. As $\rho(i)$ is bounded when $i \notin \mathcal{A}_4$, the contribution to the sum is a constant C. We also used the fact that if $i = (\{n\}, 1, t-1)$, then $\pi_i = P(W_{-t} = n)P(T = t) = \frac{1}{2}P(T = t)$.

We need to first show that condition 1 holds:

$$\lim_{n \to \infty} \frac{1}{Q_{11}^{(n)}} \sum_{r=1}^{n} \sum_{i \in \mathcal{A}_k, j \in \mathcal{A}_l} \pi_i |\rho(i) - \mu| |\rho(j) - \mu|_{\mathcal{H}} p_{ij}^{(r)} \to 0 \quad \forall k \neq l.$$

By inspection, the following transitions require visiting $\mathcal{H}$: (k, l) or $(l, k) = (1, 2), (1, 3), (2, 4), (3, 4)$. The sum is zero for these pairs. For (k, l) or $(l, k) = (1, 4), (2, 3)$, the condition is not needed due to the remark to theorem 4.2.

Condition 2 reads

$$\lim_{n\to\infty} \frac{1}{Q_{11}^{(n)}} \sum_{i,j\in\mathcal{A}_k} \pi_i(\rho(i)-c_k)(\rho(j)-c_k) \sum_{r=1}^{n} {}_{\mathcal{H}}p_{ij}^{(r)} = 0 \quad \forall k.$$

For $k = 3, 4$, ${}_{\mathcal{H}}p_{ij}^{(r)} = 0$ because these states must go to $\mathcal{H}$ in one step. For $k = 1, 2$, we have chosen c_k such that $(\rho(i) - c_k) \to 0$ by lemma 7.1. The condition holds by remark no. 2.

Condition 3 also holds for $\mathcal{A}_1$, $\mathcal{A}_2$, and $\mathcal{A}_3$ because ρ is bounded on these sets. On $\mathcal{A}_4$, it holds because ${}_{\mathcal{H}}p_{ij}^{(r)} = 0$ as argued earlier. We finally have the conclusion:

$$\lim_{n\to\infty} \frac{\sum_{r=1}^{n} \mathrm{cov}(|r_0|^d, |r_n|^d)}{Q_{11}^{(n)}/\pi_1} = \sum_{k=1}^{K} \pi_k^{\infty}(\mu - c_k)^2 > 0.$$

8. Conclusion

We have provided conditions under which the growth rate of the variance of a function of a Markov chain is identical to that of the chain itself. Although our results simplify certain proofs greatly, there is still considerable art in using them. One needs to first construct a suitable Markov chain for the problem. One also needs to choose the parameters in the theorems carefully. Although the answer will ultimately be the same, picking state 1, the partition $\{\mathcal{A}_k\}$ and $\{c_k\}$ appropriately can greatly reduce the amount of calculation required.

We don't have an answer for the case where $\rho(i)$ grows without bound as $i \to \infty$. In these cases (ρ_n) might possibly have a higher Hurst index than the chain. The proof of theorem 4.1 can provide insights for solving these questions.

The usefulness of the theorem was demonstrated by various examples in three diverse fields. Many more can be contemplated in areas like queue analysis in stochastic networks and agent based models in finance.

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References

[1] Barron, A. (1985). Logically smooth density estimation. *PhD thesis.* Stanford University Department of Electrical Engineering.

[2] Beran, J., Sherman, R., Taqqu, M. and Willinger, W. (1995). Long-range dependence in variable-bit-rate video traffic. *IEEE Transactions on Communications* **43,** 1566–1579.

[3] Bingham, N., Goldie, C. and Teugels, J. (1989). *Regular Variation.* Cambridge University Press.

[4] Carpio, K. and Daley, D. (2007). Long-range dependence of Markov chains in discrete time on countable state space. *Journal of Applied Probability* **44,** 1047–1055.

[5] Chung, K. (1967). *Markov chains with Stationary Transition Probabilities.* Springer-Verlag, Berlin.

[6] Cont, R. (2001). Empirical properties of asset returns: stylized facts and statistical issues. *Quantitative Finance* **1,** 223–236.

[7] Fitzek, F. and Reisslein, M. (2001). Mpeg4 and h. 263 video traces for network performance. *IEEE Network* **15,** 40–54.

[8] Garrett, M. and Willinger, W. (1994). Analysis, modeling and generation of self-similar VBR video traffic. *ACM SIGCOMM Computer Communication Review* **24,** 269–280.

[9] Giraitis, L., Robinson, P. and Surgailis, D. (2000). A model for long memory conditional heteroscedasticity. *Annals of Applied Probability* **10,** 1002–1024.

[10] Kontoyiannis, I. (1997). Second-order noiseless source coding theorems. *IEEE Transactions on Information Theory* **43,** 1339–1341.

[11] Mandelbrot, B. (1966). Forecasts of future prices, unbiased markets, and" martingale" models. *Journal of Business* **39,** 242–255.

[12] Markakis, M., Modiano, E. and Tsitsiklis, J. (2009). Scheduling policies for single-hop networks with heavy-tailed traffic. *Proceedings of the 47th annual Allerton conference on Communication, control, and computing* 112–120.

[13] Oğuz, B. and Anantharam, V. (2010). Compressing a long range dependent renewal process. *Information Theory Proceedings (ISIT), 2010 IEEE International Symposium on* 1443 –1447.

[14] Rose, O. (1995). Statistical properties of MPEG video traffic and their impact on traffic modeling in ATM systems. *Conference on Local Computer Networks: Proceedings* 397.