Regulating functions on partially ordered sets*

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Abstract

We study the so-called Skorokhod reflection problem (SRP) posed for real-valued functions defined on a partially ordered set (poset), when there are two boundaries, considered also to be functions of the poset. The problem is to constrain the function between the boundaries by adding and subtracting nonnegative nondecreasing (NN) functions in the most efficient way. We show existence and uniqueness of its solution by using only order theoretic arguments. The solution is also shown to obey a fixed point equation.

When the underlying poset is a σ-algebra of subsets of a set, our results yield a generalization of the classical Jordan-Hahn decomposition of a signed measure. We also study the problem on a poset that has the structure of a tree, where we identify additional structural properties of the solution, and on discrete posets, where we show that the fixed point equation uniquely characterizes the solution. Further interesting posets we consider are the poset of real n-vectors ordered by majorization, and the poset of $n \times n$ positive semidefinite real matrices ordered by pointwise ordering of the associated quadratic forms.

We say a function on a poset is of *bounded variation* if it can be written as the difference of two NN functions. The solution to the SRP when the upper and lower boundaries are the identically zero function corresponds to the most efficient or minimal such representation of a function of bounded variation. Minimal representations for several important functions of bounded variation on several of the posets mentioned above are determined in this paper.

Keywords and phrases. Skorokhod reflection, partially ordered sets, monotonicity, Jordan decomposition, signed measures, majorization, positive semidefinite matrices, bounded variation

AMS 2000 *subject classifications.* Primary 06A06,06B35,26A45,47H09,47H10 ; secondary 06B23,15A57,60K25,60J65

1 Introduction

The topic of this paper is an extension to a problem originated by A. V. Skorokhod [16]. The problem was to specify equations for the description of a random motion that was constrained to stay in a region, say the positive real line. Without constraints, the random motion was a diffusion [19] – a generalization of the famous Brownian motion [14] – that

*A version of this paper was presented at the Applied Probability Workshop, in the Mathematisches Forschungsinstitut Oberwolfach, December 2003 (organized by V. Schmidt, A. Hordijk and F. Baccelli); the authors are grateful for the invitation which gave them the opportunity to make substantial progress in finalizing this work in the uniquely wonderful research atmosphere at Oberwolfach.

†Research supported by DARPA grant N66001-00-C-8062, ONR grant N00014-1-0637 and NSF grant ECS 0123512.

‡This work was supported in part by NSF grant ANI-9903495, the INTAS-00-265 project, and a Carathéodory research award.

obeyed a "stochastic evolution equation" (stochastic differential equation [13] in a more modern terminology). With constraints, the description of the evolution equation became harder. Skorokhod observed that, instead of changing the evolution equation by introducing boundary conditions, he could, instead, apply a transformation to the unconstrained motion in order to obtain the constrained one. This observation has proved very fertile in the study of generalizations, in many directions, of the original problem of Skorokhod. Being a transformation from functions to functions, it was clear, at the outset, that the transformation introduced by Skorokhod to solve his reflection problem had nothing to do with the randomness of the motion, which somewhat explains its wide ranging significance.

To set the stage for our ideas, we briefly recall here how the classical problem is posed. Let $\mathbb{R}_+ = [0, \infty)$, and x a continuous function from $\mathbb{R}_+$ into $\mathbb{R}$. This function represents the unconstrained or free motion. To constrain it in the region $\mathbb{R}_+$, one adds another function, ℓ, so that $z(t) := x(t) + \ell(t) \geqslant 0$, for all $t \geqslant 0$, in such a way that $\ell(t)$ does not change with time t, unless $z(t) = 0$, and when $z(t) = 0$, the quantity $\ell(t)$ can only increase (i.e., it "pushes" toward the interior of $\mathbb{R}_+$). More precisely, the original Skorokhod reflection problem (SRP)[1] seeks a nonnegative nondecreasing (NN) function $\ell : \mathbb{R}_+ \to \mathbb{R}$ so that $z(t) = x(t) + \ell(t) \geqslant 0$, for all $t \geqslant 0$, such that

$$\int_{[0,\infty)} \mathbf{1}(z(t) > 0) d\ell(t) = 0.$$

If we also assume $x(0) \geqslant 0$, i.e., that the motion starts within the region we are interested in, then there should be no initial "push", namely, $\ell(0) = 0$. The integral above is a Lebesgue-Stieltjes integral against the function ℓ, and the integrand, $\mathbf{1}(z(t) > 0)$, is the indicator (or characteristic) function of the set $\{t : \; z(t) > 0\}$, namely the function that assigns value 1 to every t with $z(t) > 0$ and 0 otherwise. Skorokhod saw that ℓ is explicitly given by the formula

$$\ell(t) = \sup_{0 \leqslant s \leqslant t} (-x(s)) \vee 0, \quad t \geqslant 0, \tag{1}$$

where $a \vee b := \max(a, b)$, for reals a, b. The transformation $x \mapsto \ell$ is often called the Skorokhod reflection mapping, in this simplest case.[2]

An equivalent formulation of the problem above turned out to be this : consider the set of all NN functions $\ell : \mathbb{R}_+ \to \mathbb{R}$ such that $x(t) + \ell(t) \geqslant 0$ for all $t \geqslant 0$. This set is naturally partially ordered by pointwise ordering of functions and the problem is to find a minimal element in this set. It is easily seen that there is indeed a minimal such element and that this is the ℓ defined by (1). This equivalent formulation is the one that will form the starting point of this paper.

Building upon the work of Skorokhod, various researchers extended the problem in various directions. We indicatively mention some. For instance, it was necessary to establish such a Skorokhod reflection mapping to constrain multidimensional (perhaps random) motions so that they stay within a domain $G \subseteq \mathbb{R}^d$. This became (and still is) the primary concern of people working in areas such as constrained multidimensional stochastic differential equations, see e.g., [18, 21], and stochastic (queueing) networks, see e.g., [7]. Apparently, it was Anderson and Orey [1] who first generalized the SRP to the half-space $R_+ \times \mathbb{R}^{d-1}$ of the Euclidean d-dimensional space, with normal direction of reflection on the boundary. Indeed, when one passes from one to many dimensions, it is necessary not only

[1]The phrase "Skorokhod reflection problem" will often be abbreviated to "SRP" in the sequel. The meaning of this phrase will depend on the context during the initial discussion, and will settle down after Definition 1 in Section 2.

[2]Occasionally in the literature the Skorokhod reflection mapping refers to the transformation $x \mapsto x + \ell$.

to specify the constraining region G but also the directions, defined locally on the boundary of G, along which the reflection takes place, see e.g., [20]. We shall not enter into these details here, this being beyond the scope of our paper. It became clear in the course of such generalizations that solving the SRP for multidimensional functions raises many challenges, in that the solution may fail to exist or to be unique, see [2].

Another type of generalization in the literature proceeded by relaxing the continuity assumption on the free motion x. This was necessary because several of the stochastic processes considered in applications are discontinuous (e.g., sample paths of "Lévy processes", which are natural generalizations of Brownian motion and typically have highly discontinuous paths). The weakest type of assumption for which the problem has been studied, to-date, is that x have paths of "càdlàg" type[3], i.e., paths which have discontinuities of the first kind (jumps) and which are right-continuous everywhere, see e.g., [4, 5]. In these works it is shown that the Skorokhod reflection mapping, when it exists and is well-defined, enjoys several properties, such as the Lipschitz property; see also [3, 10, 22] for further properties of this mapping.

Specializing to dimension one again, let $\alpha < \beta$ be real numbers, let $x : \mathbb{R}_+ \to \mathbb{R}$ be such that $x(0) \in [\alpha, \beta]$ and consider the problem of finding a pair of NN functions $\ell, u : \mathbb{R}_+ \to \mathbb{R}_+$ such that $z(t) := x(t) + \ell(t) - u(t) \in [\alpha, \beta]$ for all $t \geqslant 0$, in such a way that (ℓ, u) is least. We assume that x is such that the domain of the problem is nonempty, i.e., there is at least one pair of NN functions (ℓ, u) such that $z(t) := x(t) + \ell(t) - u(t) \in [\alpha, \beta]$ for all t. It can be seen that the problem has a unique solution. Existence and uniqueness has been known for long time when x is continuous, see [6], or is càdlàg. Let us now observe that when β is very close to α (in the limit $\beta - \alpha = 0$) then $z(t) \equiv 0$, i.e., $x(t) \equiv u(t) - \ell(t)$. In other words, the problem becomes that of writing the function x as the difference of two NN functions, in the most efficient manner. From classical analysis [8] we know that the nonemptiness of the domain of the problem in the case where $\alpha = \beta = 0$ characterizes functions of *bounded variation*: these functions are precisely those that can be represented as the difference of two NN functions; this is part of the so-called *Jordan decomposition* [23]. Thus, for $\alpha < \beta$, one can view the problem posed in this paragraph as potentially leading to a generalization of the notion of a function of bounded variation.

The realization that the solution of the version of the SRP posed in the preceding paragraph depends entirely on monotonicity arguments provides the basic motivation for the following leap, which is the main concern of this paper: generalize the domain T over which x is defined (which so far has been the set $\mathbb{R}_+$) into a partially ordered set (poset). All the posets in this paper are assumed to possess a bottom (least) element, generally denoted by 0. The main result of Section 2 makes this formulation of the problem precise and constructs the solution to the problem by a recursive scheme using purely order theoretic ideas. There are two boundaries, considered also to be functions of the poset, and the problem is to constrain the function between the boundaries by adding and subtracting NN functions in the most efficient way. We say a function on a poset is of *bounded variation* if it can be written as the difference of two NN functions. The solution to the SRP when the upper and lower boundaries are the identically zero function then corresponds to the most efficient or minimal such representation of a function of bounded variation. This provides a link, for a wide range of functions on posets, to the classical Jordan decomposition.

A particularly interesting case appears when T is a σ-algebra of subsets of a set S. Here, in the case of general boundaries, one gets an extension of the classical Jordan decomposition for signed measures [23], which itself corresponds to the special case where the boundaries

[3]This is an acronym for the French phrase "*c*ontinu *à* *d*roite avec des *l*imites *à* *g*auche".

are the identically zero function. This is discussed in Section 4. We also study the problem on a poset that has the structure of a tree, where we identify additional structural properties of the solution, and on discrete posets, where we show that the fixed point equation uniquely characterizes the solution. The former is done is Subsection 2.3 and the latter in Section 3.

As a sampling of the kinds of results one can prove, the last two sections consider two different posets, of interest in several branches of mathematics and its applications. The first one is the set of equivalence classes of probability distributions on a finite set, under permutation of the elements of the set, partially ordered by the "majorization order"; the second poset is the set of positive semidefinite matrices (i.e., matrices A for which the quadratic form $x^T Ax$ is nonnegative for all vectors x), partially ordered by the pointwise ordering of the associated quadratic forms. The first poset is the subject of Section 5, and the second one of Section 6. In both cases, we consider the SRP when $\alpha = \beta = 0$, for various functions of interest, thus giving examples for the most efficient representations for certain natural functions of bounded variation, in our sense.

One motivation for studying ways to solve the SRP when $\alpha = \beta = 0$ stems from the fact that such a solution may provide bounds for a function x on a subset of T that possesses a least and a greatest element. This is briefly pointed out in Remark 10. It should also be mentioned that the problem of constraining a function between two general boundaries, even when the underlying poset is the usual one dimensional time, has important engineering applications [11]; the solution of this problem is intimately connected with certain types of traffic regulators[4] that are widely deployed in modern communication networks.

2 The Skorokhod reflection problem on a poset

2.1 Generalities

Let $(T, \leqslant)$ be a partially ordered set (poset)[5] with a least element 0, which we will call the *bottom* element, partially ordered by the relation $\leqslant$. We recall some notions. We write $s \geqslant t$ if $t \leqslant s$. If $t_1, t_2 \in T$, such that $t_1 \leqslant t_2$, the *interval* $[t_1, t_2]$ is the set of all $s \in T$ that are comparable to both t_1 and t_2, and $t_1 \leqslant s \leqslant t_2$. The interval $[0, t]$ is defined for every t and is also called the *down-set* of t, and is customarily denoted by $\downarrow t$. Note that if $t_1 \leqslant t_2 \leqslant t_3$, then $[t_1, t_2] \cup [t_2, t_3] \subseteq [t_1, t_3]$, but equality does not hold in general. The *up-set* $\uparrow t$ of a $t \in T$ contains all s such that $s \geqslant t$. Note that $[t_1, t_2] = \uparrow t_1 \cap \downarrow t_2$. A function $x : T \to \mathbb{R}$ is said to be *locally bounded* if $\sup_{s \in [0,t]} |x(s)| < \infty$ for all $t \in T$. The collection of locally bounded functions is partially ordered by the usual pointwise order: $x \leqslant y$ if $x(t) \leqslant y(t)$ for all $t \in T$. A function x is *nondecreasing* if it preserves the order, namely, $s \leqslant t \Rightarrow x(s) \leqslant x(t)$. A function x is *nonincreasing* if $-x$ is nondecreasing. Nonnegative nondecreasing functions show up very often in the sequel, and will be abbreviated as *NN functions*. For $a, b \in \mathbb{R}$, we shall write $a \vee b = \max(a, b)$, $a \wedge b = \min(a, b)$, as usual.

We say that a function on $(T, \leqslant)$ is of *bounded variation* if it can be written as the difference of two nondecreasing functions. This definition is by analogy with the case when $T = [a, b]$, a bounded interval of the real line, with the usual order – in that case we have the *Jordan decomposition theorem* [8, pg. 266] which says that a function of bounded variation on $[a, b]$ defined in the usual way [8, pg. 266], can be written as the difference of two nondecreasing functions.

[4]These are called "leaky bucket" regulators [11].

[5]For basic notions regarding partially ordered sets, see [17, Chapter 3].

We now gather a few straightforward facts about functions of bounded variation. Every such function can be written as the difference of two NN functions, because of the assumed existence of a bottom element in the poset. Indeed, if x is of bounded variation, write $x = u - \ell$, with u, ℓ both nondecreasing, and then set $\tilde{u}(t) = u(t) - m$, $\tilde{\ell}(t) = \ell(t) - m$, where $m = u(0) \wedge \ell(0)$. Clearly, $\tilde{u}$, $\tilde{\ell}$ are NN functions, and $x = \tilde{u} - \tilde{\ell}$. If x, y are of bounded variation, then $x \pm y$, $x \vee y$, $x \wedge y$ are also of bounded variation. To see that $x \vee y$ is of bounded variation, write $x = u_x - \ell_x$ and $y = u_y - \ell_y$ for the pairs of NN functions (ℓ_x, u_x) and (ℓ_y, u_y). Write

$$\begin{aligned} x \vee y + \ell_x + \ell_y &= (u_x - \ell_x) \vee (u_y - \ell_y) + \ell_x + \ell_y \\ &= (u_x + \ell_y) \vee (u_y + \ell_x) \\ &=: u_{x \vee y} \ , \end{aligned}$$

and let $\ell_{x \vee y} := \ell_x + \ell_y$. Note that $\ell_{x \vee y}$ and $u_{x \vee y}$ are NN functions, and that

$$x \vee y = u_{x \vee y} - \ell_{x \vee y} \ .$$

A similar argument shows that $x \wedge y$ is of bounded variation.

Examples of posets of interest to us in this paper are (i) the set $[0, \infty)$ of real numbers, totally ordered by the natural order, (ii) a tree T, i.e., a poset for which $\downarrow t$ is a chain[6] for each t, (iii) a σ-algebra $\mathcal{B}$ of subsets of a given set, ordered by set-inclusion, (iv) the poset of equivalence classes of probability distributions on a finite set, under the full symmetry group of the elements of the set, with majorization order, and (v) the poset of symmetric positive semidefinite real matrices of fixed size, order by the pointwise order on their associated quadratic forms. Each of these posets will appear in one or more of the examples in the sequel.

2.2 Definitions and main theorems for the SRP on a poset

Given a locally bounded function x, we associate to it the function

$$\mathcal{U}(x)(t) := \sup_{s \in [0,t]} x(s) \vee 0.$$

Owing to $[0, t_1] \subseteq [0, t_2]$ if $t_1 \leqslant t_2$, the function $\mathcal{U}(x)$ is NN. We regard $\mathcal{U}$ as an operator from the set of locally bounded functions to the set of NN functions. Note that $\mathcal{U}$ is an *increasing operator*, i.e., $x \leqslant y \Rightarrow \mathcal{U}(x) \leqslant \mathcal{U}(y)$. We also find it convenient to write

$$\mathcal{L}(x) = \mathcal{U}(-x),$$

so that $\mathcal{L}$ is a *decreasing operator*. To be explicit, given a locally bounded function x, we associate to it the function

$$\mathcal{L}(x)(t) := \sup_{s \in [0,t]} (-x(s)) \vee 0.$$

$\mathcal{L}$ is also an operator from the set of locally bounded functions to the set of NN functions.

We now give a formal definition of the Skorokhod reflection problem on a poset.

[6] A *chain* is defined as a poset in which every pair of elements is comparable. We do not require that a tree be finite.

Definition 1 (Skorokhod reflection problem on a poset). *We are given two locally bounded functions $\alpha, \beta : T \to \mathbb{R}$. with $\alpha(t) \leqslant \beta(t)$ for all t, and a locally bounded function $x : T \to \mathbb{R}$, such that $\alpha(0) \leqslant x(0) \leqslant \beta(0)$. Consider now the set of pairs of NN functions (ℓ, u) such that $z(t) := x(t) + \ell(t) - u(t)$ satisfies $\alpha(t) \leqslant z(t) \leqslant \beta(t)$, for all t. This set is ordered by the natural partial order: $(\ell, u) \leqslant (\ell', u')$ if $\ell \leqslant \ell'$ and $u \leqslant u'$. The problem is to find a minimal element of this set.*

Remark 1. The condition

$$\alpha(t) \leqslant z(t) = x(t) + \ell(t) - u(t) \leqslant \beta(t), \text{ for all } t$$

implies that the bounded variation function $y(t) := u(t) - \ell(t)$ satisfies

$$y(t) + \alpha(t) \leqslant x(t) \leqslant y(t) + \beta(t) \text{ for all } t \; . \tag{2}$$

Thus, given two locally bounded functions $\alpha, \beta : T \to \mathbb{R}$. with $\alpha(t) \leqslant \beta(t)$ for all t, *the SRP is of interest* only for those locally bounded functions $x : T \to \mathbb{R}$, with $\alpha(0) \leqslant x(0) \leqslant \beta(0)$ for which there is some function $y : T \to \mathbb{R}$ of bounded variation such that (2) holds. ■

Remark 2. Three special cases are necessary to mention:

(i) We may formally let $\beta = \infty$, and α an arbitrary locally bounded function. Then the problem becomes that of a one-sided regulator, in that we only consider those NN functions ℓ for which $x - \ell \geqslant \alpha$. We may refer to this problem as *"upward reflection at α"*. This problem is of interest for those locally bounded functions $x : T \to \mathbb{R}$, with $\alpha(0) \leqslant x(0)$ for which there exists some function $y : T \to \mathbb{R}$ of bounded variation such that $y(t)+\alpha(t) \leqslant x(t)$ for all t. But note that for any locally bounded $x : T \to \mathbb{R}$, we have that $x(t) - \alpha(t)$ is locally bounded and

$$x(t) - \alpha(t) \geqslant -\mathcal{L}(x - \alpha)(t) \text{ for all } t \; .$$

Since $\mathcal{L}(x - \alpha)$ is NN, $-\mathcal{L}(x - \alpha)$ is of bounded variation. Hence, the Skorokhod upward reflection problem at a locally bounded function is of interest for *all* locally bounded functions.

(ii) Analogously, for an arbitrary locally bounded function β, we obtain the problem of *"downward reflection at β"*, by letting $\alpha = -\infty$. As in (i), the Skorokhod downward reflection problem at a locally bounded function is of interest for *all* locally bounded functions.

(iii) By setting $\alpha = \beta$, a locally bounded function, we obtain a problem of restricting locally bounded functions x to within a one-point interval defined by the locally bounded function $\alpha = \beta$. This problem is of interest for those x, with $x(0) = \alpha(0)$, for which $x - \alpha$ is of bounded variation, as can be seen from (2). A special case of importance is when $\alpha = \beta = 0$. The SRP in this case if of interest for x of bounded variation with $x(0) = 0$. The problem can be interpreted as seeking the decomposition of a bounded variation function x in the form $x = u - \ell$, where (ℓ, u) is a pair of NN functions, in a *minimal* or "most efficient" manner. ■

It is useful to agree on some terminology for the discussion that follows. The functions α, β are called *boundary* functions, with α being the *lower boundary* and β the *upper boundary*. The function x may be called the *free* function or the *unconstrained* function, while $z = x + \ell - u$ may be called the *reflected* function, the *regulated* function, or the *constrained* function. Finally, ℓ and u are, respectively, the *lower regulator* and the *upper regulator* . We are concerned with conditions that guarantee existence and uniqueness of a solution to the SRP. This is the content of our main theorem, which follows immediately.

Theorem 1. *Let α, β, x be locally bounded with $\alpha(0) \leqslant x(0) \leqslant \beta(0)$. Assume that there is a function y of bounded variation, with $y(0) = 0$, such that*

$$y(t) + \alpha(t) \leqslant x(t) \leqslant y(t) + \beta(t), \text{ for all } t \in T. \tag{3}$$

Then the set of pairs (ℓ, u) of NN functions such that

$$\alpha(t) \leqslant x(t) + \ell(t) - u(t) \leqslant \beta(t), \text{ for all } t \in T, \tag{4}$$

possesses a least[7] *element (ℓ^*, u^*), called the solution to the SRP. Further, we have*

$$\begin{aligned} \ell^* &= \mathcal{U}(-x + u^* + \alpha) \\ u^* &= \mathcal{U}(x + \ell^* - \beta) \ , \end{aligned}$$

i.e., (ℓ^*, u^*) satisfies the fixed point equation

$$\begin{aligned} \ell &= \mathcal{U}(-x + u + \alpha) \\ u &= \mathcal{U}(x + \ell - \beta). \end{aligned} \tag{5}$$

Remark 3. It is straightforward to check that any pair of functions (ℓ, u) satisfying the fixed point equation (5) must be NN, and must satisfy the inequality (4). However, it is important to recognize that, in general, the fixed point equation (5), which is satisfied by the solution to the SRP, *does not* characterize the solution to the SRP. We illustrate this point by an example, assuming the truth of Theorem 1.

Example 1. Let $T = \mathbb{R}_+$, with the usual order, and let

$$\alpha(t) = \begin{cases} -1 & t = 0 \\ -t & t > 0 \end{cases}, \qquad \beta(t) = \begin{cases} 1 & t = 0 \\ t & t > 0 \end{cases}.$$

These are locally bounded functions, and we even have the strict inequality[8] $\alpha(t) < \beta(t)$ for all $t \in \mathbb{R}_+$. Let $x(t) = 0$ for all $t \in \mathbb{R}_+$. Then $(\ell^*, u^*) = (0, 0)$ is the solution to the SRP for x with the boundaries α (lower) and β (upper). Now write the fixed point equation (5) with $x = 0$:

$$\begin{aligned} \ell(t) &= \sup_{0 \leqslant s \leqslant t} (u(s) + \alpha(s)) \vee 0, \quad t \geqslant 0, \\ u(t) &= \sup_{0 \leqslant s \leqslant t} (\ell(s) - \beta(s)) \vee 0, \quad t \geqslant 0, \end{aligned} \tag{6}$$

and consider the NN functions

$$\ell(t) = u(t) = \begin{cases} 0 & \text{if } t = 0 \\ 1 & \text{if } t > 0 \end{cases}.$$

It is easy to see that these satisfy (6), and so, in this example, the fixed point equation (5) has at least two solutions. ■

[7] Note that our use of the words "least" and "minimal" is the traditionally accepted one: a least element is an element such that (i) it compares to every other element and (ii) it is less than any other element; hence a least element is minimal and, if it exists, it is unique.

[8] When $\alpha(t) = \beta(t)$ for some $t \in T$, it is even easier to construct examples where multiple solutions exist for the fixed point equation (5).

Remark 4. The solution (ℓ^*, u^*) announced by Theorem 1 must satisfy $\ell^*(0) = u^*(0) = 0$. We prove this by considering first the case where $\alpha(0) = \beta(0)$ and then the case where $\alpha(0) < \beta(0)$.

If $\alpha(0) = \beta(0) = c$, say, then $x(0) = c$ by hypothesis, so (4) yields $\ell^*(0) - u^*(0) = 0$. If the common value $\delta := \ell^*(0) = u^*(0)$ is strictly positive, define the functions $\tilde{\ell} := \ell^* - \delta$, $\tilde{u} := u^* - \delta$, observe that $x + \ell^* - u^* = x + \tilde{\ell} - \tilde{u}$, and note that $(\tilde{\ell}, \tilde{u})$ is strictly smaller than (ℓ^*, u^*), which contradicts the minimality of the latter. Hence it must be that $\ell^*(0) = u^*(0) = 0$.

Suppose now that $\alpha(0) < \beta(0)$ Suppose $\ell^*(0) > 0$. Since $\ell^* = \mathcal{U}(-x + u^* + \alpha)$, we have $\ell^*(0) = (-x(0) + u^*(0) + \alpha(0)) \vee 0$, and since we assumed $\ell^*(0) > 0$, we have $\ell^*(0) = -x(0) + u^*(0) + \alpha(0)$. Then, from $u^* = \mathcal{U}(x + \ell^* - \beta)$ we have $u^*(0) = (x(0) + \ell^*(0) - \beta(0)) \vee 0 = (u^*(0) + \alpha(0) - \beta(0)) \vee 0$, which gives $(\alpha(0) - \beta(0)) \vee (-u^*(0)) = 0$. Since $\alpha(0) - \beta(0) < 0$, we obtain $u^*(0) = 0$. So $\ell^*(0) = -x(0) + u^*(0) + \alpha(0) = -x(0) + \alpha(0)$. But $\alpha(0) \leqslant x(0) \leqslant \beta(0)$, by assumption. So $\ell^*(0) \leqslant 0$, which is a contradiction. We can similarly arrive at a contradiction starting from the hypothesis that $u(0) > 0$. Hence it must be that $\ell^*(0) = u^*(0) = 0$.

So the solution to the SRP is not only a pair of NN functions but they must also start from zero. It is easy to see that the same observation holds even when $\alpha \equiv -\infty$, or $\beta \equiv +\infty$. ■

Before proving Theorem 1, we collect some structural properties of the operator $\mathcal{U}$.

Lemma 1. *Let x be a locally bounded function on the poset $(T, \leqslant)$. Then:*
(i) The function $u := \mathcal{U}(x)$ is NN.
(ii) $x - u \leqslant 0$.
(iii) The operator $\mathcal{U}$ is increasing: $x \leqslant y \Rightarrow \mathcal{U}(x) \leqslant \mathcal{U}(y)$.
(iv) Suppose the function $\tilde{u}$ is NN, with $\tilde{u}(0) = 0$, and such that $x - \tilde{u} \leqslant 0$ then $u \leqslant \tilde{u}$.

Proof. (i) Let $s \leqslant t$. Since $[0, s] \subseteq [0, t]$ we have $u(s) \leqslant u(t)$.
(ii) Obvious.
(iii) If $x \leqslant y$ then $x(s) \leqslant y(s)$ for all s and so $\sup_{s \in [0,t]} x(s) \vee 0 \leqslant \sup_{s \in [0,t]} y(s) \vee 0$.
(iv) Suppose the function $\tilde{u}$ is NN, with $\tilde{u} \geqslant x$. Then, for all $s \in [0, t]$, $\tilde{u}(t) \geqslant \tilde{u}(s) \geqslant x(s)$, giving $\tilde{u}(t) \geqslant \sup_{s \in [0,t]} x(s)$. Since also $\tilde{u}(t) \geqslant 0$, we obtain $\tilde{u}(t) \geqslant \sup_{s \in [0,t]} x(s) \vee 0$. □

Corollary 1. *Let x be an arbitrary locally bounded function. Consider the downward SRP for x at the locally bounded function β, with $\alpha = -\infty$. Then the function $\mathcal{U}(x - \beta)$ is the unique solution to the SRP, i.e., it is least among the NN functions u with $x - u \leqslant \beta$. Similarly, consider the upward SRP for x at the locally bounded function α, with $\beta = \infty$. Then the function $\mathcal{U}(\alpha - x)$ is the unique solution to the SRP, in the sense that it is least among the NN functions ℓ with $x + \ell \geqslant \alpha$.*

Proof. For the first case apply Lemma 1 with x replaced by $x - \beta$. For the second case apply the same Lemma with x replaced by $\alpha - x$. □

Proof of Theorem 1. Let y be of bounded variation such that (2) holds. We may assume without loss of generality that $y(0) = 0$. We write

$$y = \overline{u} - \overline{\ell}\ ,$$

where $\overline{u}$, $\overline{\ell}$ are NN functions. We may assume without loss of generality that $\overline{u}(0) = \overline{\ell}(0) = 0$. Since $(y + \overline{\ell}) - \overline{u} \leqslant 0$, we have $\mathcal{U}(y + \overline{\ell}) \leqslant \overline{u}$. And since $(y - \overline{u}) + \overline{\ell} \geqslant 0$, we have $\mathcal{L}(y - \overline{u}) \leqslant \overline{\ell}$.

Using the fact that $\mathcal{L}$ is a decreasing operator and that $\mathcal{U}$ is an increasing operator, together with the assumptions $y \leqslant x - \alpha$, and $y \geqslant x - \beta$, the previous inequalities yield

$$\begin{aligned} \overline{\ell} &\geqslant \mathcal{L}(x - \overline{u} - \alpha) \\ \overline{u} &\geqslant \mathcal{U}(x + \overline{\ell} - \beta) \ . \end{aligned}$$

Define next sequences of functions ℓ^n, u^n, $n = 0, 1, 2, \ldots$, by

$$\begin{aligned} \ell^0 &= u^0 = 0, \\ \ell^{n+1} &= \mathcal{L}(x - u^n - \alpha), \quad n = 1, 2, \ldots, \\ u^{n+1} &= \mathcal{U}(x + \ell^n - \beta), \quad n = 1, 2, \ldots \end{aligned}$$

We have $\ell^0 \leqslant \ell^1$, $u^0 \leqslant u^1$. If, for some $n \geqslant 1$, $\ell^n \leqslant \ell^{n+1}$, then $u^{n+1} \leqslant u^{n+2}$, and if $u^n \leqslant u^{n+1}$, then $\ell^{n+1} \leqslant \ell^{n+2}$. Hence

$$\forall n \quad (\ell^n, u^n) \leqslant (\ell^{n+1}, u^{n+1}).$$

We also have $\ell^0 \leqslant \overline{\ell}$, $u^0 \leqslant \overline{u}$. If, for some $n \geqslant 1$, $\ell^n \leqslant \overline{\ell}$, then $u^{n+1} = \mathcal{U}(x + \ell^n - \beta) \leqslant \mathcal{U}(x + \overline{\ell} - \beta) \leqslant \overline{u}$. And, if $u^n \leqslant \overline{u}$, then $\ell^{n+1} = \mathcal{L}(x - u^n - \alpha) \leqslant \mathcal{L}(x - \overline{u} - \beta) \leqslant \overline{\ell}$. Hence,

$$\forall n \quad (\ell^n, u^n) \leqslant (\overline{\ell}, \overline{u}).$$

Thus, for each $t \in T$, the sequences of real numbers $\{\ell^n(t), n = 0, 1, \ldots\}$, $\{u^n(t), n = 0, 1, \ldots\}$, are nondecreasing and bounded. Let

$$\ell^*(t) = \lim_{n \to \infty} \ell^n(t), \quad u^*(t) = \lim_{n \to \infty} u^n(t),$$

be their limits. Furthermore, each of the ℓ^n, u^n are NN functions on $(T, \leqslant)$. So ℓ^*, u^* are also NN. Let now

$$\begin{aligned} \widehat{\ell} &= \mathcal{L}(x - u^* - \alpha), \\ \widehat{u} &= \mathcal{U}(x + \ell^* - \beta). \end{aligned}$$

From the recursive definitions of ℓ^n, u^n, we have

$$\begin{aligned} x - u^n + \ell^{n+1} &\geqslant \alpha, \\ x + \ell^n - u^{n+1} &\leqslant \beta. \end{aligned}$$

Taking pointwise limits we obtain

$$\begin{aligned} x - u^* + \ell^* &\geqslant \alpha, \\ x + \ell^* - u^* &\leqslant \beta \end{aligned}$$

Applying $\mathcal{L}$ to the inequality $x - u^* - \alpha \geqslant -\ell^*$ yields $\mathcal{L}(x - u^* - \alpha) \leqslant \mathcal{L}(-\ell^*) = \ell^*$. (The last equality follows from $\mathcal{L}(-\ell^*)(t) = \mathcal{U}(\ell^*)(t) = \sup_{s \in [0,t]} \ell^*(s) \vee 0 = \ell^*(t)$.) Thus

$$\widehat{\ell} \leqslant \ell^*.$$

Applying $\mathcal{L}$ to the inequality $x - u^* - \alpha \leqslant x - u^n - \alpha$ yields $\mathcal{L}(x - u^* - \alpha) \geqslant \mathcal{L}(x - u^n - \alpha)$. But $\mathcal{L}(x - u^* - \alpha) = \widehat{\ell}$, and $\mathcal{L}(x - u^n - \alpha) = \ell^{n+1}$. Thus $\widehat{\ell} \geqslant \ell^{n+1}$ and so

$$\widehat{\ell} \geqslant \ell^*.$$

We thus conclude that $\ell^* = \widehat{\ell}$. A similar line of reasoning yields $u^* = \widehat{u}$. So (ℓ^*, u^*) satisfies the fixed point equation

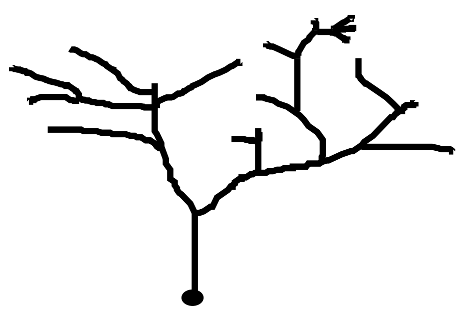
$$\begin{aligned}\ell^* &= \mathcal{L}(x - u^* - \alpha),\\ u^* &= \mathcal{U}(x + \ell^* - \beta).\end{aligned}$$

We finally need to show that (ℓ^*, u^*) is least among all pairs (ℓ, u) of NN functions, with $\ell(0) = u(0) = 0$, such that $\alpha \leqslant x + \ell - u \leqslant \beta$. Let (ℓ, u) be such a pair. We will show that $(\ell^*, u^*) \leqslant (\ell, u)$. Clearly, $(\ell^0, u^0) \leqslant (\ell, u)$. Now suppose that for some $n \geqslant 1$, we have $u^n \leqslant u$. Then $\ell^{n+1} = \mathcal{L}(x - u^n - \alpha) \leqslant \mathcal{L}(x - u - \alpha) \leqslant \ell$, where the last inequality follows from the fact that ℓ is such that $x + \ell - u \geqslant \alpha$. Similarly, if $\ell^n \leqslant \ell$ then $u^{n+1} \leqslant u$. Hence, for all n, $(\ell^n, u^n) \leqslant (\ell, u)$. Since $(\ell^*, u^*) = \lim_{n\to\infty}(\ell^n, u^n)$, we conclude that $(\ell^*, u^*) \leqslant (\ell, u)$. □

The special case where $\alpha = \beta = 0$ is of particular interest. For the SRP to make sense, it is necessary to assume that x is a function of bounded variation. The solution to the SRP then provides the minimal way to write x as the difference of two NN functions: $x = u - \ell$. Several examples along these lines will be developed in the sequel.

2.3 The SRP on a tree

As an example, consider the situation where $(T, \leqslant)$ is a tree rooted at 0, i.e., a poset with bottom element 0, such that $[0, t]$ is a chain for all $t \in T$. We examine the problem of minimal representation of a bounded variation function on this poset as a difference of two NN functions, i.e., we consider the SRP with the boundary functions $\alpha = \beta = 0$. We show that the solution to the SRP for a bounded variation function x is minimal in a much stronger sense in this example, in that, for any interval



A rooted tree.

of the poset, the increments of the NN functions solving the SRP for x on the full poset are the solutions to the SRP for the normalized restriction of x to the interval.[9] It is simple to construct examples to show that this statement is not true in general; we give one following the proof of the theorem.

Theorem 2 (minimal decomposition of a function of bounded variation on a tree). *Let $(T, \leqslant)$ be a tree rooted at 0. Let x be a function of bounded variation on $(T, \leqslant)$, with $x(0) = 0$. Consider the SRP for x with boundary functions $\alpha = \beta = 0$, and let (ℓ^*, u^*) be its solution. Let (ℓ, u) be any pair of NN functions such that $x = u - \ell$. Then, for any $t_1, t_2 \in T$ with $t_1 \leqslant t_2$, $\ell^*(t_2) - \ell^*(t_1) \leqslant \ell(t_2) - \ell(t_1)$, and $u^*(t_2) - u^*(t_1) \leqslant u(t_2) - u(t_1)$.*

[9] x is normalized to start at zero at the bottom element of the interval.

Proof. Given NN functions (ℓ, u) on $(T, \leqslant)$ with $x = u - \ell$, as in the statement of the corollary, define NN functions $(\tilde{\ell}, \tilde{u})$ on $(T, \leqslant)$ by

$$\tilde{\ell}(t) = \begin{cases} \ell^*(t_1) + \ell(t) - \ell(t_1) & \text{if } t \geqslant t_1 \\ \ell^*(t) & \text{otherwise} \end{cases}$$
$$u(t) = \begin{cases} u^*(t_1) + u(t) - u(t_1) & \text{if } t \geqslant t_1 \\ u^*(t) & \text{otherwise} \end{cases}.$$

Then we have $x = \tilde{u} - \tilde{\ell}$. By the minimality of the solution to the SRP on $(T, \leqslant)$, we must have, for all $t \geqslant t_1$, $\ell^*(t) \leqslant \tilde{\ell}(t) := \ell^*(t_1)+\ell(t)-\ell(t_1)$ and $u^*(t) \leqslant \tilde{u}(t) := u^*(t_1)+u(t)-u(t_1)$. This is just the what the theorem claims. □

Fix now a $t_1 \in T$. We observe that $(\uparrow t_1, \leqslant)$ is a tree rooted at t_1. To see this, consider $t \geqslant t_1$. Then the down-set of t in the poset $(\uparrow t_1, \leqslant)$ is a subset of the down-set of t in the poset $(T, \leqslant)$; the latter is a chain, and hence the former is a chain too. Note that t_1 is the bottom element of the poset $(\uparrow t_1, \leqslant)$. We wish to relate the SRP for functions on the tree T with that on the tree $\uparrow t_1$. Given a function $x : T \to \mathbb{R}$, define the function $\theta_{t_1} x : \uparrow t_1 \to \mathbb{R}$ by

$$\theta_{t_1} x(t) := x(t) - x(t_1), \quad t \in \uparrow t_1.$$

Corollary 2. *Let x be a function of bounded variation on the tree $(T, \leqslant)$, with $x(0) = 0$ and let (ℓ^*, u^*) be the solution to the SRP for x with boundary functions $\alpha = \beta = 0$. Fix $t_1 \in T$. Then $(\theta_{t_1}\ell^*, \theta_{t_1}u^*)$ is the solution to the SRP for $\theta_{t_1}x$ on the tree $(\uparrow t_1, \leqslant)$ with boundary functions $\alpha = \beta = 0$.*

Proof. Suppose that $(\tilde{\ell}, \tilde{u})$ is the solution to this SRP on $(\uparrow t_1, \leqslant)$. Note that $\tilde{\ell}(t_1) = \tilde{u}(t_1) = 0$, and define NN functions (ℓ, u) on $(T, \leqslant)$ by

$$\ell(t) = \begin{cases} \ell^*(t_1) + \tilde{\ell}(t) & \text{if } t \geqslant t_1 \\ \ell^*(t) & \text{otherwise} \end{cases}$$
$$u(t) = \begin{cases} u^*(t_1) + \tilde{u}(t) & \text{if } t \geqslant t_1 \\ u^*(t) & \text{otherwise} \end{cases}.$$

Notice that $x = u - \ell$ on $(T, \leqslant)$. From Theorem 2 we conclude that $\widehat{\ell}(t) := \ell^*(t) - \ell^*(t_1) \leqslant \tilde{\ell}(t)$ and $\widehat{u}(t) := u^*(t) - u^*(t_1) \leqslant \tilde{u}(t)$ for all $t \geqslant t_1$. But, since $(\tilde{\ell}, \tilde{u})$ is the solution to this SRP on $(\uparrow t_1, \leqslant)$, we have $\tilde{\ell} \leqslant \widehat{\ell}$, $\tilde{u} \leqslant \widehat{u}$. Therefore, $(\tilde{\ell}, \tilde{u}) = (\widehat{\ell}, \widehat{u})$. □

Here is another way to look at the above result. Take a function x of bounded variation on T, with $x(0) = 0$. Define the *positive* and *negative variations* of x by

$$\mathcal{V}^+(x)(t) = \sup \sum_{i=1}^{k} [x(t_i) - x(t_{i-1})]^+,$$
$$\mathcal{V}^-(x)(t) = \sup \sum_{i=1}^{k} [x(t_i) - x(t_{i-1})]^-, \tag{7}$$

respectively, where the suprema are taken over all possible subdivisions $0 = t_0 < t_1 < \cdots < t_k = t$ of $[0, t]$. Both quantities are finite, and they define NN functions on t. Define the *total variation* of x by

$$\mathcal{V}(x)(t) = \sup \sum_{i=1}^{k} |x(t_i) - x(t_{i-1})|.$$

Since $\sum_{i=1}^{k}[x(t_i) - x(t_{i-1})]^+ = \sum_{i=1}^{k}[x(t_i) - x(t_{i-1})]^- + x(t)$, for any subdivision as above, taking supremum over all such subdivisions we get $\mathcal{V}^+(x)(t) \leqslant \mathcal{V}^-(x)(t) + x(t)$. Similarly, we obtain the opposite inequality, and so

$$x = \mathcal{V}^+(x) - \mathcal{V}^-(x).$$

It is also easy to see that $\mathcal{V}(x)(t) = \mathcal{V}^+(x)(t) + \mathcal{V}^-(x)(t)$. Let (ℓ^*, u^*) be the solution to the SRP as above. Then $u^* \leqslant \mathcal{V}^+(x)$, and $\ell^* \leqslant \mathcal{V}^-(x)$. On the other hand $x(t_i) - x(t_{i-1}) \leqslant u^*(t_i) - u^*(t_{i-1})$, and so, from the definition of $\mathcal{V}(x)$, we have $\mathcal{V}^+(x) \leqslant u^*$. Similarly, $\mathcal{V}^-(x) \leqslant \ell^*$.

So we proved

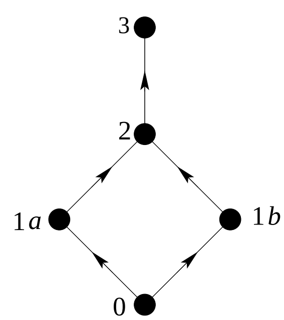
Corollary 3. *Under the assumption of Theorem 2, we have that the solution (ℓ^*, u^*) to the SRP on a tree is such that $u^* = \mathcal{V}^+(x)$, $\ell^* = \mathcal{V}^-(x)$, $u^* + \ell^* = \mathcal{V}(x)$, where $\mathcal{V}^+, \mathcal{V}^-$ are defined by* (7).

In particular, this decomposition of a function x of bounded variation on a tree T reduces to the so-called Jordan decomposition of x when $T = [0, \infty)$.

Example 2. We now give a simple example where the conclusion of Theorem 2 fails when $(T, \leqslant)$ is not a tree. Let T be the set with 5 elements $T = \{0, 1a, 1b, 2, 3\}$ with bottom element 0 and with the partial order generated by the relations

$$0 \leqslant 1a\ ,\quad 0 \leqslant 1b\ ,\quad 1a \leqslant 2\ ,\quad 1b \leqslant 2\ ,\quad 2 \leqslant 3\ .$$

See the figure.



The poset of Example 2.

Note that $1a$ and $1b$ are not comparable in $(T, \leqslant)$. Consider the following functions on T :

t	0	$1a$	$1b$	2	3
x	0	10	3	6	8
ℓ^*	0	0	0	4	4
u^*	0	10	3	10	12
ℓ	0	0	4	4	4
u	0	10	7	10	12

.

Note that the pair of NN functions (ℓ^*, u^*) is the solution to the SRP for the function x with boundary functions $\alpha = \beta = 0$. Hence $x = u^* - \ell^*$. One can also check that (ℓ, u) is a pair of NN functions with $x = u - \ell$.

Let $t_1 = 1b$. Then $\uparrow t_1 = \{1b, 2, 3\}$ and the bottom element of $(\uparrow t_1, \leqslant)$ is $1b$. We have :

t	$1b$	2	3
$x(t) - x(1b)$	0	3	5
$\ell^*(t) - \ell^*(1b)$	0	4	4
$u^*(t) - u(1b)$	0	7	9
$\ell(t) - \ell(1b)$	0	0	0
$u(t) - u(1b)$	0	3	5

.

Note that, for instance, $u^*(2) - u^*(1b) > u(2) - u(1b)$. Theorem 2 guarantees that this phenomenon cannot occur when $(T, \leqslant)$ is a tree. ■

3 Discrete posets

Definition 2 (Discrete poset). *Let $(T, \leqslant)$ be a poset with a bottom element, denoted 0. We say the poset is* discrete *if, for all $t \in T$, the interval $[0, t]$ is a finite set.*

Note that a discrete poset is not necessarily a countable set. As an example of an uncountable discrete poset, let S be an uncountable set, let $\mathcal{P}_n(S)$ be the collection of its subsets of cardinality n, $n \in \mathbb{N}$, with $\mathcal{P}_0(S) := \{\emptyset\}$, and let $T := \cup_{n=0}^{\infty} \mathcal{P}_n(S)$ be the collection of all finite subsets of S; order T by set inclusion. Then for each set $t \subseteq S$ with cardinality n, the interval $[0, t]$ is the collection of all subsets of t, so $[0, t]$ has cardinality 2^n. Thus T is a discrete poset with $\emptyset$ as the bottom element.

Definition 3 (Bulk of an element in a discrete poset). *Let $(T, \leqslant)$ be a discrete poset with bottom element 0. Given $t \in T$, we define its* bulk, *denoted $B(t)$, to be the cardinality of the interval $[0, t]$. Thus, $B(\cdot)$ is a positive integer valued function on T, with $B(0) = 1$, and 0 is the unique element with bulk 1.*

Lemma 2. *Let $(T, \leqslant)$ be a discrete poset with bottom element 0. Let $x : T \to \mathbb{R}$ be an arbitrary function. Then x is locally bounded. Also, x is of bounded variation.*

Proof. That x is locally bounded is immediate from the definition. To see that x is of bounded variation, we need to construct NN functions $u : T \to \mathbb{R}$ and $\ell : T \to \mathbb{R}$ such that

$$x(t) = u(t) - \ell(t) \text{ for all } t \in T \ .$$

We may assume without loss of generality that $x(0) = 0$ and we will construct the required functions with the additional property that $u(0) = \ell(0) = 0$. We proceed to do this by induction on the bulk.

To begin with, let $u(0) = \ell(0) = 0$. Thus $x(0) = u(0) - \ell(0)$. If $T = \{0\}$ there is nothing to prove. So let T contain an element different from the bottom element. Let $t \in T$ be such that $B(t) = 2$ (it is straightforward to see that there is at least one such element in T). Thus $[0, t] = \{0, t\}$. We define $u(t)$ and $\ell(t)$ as follows :

$$\begin{array}{ll} u(t) = x(t) \text{ and } \ell(t) = 0 & \text{if } x(t) \geqslant 0 \\ u(t) = 0 \text{ and } \ell(t) = -x(t) & \text{if } x(t) \leqslant 0 \end{array} .$$

Suppose now that $u(\cdot)$ and $\ell(\cdot)$ have been defined for all $s \in T$ with $1 \leqslant B(s) \leqslant b$. Let $\bar{b} = \min\{n \in \mathbb{N} : \ n > b \text{ and } \exists t \in T \ B(t) = n\}$.[10] We now demonstrate how to define $u(t)$ and $\ell(t)$ for all t with $B(t) = \bar{b}$.

[10] It is easy to construct examples where there may not be any element in T with bulk $b+1$, which is why it is necessary to introduce $\bar{b}$.

Given any t with $B(t) = \bar{b}$, note that for every $s \in [0,t] - \{t\}$ we have $B(s) \leqslant b$. This allows us to temporarily define nonnegative real numbers :

$$\begin{aligned} \bar{z}_\ell &:= \max_{s \in [0,t]-\{t\}} \ell(s) \\ \bar{z}_u &:= \max_{s \in [0,t]-\{t\}} u(s) \ . \end{aligned}$$

Now consider the set

$$\{(z_\ell, z_u) \ : \ z_\ell \geqslant \bar{z}_\ell \ , \quad z_u \geqslant \bar{z}_u \ , \quad z_u - z_\ell = x(t)\} \ .$$

It is straightforward to see that this set is nonempty and has a least element in the sense of pointwise partial order on $\mathbb{R}^2_+$. We define the pair $(\ell(t), u(t))$ to be this minimal element.

We finally observe that the functions $u : T \to \mathbb{R}$ and $\ell : T \to \mathbb{R}$ defined in this way are NN functions, and that

$$x(t) = u(t) - \ell(t) \text{ for all } t \in T \ .$$

□

Corollary 4. *Let $(T, \leqslant)$ be a discrete poset with least element 0. Then for every triple of functions α, β, and x on T, with $\alpha(t) \leqslant \beta(t)$ for all $t \in T$ and $\alpha(0) \leqslant x(0) \leqslant \beta(0)$, the SRP for $x(\cdot)$ with lower boundary $\alpha(\cdot)$ and upper boundary $\beta(\cdot)$ is of interest (in the sense defined in Remark 1 above).*

Proof. First note that since all functions on T are locally bounded, it makes sense to even talk about whether the SRP is of interest. To verify the truth of the corollary, we need to demonstrate the existence of a function of bounded variation $y : T \to \mathbb{R}$ such that

$$y(t) + \alpha(t) \leqslant x(t) \leqslant y(t) + \beta(t), \text{ for all } t \in T. \tag{8}$$

Let $y(t) = x(t) - \frac{\alpha(t)+\beta(t)}{2}$. Then $y(\cdot)$ satisfies the conditions in (8). Further, since every function is of bounded variation, so is $y(\cdot)$. □

We have now established the truth of the following general theorem :

Theorem 3. *Let $(T, \leqslant)$ be a discrete poset with least element 0. Let α, β, and x be an arbitrary triple of functions on T, with $\alpha(t) \leqslant \beta(t)$ for all $t \in T$ and $\alpha(0) \leqslant x(0) \leqslant \beta(0)$. Then the set of pairs (ℓ, u) of NN functions on T such that*

$$\alpha(t) \leqslant x(t) + \ell(t) - u(t) \leqslant \beta(t), \text{ for all } t \in T,$$

possesses a least element (ℓ^*, u^*) which satisfies the fixed point equation

$$\begin{aligned} \ell^* &= \mathcal{U}(-x + u^* + \alpha) \\ u^* &= \mathcal{U}(x + \ell^* - \beta). \end{aligned} \tag{9}$$

In particular, specializing to the case where $\alpha(t) = \beta(t) = 0$ for all $t \in T$, we have that every function $x : T \to \mathbb{R}$ with $x(0) = 0$ has a representation

$$x(t) = u^*(t) - \ell^*(t) \text{ for all } t \in T,$$

where $\ell^*(\cdot)$ and $u^*(\cdot)$ are NN functions on T, and the pair (ℓ^*, u^*) is the least among all pairs (ℓ, u) of NN functions on T satisfying $x(t) = u(t) - \ell(t)$ for all $t \in T$. Further,

$$\begin{aligned} \ell^* &= \mathcal{U}(-x + u^*) \\ u^* &= \mathcal{U}(x + \ell^*). \end{aligned}$$

In the Remark 3 following the statement of Theorem 1 we gave an example where the fixed point equation (5) has multiple solutions, even when the boundary functions α and β are strictly separated in the sense that $\alpha(t) < \beta(t)$ for all $t \in T$. If $(T, \leqslant)$ be a discrete poset, however, such a phenomenon cannot arise.

Theorem 4. *Let $(T, \leqslant)$ be a discrete poset with least element 0. Let α, β, and x be an arbitrary triple of functions on T, with $\alpha(t) < \beta(t)$ for all $t \in T$ and $\alpha(0) \leqslant x(0) \leqslant \beta(0)$. Then there is a* unique pair *of NN functions (ℓ, u) on T which satisfy the fixed point equation (5). This pair necessarily satisfies*

$$\alpha(t) \leqslant x(t) + \ell(t) - u(t) \leqslant \beta(t), \text{ for all } t \in T, \tag{10}$$

and, by virtue of equation (9) is the pair (ℓ^*, u^*) that solves the SRP for x with boundary functions α and β.

Proof. It is straightforward to verify that any pair of NN functions (ℓ, u) on T which satisfy the fixed point equation (5) must satisfy the inequality (10). Theorem 3 verifies that the pair (ℓ^*, u^*) satisfies the fixed point equation (5). Let (ℓ, u) be a pair of NN functions on T which satisfy the fixed point equation (5). We shall prove that $(\ell, u) = (\ell^*, u^*)$.

We prove this by induction on the bulk of $t \in T$. To begin with, consider $t = 0$, which has bulk $B(0) = 1$. Suppose $u(0) > 0$. Then, since $u = \mathcal{U}(x + \ell - \beta)$, we must have

$$u(0) = x(0) + \ell(0) - \beta(0),$$

which implies that

$$\ell(0) = u(0) + \beta(0) - x(0) > 0,$$

which, since $\ell = \mathcal{U}(-x + u + \alpha)$, implies that

$$\ell(0) = -x(0) + u(0) + \alpha(0).$$

But we have arrived at the contradiction that $x(0) + \ell(0) - u(0) = \alpha(0) = \beta(0)$, where we know that $\alpha(0) < \beta(0)$. We can arrive at a similar contradiction starting with the assumption that $\ell(0) > 0$. Therefore, it must be the case that $\ell(0) = 0$, and $u(0) = 0$. But we know (see Remark 4 above) that $\ell^*(0) = 0$, and $u^*(0) = 0$. Thus, $(\ell(0), u(0)) = (\ell^*(0), u^*(0))$. This is the initial step of the inductive process.

Assume $T \neq \{0\}$, and let $b \geqslant 2$. Suppose we have proved, for all $t \in T$ with $B(t) \leqslant b$, that $(\ell(s), u(s)) = (\ell^*(s), u^*(s))$ for all $s \leqslant t$. Let $t \in T$ be such that $B(t)$ has the smallest possibly bulk that is strictly bigger than b.

Suppose $\ell(t) = 0$. Since $\ell \geqslant \ell^*$, it then follows that $\ell^*(t) = 0$. Also, since ℓ and ℓ^* are NN, it follows that both $\ell(s) = 0$ for all $s \leqslant t$ and $\ell^*(s) = 0$ for all $s \leqslant t$. Since $u = \mathcal{U}(x + \ell - \beta)$ and $u^* = \mathcal{U}(x + \ell^* - \beta)$ it follows that $u(s) = u^*(s)$ for all $s \leqslant t$. Similarly, if $u(t) = 0$, we can conclude that $u(s) = u^*(s) = 0$ for all $s \leqslant t$, and $\ell(s) = \ell^*(s)$ for all $s \leqslant t$. In either case, this suffices to propagate the induction.

Let us therefore assume that $\ell(t) > 0$ and $u(t) > 0$. Since $\ell = \mathcal{U}(-x + u + \alpha)$ and $u = \mathcal{U}(x + \ell - \beta)$, we then have

$$\begin{aligned} \ell(t) &= -x(s) + u(s) + \alpha(s), \quad \text{for some } s \leqslant t \\ u(t) &= x(\bar{s}) + \ell(\bar{s}) - \beta(\bar{s}), \quad \text{for some } \bar{s} \leqslant t. \end{aligned}$$

Since $\alpha(t) < \beta(t)$, we cannot have $s = \bar{s} = t$. Thus, $s \neq t$ or $\bar{s} \neq t$.

Suppose $s \neq t$. Since also $s \leqslant t$, we have $B(s) < B(t)$. Since

$$-x(s) + u(s) + \alpha(s) \leqslant \ell(s) \leqslant \ell(t) = -x(s) + u(s) + \alpha(s) \ ,$$

we conclude that $\ell(s) = \ell(t)$. By inductive hypothesis we had $\ell(s) = \ell^*(s)$. Hence we have

$$\ell^*(t) \leqslant \ell(t) = \ell(s) = \ell^*(s) \leqslant \ell^*(t) \ ,$$

from which we conclude that $\ell(t) = \ell^*(t)$. Together with the inductive hypothesis, we now conclude that $\ell(s) = \ell^*(s)$ for all $s \leqslant t$. Since $u = \mathcal{U}(x + \ell - \beta)$ and $u^* = \mathcal{U}(x + \ell^* - \beta)$ we now conclude that $u(s) = u^*(s)$ for all $s \leqslant t$. This suffices to propagate the induction in this case. A similar argument suffices to propagate the induction in the remaining case where $\bar{s} \neq t$, which completes the proof. □

4 The Skorokhod reflection problem on a $\boldsymbol{\sigma}$-algebra of sets

Consider a set S and a σ-algebra $\mathcal{B}$ of subsets of S, ordered by set inclusion. Recall that $\mathcal{B}$ is a σ-algebra if $\emptyset \in \mathcal{B}$, if $A \in \mathcal{B} \Rightarrow A^c = S - A \in \mathcal{B}$, and if $\{A_n, n \in \mathbb{N}\} \subseteq \mathcal{B} \Rightarrow \bigcup_{n\in\mathbb{N}} A_n \in \mathcal{B}$. In this section, $(\mathcal{B}, \subseteq)$ is our poset, with bottom element the empty set $\emptyset$. We will consider functions $x : \mathcal{B} \to \mathbb{R}$ with $x(\emptyset) = 0$ that are σ-additive, i.e.,

$$x\left(\bigcup_{j=1}^{\infty} A_j\right) = \sum_{j=1}^{\infty} x(A_j), \text{ for any disjoint sequence } \{A_j\} \text{ of sets in } \mathcal{B},$$

where equality means that the series on the right converges absolutely to the number on the left. Such a function on $(\mathcal{B}, \subseteq)$ may equivalently be viewed as a *signed measure*[11] on $(S, \mathcal{B})$ [23, pg. 35].

We write $\mathfrak{A}(S, \mathcal{B})$ for the collection of all signed measures on the σ-algebra $\mathcal{B}$ of subsets of the set S Equivalently, these correspond to σ-additive functions on the poset $(\mathcal{B}, \subseteq)$. We recall some terms and facts about signed measures. Two measures[12] μ, ν are said to be *mutually singular* (we write $\mu \perp \nu$) if there are disjoint sets $A, B \in \mathcal{B}$ with $S = A \cup B$ such that $\mu(B) = \nu(A) = 0$. Any signed measure $x \in \mathfrak{A}(S, \mathcal{B})$ induces a *Hahn decomposition* (S_+, S_-) of S, i.e., there are sets S_+, S_- such that

$$\begin{aligned} (i) &\quad S = S_+ \cup S_-, \quad S_+ \cap S_- = \emptyset \\ (ii) &\quad x(\cdot \cap S_+), \quad -x(\cdot \cap S_-) \text{ are both (nonnegative) measures.} \end{aligned}$$

See, e.g., [23, Thm. 3, pg. 36]. The Hahn decomposition (S_+, S_-) is not unique. However, the pair of measures[13] $(x^{\oplus}, x^{\ominus})$ defined by

$$x^{\oplus} := x(\cdot \cap S_+), \quad x^{\ominus} := -x(\cdot \cap S_-)$$

form a unique decomposition of x, called the *Jordan decomposition* [23, Thm. 2, pg. 36], in that

$$\begin{aligned} (i) &\quad x = x^{\oplus} - x^{\ominus}, \\ (ii) &\quad x = \mu - \nu, \quad \mu, \nu \text{ measures}, \quad \mu \perp \nu \quad \Rightarrow \quad \mu = x^{\oplus}, \quad \nu = x^{\ominus}. \end{aligned}$$

[11] Some authors, e.g., [15, pg. 235], allow signed measure to explicitly take on the value $+\infty$ or $-\infty$ (one of them–never both). We will not allow this, but extensions of our results to cover this case are possible.

[12] "Measure" means "nonnegative signed measure".

[13] We choose the notation $x^{\oplus}, x^{\ominus}$ instead of the more traditional notation x^+, x^- of measure theory, to avoid confusion with similar symbols elsewhere in this paper.

The *total variation* $||x||$ of a signed measure x is the measure defined by

$$||x|| := x^{\oplus} + x^{\ominus}.$$

One consequence of the Jordan decomposition is that any signed measure, viewed as a σ-additive function $x : \mathcal{B} \to \mathbb{R}$, is automatically locally bounded, in the sense defined in this paper [23, Lemma 2, pg. 35]. Secondly, since any signed measure can be written as the difference of two measures, when viewed as a function on the poset $(\mathcal{B}, \subseteq)$, it is a function of bounded variation, as defined in this paper.

4.1 The case $\alpha = \beta = 0$

We first consider the SRP of a signed measure x for the case $\alpha = \beta = 0$. The problem is of interest because x is of bounded variation, as discussed above. We find that the solution to the SRP for x is just the Jordan decomposition of x.

Theorem 5 (Jordan decomposition). *Let $x \in \mathfrak{A}(S, \mathcal{B})$ be a signed measure. Let (ℓ^*, u^*) be the solution to the SRP with $\alpha = \beta = 0$. Then $u^* = x^{\oplus}$, $\ell^* = x^{\ominus}$.*

Proof. Since (ℓ^*, u^*) is least among pairs of NN functions $(\overline{\ell}, \overline{u})$ for which $x = \overline{u} - \overline{\ell}$, and since $(x^{\ominus}, x^{\oplus})$ is such a pair, we have

$$\ell^* \leqslant x^{\ominus}, \quad u^* \leqslant x^{\oplus}.$$

However $x^{\oplus} \perp x^{\ominus}$, i.e., there is a partition $S = A \cup B$, with $A, B \in \mathcal{B}$, such that $x^{\oplus}(B) = x^{\ominus}(A) = 0$. Hence $u^*(B) \leqslant x^{\oplus}(B) = 0$, and $\ell^*(A) \leqslant x^{\ominus}(A) = 0$. So $u^* \perp \ell^*$. It is an easy standard argument in measure theory that if $u^* - \ell^* = x^{\oplus} - x^{\ominus}$, $u^* \perp \ell^*$, $x^{\ominus} \perp x^{\oplus}$ actually imply that $u^* = x^{\oplus}$, $\ell^* = x^{\ominus}$. □

Consider now the fixed point equation (5) of the general Theorem 1:

$$\begin{aligned} \ell &= \mathcal{U}(-x + u), \\ u &= \mathcal{U}(x + \ell). \end{aligned}$$

This, written explicitly, reads:

$$\begin{aligned} \ell(B) &= \sup_{C \subseteq B} (-x(C) + u(C)), \quad B \in \mathcal{B} \\ u(B) &= \sup_{C \subseteq B} (x(C) + \ell(C)), \quad B \in \mathcal{B}. \end{aligned} \tag{11}$$

It is easy to see that (11) is satisfied with $(\ell, u) = (\mathcal{L}(x), \mathcal{U}(x))$. Indeed, substituting $\mathcal{U}(x)(C) = \sup_{C' \subseteq C} x(C')$ in the right-hand side of the first of the above gives

$$\begin{aligned} \sup_{C \subseteq B} (-x(C) + \mathcal{U}(x)(C)) &= \sup_{C \subseteq B} (-x(C) + \sup_{C' \subseteq C} x(C')) \\ &= \sup_{C' \subseteq C \subseteq B} (-x(C - C')) \\ &= \sup_{A \subseteq B} (-x(A)) = \mathcal{L}(x)(B), \end{aligned}$$

which is its left-hand side. One can handle the second equation in (11) similarly. These calculation show that the following holds:

Corollary 5. *Let $x \in \mathfrak{A}(S,\mathcal{B})$ be a signed measure. Consider three SRP's: (i) the upward reflection of x at $\alpha = 0$, and let $\bar{\ell}$ be its solution; (ii) the downward reflection of x at $\beta = 0$, and let $\bar{u}$ be its solution; (iii) the reflection of x at $\alpha = \beta = 0$, and let (ℓ^*, u^*) be its solution. Then $x^\ominus = \ell^* = \mathcal{L}(x) = \bar{\ell}$, and $x^\oplus = u^* = \mathcal{U}(x) = \bar{u}$.*

Proof. In the proof of Theorem 5 we showed that $x^\ominus = \ell^*$, and $x^\oplus = u^*$. That $\mathcal{L}(x) = \bar{\ell}$ and $\mathcal{U}(x) = \bar{u}$ follows from the Corollary 1. We next show that $\ell^* = \mathcal{L}(x)$ and $u^* = \mathcal{U}(x)$. Let ℓ, u solve the fixed point equation (5), which, in our case is written as (11). Then, for all $B \in \mathcal{B}$, $u(B) = \sup_{C \subseteq B}(x(C) + \ell(C)) \geqslant \sup_{C \subseteq B} x(C) = \mathcal{U}(x)(B)$. Similarly, $\ell(B) \geqslant \mathcal{L}(x)(B)$, for all $B \in \mathcal{B}$. But the solution (ℓ^*, u^*) to the third SRP certainly satisfies (11). Hence $u^* \geqslant \mathcal{U}(x)$, $\ell^* \geqslant \mathcal{L}(x)$. On the other hand, as shown in the remarks preceding the statement of the corollary, $\mathcal{U}(x), \mathcal{L}(x)$ satisfy the (11). Hence $x = \mathcal{U}(x) - \mathcal{L}(x)$. (Any solution to the fixed point equation (5) must satisfy the inequality (10)–as observed in Remark 3.) Thus, by the minimality of (ℓ^*, u^*) we have $\ell^* \leqslant \mathcal{L}(x)$ and $u^* \leqslant \mathcal{U}(x)$. □

Remark 5. The meaning of Corollary 5 is that the solutions to the one-sided reflection problems at 0 also give the solution to the simultaneous reflection problem when the upper and lower boundary are both 0. This is *not* true in general for the SRP with boundaries 0 on a general poset; the reader should have no difficulty constructing examples on discrete posets, for instance, where this is not true. It is a peculiar feature of the SRP for a signed measure on the poset of a σ-algebra, when the boundaries are the zero measure. ■

Remark 6. We further remark that this corollary provides an alternative proof of the fact that the Jordan decomposition $(x^\oplus, x^\ominus)$ of the signed measure x is given by $x^\oplus(B) := \sup_{C \subseteq B} x(C)$, $x^\ominus(B) := -\inf_{C \subseteq B} x(C)$, for $B \in \mathcal{B}$. (See, e.g., [15] for a standard measure-theoretic proof.)

4.2 The case $\alpha \leqslant \beta$, with α, β being measures

Theorem 6 (generalized Jordan decomposition). *Let $x, \alpha, \beta \in \mathfrak{A}(S,\mathcal{B})$, where $\alpha \leqslant \beta$. We view these as functions on the poset $(\mathcal{B}, \subseteq)$. Let (ℓ^*, u^*) be the solution to the SRP for x with the boundary functions α and β on this poset. Then ℓ^*, u^* are σ-additive, so that they can be viewed as measures*[14] *in $\mathfrak{A}(S,\mathcal{B})$. As measures, we have $\ell^* \perp u^*$. Further, we have the explicit expressions:*

$$\ell^* = (\alpha - x)^\oplus,$$
$$u^* = (x - \beta)^\oplus.$$

Proof. First note that x, α, and β are locally bounded as functions on the poset $(\mathcal{B}, \subseteq)$, by virtue of being signed measures on $(S, \mathcal{B})$. Further, since $x - \alpha$ is a signed measure on $(S, \mathcal{B})$, it is of bounded variation as a function on the poset $(\mathcal{B}, \subseteq)$. Setting $y := x - \alpha$, we have

$$y + \alpha \leqslant x \leqslant y + \beta\ ,$$

so it makes sense to talk about the SRP for x with the boundary functions α and β.

We will construct the solution (ℓ^*, u^*) to this SRP by going through the steps of the iteration that was used in the proof of Theorem 1 in order to argue its existence and uniqueness.

[14] since they are nonnegative signed measures

We start with $\ell^0 = u^0 = 0$. We have

$$\begin{aligned}\ell^1 &= \mathcal{U}(\alpha + u^0 - x) \\ &= \mathcal{U}(\alpha - x) \\ &= (\alpha - x)^{\oplus},\end{aligned}$$

where the expression $(\alpha - x)^{\oplus}$ is viewed as the function on $(\mathcal{B}, \subseteq)$ corresponding to the measure $(\alpha - x)^{\oplus}$. We also have

$$\begin{aligned}u^1 &= \mathcal{U}(x + \ell^0 - \beta) \\ &= \mathcal{U}(x - \beta) \\ &= (x - \beta)^{\oplus}.\end{aligned}$$

At this point we know that $(\alpha - x)^{\oplus} \leqslant \ell^*$ and $(x - \beta)^{\oplus} \leqslant u^*$. If we could show that

$$\beta \geqslant x + (\alpha - x)^{\oplus} - (x - \beta)^{\oplus} \geqslant \alpha \ , \tag{12}$$

then the proof of the theorem would be complete, because equation (12) and the minimality of (ℓ^*, u^*) imply that $(\alpha - x)^{\oplus} \geqslant \ell^*$ and $(x - \beta)^{\oplus} \geqslant u^*$. To show the truth of equation (12) we need to show that

$$x + (\alpha - x)^{\oplus} - (x - \beta)^{\oplus} \geqslant \alpha \ , \tag{13}$$

and

$$\beta \geqslant x + (\alpha - x)^{\oplus} - (x - \beta)^{\oplus} \ . \tag{14}$$

To show (13), write

$$\begin{aligned}(x - \alpha) + (\alpha - x)^{\oplus} &= (x - \alpha) + (x - \alpha)^{\ominus} \\ &= (x - \alpha)^{\oplus} \\ &\geqslant (x - \beta)^{\oplus}\end{aligned}$$

where the last step comes from $(x - \alpha) \geqslant (x - \beta)$. To show (14), write

$$\begin{aligned}(\beta - x) + (x - \beta)^{\oplus} &= (\beta - x) + (\beta - x)^{\ominus} \\ &= (\beta - x)^{\oplus} \\ &\geqslant (\alpha - x)^{\oplus}\end{aligned}$$

where the last step comes from $(\beta - x) \geqslant (\alpha - x)$. This completes the proof. □

Remark 7. What is interesting in the above is that, *a priori*, there is no reason for the solution (ℓ^*, u^*) of the SRP on the poset $(\mathcal{B}, \subseteq)$ to possess additional structure besides monotonicity. It turned out that both ℓ^* and u^* are σ-additive and mutually singular, just as in the standard Jordan decomposition, whence the adjective "generalized". In a sense, this is a Jordan decomposition with "slackness" defined by the boundary functions α, β, or, rather, by $\beta - \alpha$. It is interesting that "explicit" formulas exist for the solution to the SRP of a signed measure on a σ-algebra where the boundary functions are also signed measure. ■

Remark 8. Let $S = \mathbb{R}_+ = [0, \infty)$, and $\mathcal{B}$ its Borel σ-algebra. Consider a signed measure x on $\mathcal{B}$ and the SRP with $\alpha = \beta = 0$. We know that its solution is given by $(\ell^*, u^*) = (x^{\ominus}, x^{\oplus})$. Now consider the function $X : S \to \mathbb{R}$ defined by

$$X(t) := x([0, t]),$$

and consider the SRP with for the function X on the poset $(S, \leqslant)$ with boundaries both at zero. We know that its solution is given by $(L^*, U^*) = (\mathcal{V}^-(X), \mathcal{V}^+(X))$, as defined by (7). Observe that the solutions of the two different SRP's are related, namely,

$$\mathcal{V}^+(X)(t) = x^{\oplus}([0,t]), \quad \mathcal{V}^-(X)(t) = x^{\ominus}([0,t]), \quad t \in \mathbb{R}_+.$$

Thus the Jordan decomposition of a signed measure x, which results from solving the SRP for x with boundaries $\alpha = \beta = 0$ on the poset $(\mathcal{B}, \subseteq)$ is directly related to the Jordan decomposition of the function X, which results from solving the SRP for X with boundaries $\alpha = \beta = 0$ on the poset $(\mathbb{R}_+, \leqslant)$. ■

Remark 9. One might conjecture that the observation above carries through to the case where $\alpha \neq \beta$. Specifically, Consider the SRP for the signed measure x on $(\mathcal{B}, \subseteq)$, as above, but with boundaries α, β, where $\alpha \leqslant \beta$ are signed measures. Let (ℓ^*, u^*) be its solution. Next, let

$$X(t) := x([0,t]), \quad A(t) := \alpha([0,t]), \quad B(t) := \beta([0,t]), \quad t \in \mathbb{R}_+,$$

and consider the SRP for X, with boundary functions A and B, on the poset $(\mathbb{R}_+, \leqslant)$. One might conjecture that $L^*(t) = \ell^*([0,t])$, and $U^*(t) = u^*([0,t])$, Some motivation for this conjecture is provided by the previous remark. Unfortunately, this conjecture is false in general. Indeed, such a conjecture is quite naïve, as there is no particular relation between the order structure of $(\mathcal{B}, \subseteq)$, with bottom element $\emptyset$, and the order structure of $(\mathbb{R}_+, \leqslant)$, with bottom element 0; this perhaps makes the result of Corollary 3 seem surprising. It is probably illustrative to see a concrete counterexample to this conjecture. Let α be the discrete measure assigning mass 1 to the point $\{1\}$, and β the discrete measure assigning mass 2 to the point $\{1\}$. Then

$$A(t) = \begin{cases} 0 & \text{if } 0 \leqslant t < 1 \\ 1 & \text{if } t \geqslant 1 \end{cases},$$

and

$$B(t) = \begin{cases} 0 & \text{if } 0 \leqslant t < 1 \\ 2 & \text{if } t \geqslant 1 \end{cases}.$$

Let x be the signed measure assigning mass $\frac{3}{2}$ to the point $\{1\}$, having uniform density -1 over the interval $(1,2)$, and uniform density 1 over the interval $(2,3)$. Then

$$X(t) = \begin{cases} 0 & \text{if } 0 \leqslant t < 1 \\ \frac{5}{2} - t & \text{if } 1 \leqslant t < 2 \\ -\frac{3}{2} + t & \text{if } 2 \leqslant t < 3 \\ \frac{3}{2} & \text{if } t \geqslant 3 \end{cases}.$$

Note that $\ell^* = (\alpha - x)^{\oplus}$ is the measure with uniform density 1 over the interval $(1,2)$, so, for instance, $\ell^*([0, \frac{3}{2}]) = \frac{1}{2}$. However $L^*(\frac{3}{2}) = 0$. Note that in this example both α and β are actually measures. ■

5 Majorization order

Given $p = (p_1, \ldots, p_n) \in \mathbb{R}^n$, let $p_{[1]} \geqslant p_{[2]} \geqslant \ldots \geqslant p_{[n]}$ be defined as the components of p in decreasing order [12]. Now define the relation $\prec$ on $\mathbb{R}^n$ by:

$$p \prec q \quad \text{iff} \quad \sum_{i=1}^{j} p_{[i]} \leqslant \sum_{i=1}^{j} q_{[i]} \quad \text{for all } 1 \leqslant j \leqslant n-1, \quad \text{and} \quad \sum_{i=1}^{n} p_{[i]} = \sum_{i=1}^{n} q_{[i]};$$

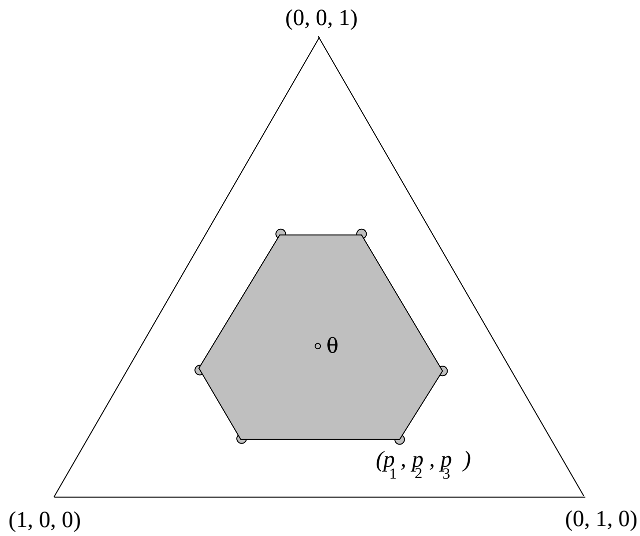
in this case, we say that p is *majorized by* q [12, pg. 7]. This relation is a *preorder*, namely, for all $p, q, r \in \mathbb{R}^n$, $p \prec q, q \prec r \Rightarrow p \prec r$ (transitivity), and $p \prec p$ (reflexivity). However, if $p \prec q$ and $q \prec p$ then the entries of q are a permutation of those of p [12, pp. 12-13].

Let $\mathcal{P}_n$ denote the set of probability distributions on $[n] := \{1, \ldots, n\}$. We may think of the element p of $\mathcal{P}_n$ as a vector in $\mathbb{R}^n$ with nonnegative entries p_i, $i = 1, \ldots, n$ such that $\sum_{i=1}^n p_i = 1$.

Let $\overline{\mathcal{P}}_n$ denote the set of equivalence classes of probability distributions on $[n]$, up to permutation. This can be thought of as comprised of *sets* of n nonnegative numbers that sum to 1. Then $(\overline{\mathcal{P}}_n, \prec)$ is a poset with bottom element the equivalence class of the *uniform distribution*

$$\theta := (1/n, \ldots, 1/n)$$

on $[n]$ [12, pg. 13]. Elements of $\overline{\mathcal{P}}_n$ will be denoted by $p, q, r, \ldots$.[15] See the figure.



The unit simplex in $\mathbb{R}^3$*. The shaded region represents those vectors* q *such that* $q \prec p$*. The center* θ *is the bottom element.*

The increasing functions on $(\overline{\mathcal{P}}_n, \prec)$ can be identified with the *Schur convex* functions on $\mathcal{P}_n$ [12, pp. 13-14]. The negative of a Schur convex function is called a *Schur concave* function. Thus the functions of bounded variation on $(\overline{\mathcal{P}}_n, \prec)$ are precisely those that can be expressed as the sum of a Schur convex function and a Schur concave function.

Theorem 1, applied to the poset $(\overline{\mathcal{P}}_n, \prec)$ with upper and lower boundaries $\alpha = \beta = 0$ tells us that every bounded variation function $x : \overline{\mathcal{P}}_n \to \mathbb{R}$ with $x(\theta) = 0$ has a minimal decomposition of the form $x = u^* - \ell^*$ where u^* and ℓ^* are nonnegative Schur convex functions, in the sense that for any pair of nonnegative Schur convex functions (ℓ, u) with $x = u - \ell$, we have $\ell^* \leqslant \ell$ and $u^* \leqslant u$.

Remark 10. Suppose $A \subset \overline{\mathcal{P}}_n$ has a Schur minimal element, denoted p_m, and a Schur maximal element, denote p_M, i.e., for all $p \in A$ we have $p_m \prec p \prec p_M$. Many interesting sets of probability distributions are symmetric and are such that the set of equivalence classes under symmetry (which is a subset of $\overline{\mathcal{P}}_n$) admits both a Schur minimal and a Schur maximal element. Suppose x is a function of bounded variation on $\overline{\mathcal{P}}_n$, and let $x = u^* - \ell^*$

[15] We abuse notation and use p also for equivalence classes of probability distributions under symmetry in order to not overload the notation with overlines.

be its minimal representation as the difference of two nonnegative Schur convex functions. Then one has the bounds

$$\begin{aligned} x(p) &\leqslant u^*(p_M) - \ell^*(p_m) \quad \text{and} \\ x(p) &\geqslant u^*(p_m) - \ell^*(p_M) \end{aligned}$$

for all $p \in A$. The solution to the SRP also gives the best bounds of this form among all representation of x as the difference of two nonnegative Schur convex functions on $(\overline{\mathcal{P}}_n, \prec)$. The analogous claim is also true in any poset: for a function of bounded variation the solution to the SRP gives uniform upper and lower bounds over any subset of the poset that admits both a maximum and a minimum element, and the bounds are the best possible among bounds of this kind (i.e., those that result from representing the function as a difference of two NN functions). ■

5.1 Regulating an order statistic

It is interesting to explore the solution to the SRP with $\alpha = \beta = 0$ for some natural functions on the poset $(\overline{\mathcal{P}}_n, \prec)$ of equivalence classes of probability distributions on $[n]$ with majorization order. We work out an example here.

For $1 \leqslant k \leqslant n$, let $\boldsymbol{\delta}_k, \boldsymbol{\sigma}_k$ denote the functions on $\overline{\mathcal{P}}_n$ given by

$$\begin{aligned} \boldsymbol{\delta}_k(p) &:= p_{[k]}, \\ \boldsymbol{\sigma}_k(p) &:= p_{[1]} + \cdots + p_{[k]}, \end{aligned}$$

where $p_{[1]} \geqslant p_{[2]} \geqslant \ldots \geqslant p_{[n]}$ denote the components of $p \in \overline{\mathcal{P}}_n$ in decreasing order. Thus, $\boldsymbol{\delta}_k(p)$ gives the k-th *order statistic*[16] of p. We seek the solution to the SRP for the function

$$x_k := \boldsymbol{\delta}_k - \frac{1}{n},$$

which equals 0 at θ, the bottom element of $(\overline{\mathcal{P}}_n, \prec)$. This function is of bounded variation, since we can write

$$x_k = \boldsymbol{\sigma}_k - \left(\boldsymbol{\sigma}_{k-1} + \frac{1}{n}\right) \tag{15}$$

where $\boldsymbol{\sigma}_k$ is seen to be Schur convex on $\overline{\mathcal{P}}_n$, from the definition of majorization.

Notice that $\boldsymbol{\sigma}_{k-1}(\theta) = (k-1)/n$. It is therefore easy to improve the representation in equation (15) and write

$$x_k = \left(\boldsymbol{\sigma}_k - \frac{k}{n}\right) - \left(\boldsymbol{\sigma}_{k-1} - \frac{k-1}{n}\right), \tag{16}$$

where we observe that $\boldsymbol{\sigma}_{k-1} - \frac{k-1}{n}$ and $\boldsymbol{\sigma}_k - \frac{k}{n}$ are nonnegative Schur convex functions (NN functions).

We now argue that the representation in equation (16) is the solution to the SRP for x_k, i.e., that $(\ell^*, u^*) := (\boldsymbol{\sigma}_{k-1} - \frac{k-1}{n}, \boldsymbol{\sigma}_k - \frac{k}{n})$ is the minimal pair of nonnegative Schur convex functions among all pairs (ℓ, u) with $\boldsymbol{\delta}_k - \frac{1}{n} = u - \ell$. This can be seen by going through the steps in the proof of Theorem 1, where the solution to the SRP was constructed as the limit of a sequence of iterations. The following lemma is useful in the computations below.

[16]The terminology is borrowed from Statistics.

Lemma 3. *Fix $p \in \overline{\mathcal{P}}_n$ and $c_1, c_2 \geqslant 0$. For $m \in [n]$ let $\boldsymbol{\tau}^m(p)$ denote the element of $\overline{\mathcal{P}}_n$ given by*

$$\boldsymbol{\tau}^m(p)_i = \begin{cases} \frac{\boldsymbol{\sigma}_m(p)}{m}, & 1 \leqslant i \leqslant m \\ \frac{1-\boldsymbol{\sigma}_m(p)}{n-m}, & m+1 \leqslant i \leqslant n. \end{cases}$$

Then:
(i)

$$\sup\{c_1\boldsymbol{\sigma}_k(r) + c_2\boldsymbol{\delta}_k(r) : \ r \prec p\} = c_1\boldsymbol{\sigma}_k(p) + c_2\frac{\boldsymbol{\sigma}_k(p)}{k},$$

and the supremum is achieved at $r = \boldsymbol{\tau}^k(p)$.
(ii)

$$\sup\{c_1\boldsymbol{\sigma}_{k-1}(r) - c_2\boldsymbol{\delta}_k(r) : \ r \prec p\} = c_1\boldsymbol{\sigma}_{k-1}(p) - c_2\frac{1-\boldsymbol{\sigma}_{k-1}(p)}{n-k+1},$$

supremum is achieved at $r = \boldsymbol{\tau}^{k-1}(p)$.

Proof. (i) For $r \prec p$ we have $c_1\boldsymbol{\sigma}_k(r) \leqslant c_1\boldsymbol{\sigma}_k(p)$. Also, we have $c_2\boldsymbol{\delta}_k(r) \leqslant c_2\frac{\boldsymbol{\sigma}_k(r)}{k} \leqslant c_2\frac{\boldsymbol{\sigma}_k(p)}{k}$. It is also straightforward to check that $c_1\boldsymbol{\sigma}_k(r) + c_2\boldsymbol{\delta}_k(r)$ evaluates to $c_1\boldsymbol{\sigma}_k(p) + c_2\frac{\boldsymbol{\sigma}_k(p)}{k}$ at $r = \boldsymbol{\tau}^k(p)$. (ii) For $r \prec p$ we have $c_1\boldsymbol{\sigma}_(k-1)(r) \leqslant c_1\boldsymbol{\sigma}_(k-1)(p)$. Also, we have $c_2\boldsymbol{\delta}_(k)(r) \geqslant c_2\frac{1-\boldsymbol{\sigma}_(k-1)(r)}{n-k+1} \geqslant c_2\frac{1-\boldsymbol{\sigma}_(k-1)(p)}{n-k+1}$. It is again straightforward to check that $c_1\boldsymbol{\sigma}_{k-1}(r) - c_2\boldsymbol{\delta}_k(r)$ evaluates to $c_1\boldsymbol{\sigma}_{k-1}(p) - c_2\frac{1-\boldsymbol{\sigma}_{k-1}(p)}{n-k+1}$ at $r = \boldsymbol{\tau}^{k-1}(p)$. □

We continue with our example. Recall our nondecreasing recursive scheme:

$$\ell^{j+1}(p) = \mathcal{U}(-x_k + u^j)(p) = \sup_{r \prec p}(-\boldsymbol{\delta}_k(r) + (1/n) + u^j(r)) \vee 0,$$
$$u^{j+1}(p) = \mathcal{U}(x_k + \ell^j)(p) = \sup_{r \prec p}(\boldsymbol{\delta}_k(r) - (1/n) + \ell^j(r)) \vee 0.$$

We start with $\ell^0 = u^0 = 0$. It is easy to see that, since $x_k(\theta) = 0$, and $\ell^{j+1}(\theta) = u^j(\theta) \geqslant 0$, $u^{j+1}(\theta) = \ell^j(\theta) \vee 0$, for all j, we have $\ell^j(\theta) = u^j(\theta) = 0$ for all j, and hence the maximum with 0 in the above recursion can be dropped. We have $\ell^1 = \mathcal{U}(-x_k) = 0$, because $x_k \geqslant 0$. Then

$$u^1(p) = \sup_{r \prec p} \boldsymbol{\delta}_k(r) - \frac{1}{n} = \frac{\boldsymbol{\sigma}_k(p)}{k} - \frac{1}{n},$$

by (i) of Lemma 3. Next,

$$\ell^2(p) = \sup_{r \prec p}\left(-\boldsymbol{\delta}_k(r) + \frac{\boldsymbol{\sigma}_k(r)}{k}\right) = \sup_{r \prec p}\left(\frac{\boldsymbol{\sigma}_{k-1}(r)}{k} - \frac{k-1}{k}\boldsymbol{\delta}_k(r)\right)$$
$$= \frac{\boldsymbol{\sigma}_{k-1}(p)}{k} - \frac{k-1}{k}\ \frac{1-\boldsymbol{\sigma}_{k-1}(p)}{n-k+1} = c_2\boldsymbol{\sigma}_{k-1}(p) - d_2,$$

where we used (ii) of Lemma 3 and define c_2 and d_2 in this equation. Also,

$$u^2(p) = \sup_{r \prec p} \delta_k(r) - \frac{1}{n} = u^1(p) = \frac{1}{k}\boldsymbol{\sigma}_k(p) - \frac{1}{n} = a_2\boldsymbol{\sigma}_k(p) - b_2,$$

where we define a_2 and b_2 in this equation. We now argue by induction that, for all $j \geqslant 2$, we have

$$\ell^j \ = \ c_j\boldsymbol{\sigma}_{k-1} - d_j\ , \quad \text{and}$$
$$u^j \ = \ a_j\boldsymbol{\sigma}_k - b_j,$$

for nonnegative constants a_j, b_j, c_j, d_j with $a_j \leqslant 1$, $b_j \geqslant \frac{1}{n}$, and $c_j \leqslant 1$ for all $j \geqslant 2$. The induction starts with the case $j = 2$, where have

$$a_2 = \frac{1}{k}, \quad b_2 = \frac{1}{n}, \quad c_2 = \frac{n}{k(n-k+1)}, \quad d_2 = \frac{k-1}{k(n-k+1)}.$$

For the inductive step, first consider

$$\begin{aligned}
\ell^{j+1}(p) &= \sup_{r \prec p}(-\boldsymbol{\delta}_k(r) + (1/n) + a_j \boldsymbol{\sigma}_k(r) - b_j) \\
&= \sup_{r \prec p}[a_j \boldsymbol{\sigma}_{k-1}(r) - (1-a_j)\boldsymbol{\delta}_k(r)] - [b_j - (1/n)] \\
&= a_j \boldsymbol{\sigma}_{k-1}(p) - (1-a_j)\frac{1-\boldsymbol{\sigma}_{k-1}(p)}{n-k+1} - [b_j - (1/n)] \\
&= c_{j+1}\boldsymbol{\sigma}_{k-1}(p) - d_{j+1},
\end{aligned}$$

where we used (i) of Lemma 3, and where

$$c_{j+1} = \frac{(n-k)a_j + 1}{n-k+1}, \quad d_{j+1} = b_j - \frac{1}{n} + \frac{1-a_j}{n-k+1}. \tag{17}$$

We also observe that $0 \leqslant c_{j+1} \leqslant 1$ and $d_{j+1} \geqslant 0$. To complete the inductive step, consider

$$\begin{aligned}
u^{j+1}(p) &= \sup_{r \prec p}(\boldsymbol{\delta}_k(r) - (1/n) + c_j \boldsymbol{\sigma}_{k-1}(r) - d_j) \\
&= \sup_{r \prec p}[c_j \boldsymbol{\sigma}_k(r) + (1-c_j)\boldsymbol{\delta}_k(r)] - [d_j - (1/n)] \\
&= c_j \boldsymbol{\sigma}_k(p) + (1-c_j)\frac{\boldsymbol{\sigma}_k(p)}{k} - [d_j - (1/n)] \\
&= a_{j+1}\boldsymbol{\sigma}_k(p) - b_{j+1},
\end{aligned}$$

where we used (i) of Lemma 3, and where

$$a_{j+1} = \frac{1+(k-1)c_j}{k}, \quad b_{j+1} = d_j + \frac{1}{n}. \tag{18}$$

We also observe that $0 \leqslant a_{j+1} \leqslant 1$ and $b_{j+1} \geqslant \frac{1}{n}$. This completes the inductive proof of the claim regarding the structure of ℓ^j and u^j for $j \geqslant 2$.

Now notice that both a_j and c_j are nondecreasing in j. This is apparent, by induction on j, from equations (17) and (18). Let a_∞ and c_∞ denote the respective limits of these increasing sequences. Then

$$\begin{aligned}
a_\infty &= \frac{1+(k-1)c_\infty}{k}, \\
c_\infty &= \frac{(n-k)a_\infty + 1}{n-k+1}.
\end{aligned}$$

This pair of equations has the unique solutions $a_\infty = c_\infty = 1$. From equation (17) we have $d_3 = b_2 - \frac{1}{n} + \frac{1-a_2}{n-k+1} = \frac{k-1}{k(n-k+1)} = d_2$. It then follows from equations (17) and (18) that the sequence d_j is nondecreasing in j. Let d denote the limit of this sequence. As apparent from equation (18), the limit of the sequence b_j exists and equals $d + \frac{1}{n}$.

We conclude that the solution to the SRP for $x_k := \boldsymbol{\delta}_k - \frac{1}{n}$ is $(\boldsymbol{\sigma}_{k-1} - d, \boldsymbol{\sigma}_k - d - \frac{1}{n})$. However, it is clear that for any real number d for which $\boldsymbol{\sigma}_{k-1} - d$ is everywhere nonnegative, the pair $(\ell, u) = (\boldsymbol{\sigma}_{k-1} - d, \boldsymbol{\sigma}_k - d - \frac{1}{n})$ gives a representation of $\boldsymbol{\delta}_k - \frac{1}{n}$ as the difference of two nonnegative Schur convex functions. The largest d for which this holds will give the minimal representation in this class of representations, and this is the choice $\frac{k-1}{n}$. It must then be the case that d is actually $\frac{k-1}{n}$, and we have proved that the solution to the SRP for $x_k := \boldsymbol{\delta}_k - \frac{1}{n}$ is $(\ell^*, u^*) = (\boldsymbol{\sigma}_{k-1} - \frac{k-1}{n}, \boldsymbol{\sigma}_k - \frac{k}{n})$, as claimed. ■

Remark 11. It will be of interest to explicitly work out the solution to the SRP for arbitrary[17] *monomials*. A monomial is a function on $\overline{\mathcal{P}}_n$ of the type

$$\boldsymbol{\delta}_{k_1}^{m_1}\boldsymbol{\delta}_{k_2}^{m_2}\dots\boldsymbol{\delta}_{k_d}^{m_d},$$

where $d \geqslant 1$, $1 \leqslant k_1 < k_2 < \dots < k_d \leqslant n$, and $m_j \geqslant 1$, $1 \leqslant j \leqslant d$. It is straightforward to see that any such function is of bounded variation (simply use the formula $\boldsymbol{\delta}_k = \boldsymbol{\sigma}_k - \boldsymbol{\sigma}_{k-1}$, which was also used in the example above, expand out the product, and gather all terms with positive coefficients on one side and all terms with negative coefficients on the other side). However: what is the *minimal representation* of such a function as the difference of nonnegative Schur convex functions? ■

6 Positive semidefinite symmetric matrices

Let $\mathcal{S}$ denote the set of positive semidefinite symmetric $d \times d$ real matrices. Let $\prec$ be the partial order defined on $\mathcal{S}$ as follows: given $A, B \in \mathcal{S}$, we say $B \prec A$ iff $A - B$ is a positive semidefinite matrix, i.e., iff

$$x^T Bx \leqslant x^T Ax \quad \text{for all } x \in \mathbb{R}^d.$$

Then $(\mathcal{S}, \prec)$ is a partially ordered set with minimal element the zero matrix, which will be denoted $\underline{0}$ in the sequel. A real valued function on $(\mathcal{S}, \prec)$ will be called a *matrix function*. Theorem 1, applied to the poset $(\mathcal{S}, \prec)$ with upper and lower boundaries the matrix functions $\alpha = \beta = 0$ tells us that every bounded variation matrix function x with $x(\underline{0}) = 0$ has a minimal decomposition of the form $x = u^* - \ell^*$ where u^* and ℓ^* are NN matrix functions, in the sense that for any pair of NN matrix functions (ℓ, u) with $x = u - \ell$, we have $\ell^* \leqslant \ell$ and $u^* \leqslant u$.

Remark 12. Every positive semidefinite symmetric $d \times d$ real matrix A has d real eigenvalues, which we enumerate as $\lambda_1(A) \geqslant \lambda_2(A) \geqslant \dots \geqslant \lambda_d(A) \geqslant 0$. It is well known that each of the matrix functions λ_j, $1 \leqslant j \leqslant d$, is nonnegative and nondecreasing on the poset $(\mathcal{S}, \prec)$. This is most easily seen from the variational characterization of $\lambda_j(A)$ [9, pg. 179]. Given vectors $y_1, \dots, y_l \in \mathbb{R}^d$, let $V^{\perp}_{y_1,\dots,y_l}$ denote the orthogonal complement of the subspace spanned by these vectors. Then

$$\lambda_j(A) = \inf_{y_1,\dots,y_{j-1}} \sup_{x \in V^{\perp}_{y_1,\dots,y_{j-1}},\ x^T x = 1} x^T Ax. \tag{19}$$

Here, the interpretation for $j = 1$ is that there is no outer infimum and the condition $x \in V^{\perp}_{y_1,\dots,y_{j-1}}$ is vacuous. Now suppose $B \prec A$. Then, for every $y_1, \dots, y_{j-1}$ we have

$$\sup_{x \in V^{\perp}_{y_1,\dots,y_{j-1}},\ x^T x = 1} x^T Bx \leqslant \sup_{x \in V^{\perp}_{y_1,\dots,y_{j-1}},\ x^T x = 1} x^T Ax,$$

so we may next take the outer infimum in equation (19) to conclude that $\lambda_j(B) \leqslant \lambda_j(A)$. ■

6.1 Examples

It is interesting to explore the solution to the SRP with $\alpha = \beta = 0$ for some natural matrix functions on the poset $(\mathcal{S}, \prec)$. We work out some examples here.

[17] with a suitable constant subtracted

6.1.1 Differences of eigenvalues

For $1 \leqslant i < j \leqslant d$, let x_{ij} denote the matrix function

$$x_{ij} := \lambda_i - \lambda_j.$$

This is obviously of bounded variation, being the difference of the NN matrix functions λ_i and λ_j. We claim that this representation is the solution to the SRP for x_{ij}, i.e., that (λ_j, λ_i) is the minimal pair of NN matrix functions among all pairs (ℓ, u) with $x_{ij} = u - \ell$. This can be seen by going through the steps of the iteration in the proof of Theorem 1. Before beginning, let us note that $x_{ij}(\underline{0}) = \lambda_i(\underline{0}) = \lambda_j(\underline{0}) = 0$.

We start with $\ell^0 = u^0 = 0$. We have $\ell^1 = \mathcal{U}(-x_{ij}) = 0$, because $x_{ij} \geqslant 0$. Then

$$\begin{aligned} u^1(A) &= \mathcal{U}(x_{ij})(A) \\ &= \sup_{B \prec A} x_{ij}(B) \vee 0 \\ &= \sup_{B \prec A} \lambda_i(B) - \lambda_j(B) \\ &= \lambda_i(A)\ , \end{aligned}$$

which means $u^1 = \lambda_i$. Here the last step is because one can find $B \prec A$ with $\lambda_i(B) = \lambda_i(A)$ and $\lambda_j(B) = 0$. Now note that this already completes the proof. This is because we know from the proof of Theorem 1 that $u^* \geqslant u^1$, where (ℓ^*, u^*) denotes the solution to the SRP for x_{ij}. Hence we must have $u^* \geqslant \lambda_i$, and then trivially also $\ell^* \geqslant \lambda_j$, so it now follows that $(\ell^*, u^*) = (\lambda_j, \lambda_i)$.

6.1.2 Powers of differences of eigenvalues

For $1 \leqslant i < j \leqslant d$ and $m \geqslant 1$, let x_{ij}^m denote the matrix function

$$x_{ij}^m := (\lambda_i - \lambda_j)^m.$$

This may be seen to be of bounded variation by expanding out the power, and gathering the terms with positive coefficients to one side and the terms with negative coefficients to the other side, which gives a representation of x_{ij}^m as the difference of two NN matrix functions.

We now observe that the matrix function $\lambda_i^m - (\lambda_i - \lambda_j)^m$ is nonnegative and nondecreasing. Thus we have the representation

$$x_{ij}^m = \lambda_i^m - [\lambda_i^m - (\lambda_i - \lambda_j)^m]\ ,$$

of x_{ij}^m in terms of the pair of NN matrix functions $(\lambda_i^m - (\lambda_i - \lambda_j)^m, \lambda_i^m)$. We claim that this representation is the solution to the SRP for x_{ij}^m. This can also be seen by going through the steps of the iteration in the proof of Theorem 1.

We start with $\ell^0 = u^0 = 0$. We have $\ell^1 = \mathcal{U}(-x_{ij}^m) = 0$, because $x_{ij}^m \geqslant 0$. Then

$$\begin{aligned} u^1(A) &= \mathcal{U}(x_{ij}^m)(A) \\ &= \sup_{B \prec A} x_{ij}^m(B) \vee 0 \\ &= \sup_{B \prec A} (\lambda_i(B) - \lambda_j(B))^m \\ &= \lambda_i^m(A)\ , \end{aligned}$$

which means $u^1 = \lambda_i^m$. Here the last step, as in the case $m = 1$ considered earlier, is because one can find $B \prec A$ with $\lambda_i(B) = \lambda_i(A)$ and $\lambda_j(B) = 0$. As in the case $m = 1$, this already completes the proof.

6.1.3 Ratios

For $1 \leqslant i < j \leqslant d$, and an arbitrary real number $a > 0$, let y_{ij} denote the matrix function.[18]

$$y_{ij} := \frac{\lambda_i}{\lambda_j + a}$$

This is of bounded variation, since we may write

$$\frac{\lambda_i}{\lambda_j + a} = \frac{1}{a}\lambda_i - \left(\frac{1}{a}\lambda_i - \frac{\lambda_i}{\lambda_j + a}\right),$$

which gives a representation of y_{ij} in terms of the pair of NN functions $(\frac{1}{a}\lambda_i - \frac{\lambda_i}{\lambda_j+a}, \frac{1}{a}\lambda_i)$. We claim that this representation is the solution to the SRP for y_{ij}. This can be seen by going through the steps of the iteration in the proof of Theorem 1.

We start with $\ell^0 = u^0 = 0$. We have $\ell^1 = \mathcal{U}(-y_{ij}) = 0$, because $y_{ij} \geqslant 0$. Then

$$\begin{aligned} u^1(A) &= \mathcal{U}(y_{ij})(A) \\ &= \sup_{B \prec A} y_{ij}(B) \vee 0 \\ &= \sup_{B \prec A} \frac{\lambda_i(B)}{\lambda_j(B) + a} \\ &= \frac{1}{a}\lambda_i(A) , \end{aligned}$$

which means $u^1 = \frac{1}{a}\lambda_i$. This completes the proof.

6.1.4 Powers of ratios

For $1 \leqslant i < j \leqslant d$, $m \geqslant 1$, and an arbitrary real number $a > 0$, consider

$$y_{ij}^m := \left(\frac{\lambda_i}{\lambda_j + a}\right)^m .$$

This is of bounded variation, since we may write

$$\left(\frac{\lambda_i}{\lambda_j + a}\right)^m = \frac{1}{a^m}\lambda_i^m - \left(\frac{1}{a^m}\lambda_i^m - \left(\frac{\lambda_i}{\lambda_j + a}\right)^m\right),$$

which gives a representation of y_{ij}^m in terms of the pair of NN functions $(\frac{1}{a^m}\lambda_i^m - (\frac{\lambda_i}{\lambda_j+a})^m, \frac{1}{a^m}\lambda_i^m)$. We claim that this representation is the solution to the SRP for y_{ij}. This can be seen by going through the steps of the iteration in the proof of Theorem 1.

We start with $\ell^0 = u^0 = 0$. We have $\ell^1 = \mathcal{U}(-y_{ij}^m) = 0$, because $y_{ij}^m \geqslant 0$. Then

$$\begin{aligned} u^1(A) &= \mathcal{U}(y_{ij}^m)(A) \\ &= \sup_{B \prec A} y_{ij}^m(B) \vee 0 \\ &= \sup_{B \prec A} (\frac{\lambda_i(B)}{\lambda_j(B) + a})^m \\ &= \frac{1}{a^m}\lambda_i^m(A) , \end{aligned}$$

which means $u^1 = \frac{1}{a^m}\lambda_i^m$. This completes the proof.

[18]We add a constant to the denominator to make the ratio well defined when $\lambda_j(A) = 0$.

6.1.5 Matrix entries

On the poset $(\mathcal{S}, \prec)$, with bottom element the zero matrix, consider the matrix function

$$z_{ij}(A) := e_i^T A e_j,$$

where e_i is the unit vector for coordinate i. Thus z_{ij} gives the (i,j) coordinate of its argument. It is interesting to study the solution to the SRP for this function with boundary functions $\alpha = \beta = 0$, i.e., to determine the minimal representation of z_{ij} as a difference of NN functions. For $i = j$ this function is already NN, since it equals $e_i^T A e_i$. For all $i \neq j$, z_{ij} is a function of bounded variation, because

$$z_{ij}(A) = \left[\frac{1}{2}(e_i + e_j)^T A (e_i + e_j)\right] - \left[\frac{1}{2}(e_i^T A e_i + e_j^T A e_j)\right] .$$

The solution to the SRP for all the z_{ij}, $i \neq j$, should be the same, up to permutation, by symmetry considerations. Hence we focus on z_{12}, and economize on subscripts by writing it as z, i.e.,

$$z(A) := e_1^T A e_2.$$

We first describe how one might guess the solution to the SRP for z with boundary functions $\alpha = \beta = 0$. Observe that, for any real number $\gamma > 0$, we have

$$z(A) = e_1^T A e_2 = \left[(\gamma e_1 + \frac{1}{2\gamma} e_2)^T A (\gamma e_1 + \frac{1}{2\gamma} e_2)\right] - \left[\gamma^2 e_1^T A e_1 + \frac{1}{4\gamma^2} e_2^T A e_2\right] .$$

This suggests to consider the NN function:

$$\inf_{\gamma > 0} \left\{ \gamma^2 e_1^T A e_1 + \frac{1}{4\gamma^2} e_2^T A e_2 \right\} .$$

Straightforward algebra reveals that this function is $\sqrt{(e_1^T A e_1)(e_2^T A e_2)}$. Next, we recognize that, for any $A \in \mathcal{S}$,

$$\det{}_{12}(A) = (e_1^T A e_1)(e_2^T A e_2) - (e_1^T A e_2)^2 \geqslant 0 ,$$

where $\det_{12}(A)$ denotes the determinant of the 2×2 submatrix of A defined by the first two rows and the first two columns of A (i.e., we focus attention on the subspace spanned by e_1 and e_2). We see from this that the function:

$$A \mapsto \sqrt{(e_1^T A e_1)(e_2^T A e_2)} - e_1^T A e_2 , \tag{20}$$

is a nonnegative function, as is the function:

$$A \mapsto \sqrt{(e_1^T A e_1)(e_2^T A e_2)} + e_1^T A e_2 . \tag{21}$$

Further, each of these functions is strictly positive precisely where $\det_{12}(A) \neq 0$. We now claim that each of these functions is also nondecreasing on the poset $(\mathcal{S}, \prec)$. To see this for the function (20), let $v \in \mathbb{R}^d$ be an arbitrary vector, let $\delta \geqslant 0$ be a real number, let A_δ denote $A + \delta v v^T$, and consider the function

$$\delta \mapsto \sqrt{(e_1^T A_\delta e_1)(e_2^T A_\delta e_2)} - e_1^T A_\delta e_2 .$$

The first derivative of this function at $\delta = 0$ exists for all A with $\det_{12}(A) \neq 0$ and equals

$$\frac{1}{2}(e_1^T v)^2 \sqrt{\frac{e_2^T A e_2}{e_1^T A e_1}} + \frac{1}{2}(e_2^T v)^2 \sqrt{\frac{e_1^T A e_1}{e_2^T A e_2}} - (e_1^T v)(e_2^T v) \ ,$$

which is nonnegative. This demonstrates that the function (20) is nondecreasing on the poset $(\mathcal{S}, \prec)$. The proof that the function (21) is nondecreasing is similar.

Let

$$\begin{aligned} \widehat{\ell}(A) &= \frac{1}{2}\left[\sqrt{(e_1^T A e_1)(e_2^T A e_2)} - e_1^T A e_2\right] \ , \\ \widehat{u}(A) &:= \frac{1}{2}\left[\sqrt{(e_1^T A e_1)(e_2^T A e_2)} + e_1^T A e_2\right] \ . \end{aligned} \tag{22}$$

Then we have

$$z = \widehat{u} - \widehat{\ell} \ .$$

Also, we have just demonstrated that $(\widehat{\ell}, \widehat{u})$ is a pair of NN function on the poset $(\mathcal{S}, \prec)$. We will next demonstrate that this representation is the solution to the SRP for z with boundary functions $\alpha = \beta = 0$.

We follow the steps of the iterative procedure by which the existence and uniqueness of the solution to the SRP was demonstrated in Theorem 1. We start with $\ell^0 = u^0 = 0$.

We have

$$u^1(A) = \mathcal{U}(z)(A) = \sup_{B \prec A} e_1^T B e_2 \ .$$

To determine $u^1(A)$ it is enough to restrict attention to the subspace spanned by e_1 and e_2, so we will assume, for purposes of determining $u^1(A)$, that A is a 2×2 matrix. Further, since $u_1(\underline{0}) = 0$, we will assume that $A \neq \underline{0}$. We next observe that in order to determine $u^1(A)$ it suffices to consider $B \prec A$ of rank 1. Indeed, any $B \in \mathcal{S}$ can be written as $B = \eta_1 w_1 w_1^T + \eta_2 w_2 w_2^T$, with $\eta_1 \geqslant \eta_2 \geqslant 0$, $w_1^T w_1 = w_2^T w_2 = 1$, and $w_1^T w_2 = 0$, and then we have

$$\begin{aligned} e_1^T B e_2 &= e_1^T(\eta_1 w_1 w_1^T + \eta_2 w_2 w_2^T) e_2 \\ &= (\eta_1 - \eta_2) e_1^T (w_1 w_1^T) e_2 + \eta_2 e_1^T (w_1 w_1^T + w_2 w_2^T) e_2 \\ &= (\eta_1 - \eta_2) e_1^T (w_1 w_1^T) e_2 \ , \end{aligned}$$

because $w_1 w_1^T + w_2 w_2^T$ is the 2×2 identity matrix, and $e_1^T e_2 = 0$. Since $B \prec A$ implies that $(\eta_1 - \eta_2) w_1 w_1^T \prec A$, we might as well have taken B to equal $(\eta_1 - \eta_2) w_1 w_1^T$, for purposes of determining $u_1(A)$.

Thus, fix $w \in \mathbb{R}^2$ with $w^T w = 1$, and ask how large η can be so that we still have

$$\eta w w^T \prec A \ . \tag{23}$$

Recall that we assume $A \neq 0$. It is convenient to express A as $A = \lambda_1 v_1 v_1^T + \lambda_2 v_2 v_2^T$ with $\lambda_1 \geqslant \lambda_2 \geqslant 0$, $v_1^T v_1 = v_2^T v_2 = 1$, and $v_1^T v_2 = 0$, and to write

$$w = \gamma_1 v_1 + \gamma_2 v_2 \ .$$

Then, if we write

$$\bar{w} := -\gamma_2 v_1 + \gamma_1 v_2 \ ,$$

we have $\bar{w}^T\bar{w} = 1$ and $\bar{w}^T w = 0$. Consider an arbitrary vector in $\mathbb{R}^2$, written as $aw + \bar{a}\bar{w}$. The condition in equation (23) requires that, for every choice of $a, \bar{a}$, we have

$$\eta(aw + \bar{a}\bar{w})^T ww^T(aw + \bar{a}\bar{w}) \leqslant (aw + \bar{a}\bar{w})^T A(aw + \bar{a}\bar{w}) \ ,$$

i.e., that

$$\eta a^2 \leqslant \lambda_1(a\gamma_1 - \bar{a}\gamma_2)^2 + \lambda_2(a\gamma_2 + \bar{a}\gamma_1)^2 \ .$$

We wish to determine the largest value of η satisfying this condition for all $a, \bar{a}$. Some algebra shows that this is:

$$\eta^*(w) := \frac{\lambda_1\lambda_2}{\lambda_1\gamma_2^2 + \lambda_2\gamma_1^2} \ . \tag{24}$$

This expression holds for all $A \neq 0$ (including those of rank 1). Note that the denominator of the expression on the right hand side of equation (24) is uniquely defined in terms of w even when A has rank 1, and even when $\lambda_1 = \lambda_2$ so this formula makes sense.[19]

So now, to determine $u_1(A)$ for $A \neq 0$, we may write

$$u_1(A) = \sup_{w \in \mathbb{R}^2 \ : \ w^T w = 1} \eta^*(w) e_1^T(ww^T)e_2 \ ,$$

where $\eta^*(w)$ is given by equation (24). Let $w := [w_1 \ w_2]^T$, $v_1 := [v_{11} \ v_{12}]^T$, and $v_2 := [v_{21} \ v_{22}]^T$ be the expression of w, v_1, and v_2 respectively in coordinates in the basis given by e_1 and e_2. Then $\gamma_1 = w^T v_1 = w_1 v_{11} + w_2 v_{12}$ and $\gamma_2 = w^T v_2 = w_1 v_{21} + w_2 v_{22}$. We conclude that

$$u_1(A) = \sup_{w_1, w_2 \ : \ w_1^2 + w_2^2 = 1} \frac{\lambda_1\lambda_2 w_1 w_2}{\lambda_1(w_1 v_{21} + w_2 v_{22})^2 + \lambda_2(w_1 v_{11} + w_2 v_{12})^2} \ .$$

Some algebra shows that the optimal solution has w_1 and w_2 in the ratio

$$\frac{w_1}{w_2} = \frac{\sqrt{\lambda_1 v_{22}^2 + \lambda_2 v_{12}^2}}{\sqrt{\lambda_1 v_{21}^2 + \lambda_2 v_{11}^2}} = \frac{\sqrt{e_1^T A e_1}}{\sqrt{e_2^T A e_2}} \ .$$

Substituting for this gives

$$u_1 = \widehat{u} \ ,$$

where $\widehat{u}$ is defined in equation (22).

Now, we also have

$$\ell^1(A) = \mathcal{U}(-z)(A) = \sup_{B \prec A} -e_1^T B e_2 \ .$$

Mimicking the preceding calculation (the only difference is the use of $-e_1$ instead of e_1) we arrive at the conclusion that

$$\ell^1 = \widehat{\ell} \ ,$$

where $\widehat{\ell}$ is defined in equation (22).

But now we have verified that the pair of NN functions $(\widehat{\ell}, \widehat{u})$ is the solution to the SRP for z with boundary functions $\alpha = \beta = 0$. Indeed if (ℓ^*, u^*) denotes the solution to this SRP, then the fact that $\widehat{\ell}$ and $\widehat{u}$ are NN and that $z = \widehat{u} - \widehat{\ell}$ implies that $\widehat{\ell} \geqslant \ell^*$ and $\widehat{u} \geqslant u^*$, while, since we have $\ell^1 \leqslant \ell^*$ and $u^1 \leqslant u^*$, the preceding calculation has demonstrated that $\widehat{\ell} \leqslant \ell^*$ and $\widehat{u} \leqslant u^*$.

[19]Note that the sign of γ_2 is not uniquely defined when A has rank 1; further, γ_1 and γ_2 are not uniquely defined when $\lambda_1 = \lambda_2$.

Remark 13. 1. It will be of interest to explicitly work out the solution to the SRP for the matrix function $\frac{1}{\lambda_i - \lambda_j + a}$, where $a > 0$ is an arbitrary real number and $1 \leqslant i < j \leqslant$ are fixed.

2. It will be of interest to explicitly work out the solution to the SRP for the matrix function $\prod_{1 \leqslant i < j \leqslant d} (\lambda_i - \lambda_j)$

■

References

[1] Anderson, R. and Orey, S. (1976) Small random perturbations of dynamical systems with reflecting boundary. *Nagoya Math. J.* **60**, 189-216.

[2] Bernard, A.K. and El Kharroubi (1991) Régulation déterministes et stochastiques dans le premier "orthant" de $\mathbb{R}^n$. *Stochastics and Stochastics Reports* **34**, 149-167.

[3] Cépa, E. (1998) Problème de Skorokhod multivoque. *The Annals of Probability* **26**, 500-532.

[4] Dupuis, P. and Ishii, H. (1991) On the Lipschitz continuity of the solution mapping to the Skorokhod problem, with applications. *Stochastics and Stochastics Reports* **35**, 31-62.

[5] Dupuis, P. and Ramanan, K. (1999) Convex duality and the Skorokhod problem, parts I & II. *Prob. Th. and Rel. Fields* **115** (1999), 153-195; 197-236.

[6] Harrison, J.M. (1985) *Brownian Motion and Stochastic Flow Systems.* Wiley, New York.

[7] Harrison, J.M. and Reiman, M.I. (1981) Reflected Brownian motion on an orthant. *The Annals of Probability* **9**, 302-308

[8] Hewitt, E. and Stromberg, K. (1955) *Real and Abstract Analysis.* Springer-Verlag, Berlin.

[9] Horn, R.A. and Johnson, C.R. (1985) *Matrix Analysis.* Cambridge Univ. Press, Cambridge.

[10] Konstantopoulos, T. (1999) The Skorokhod reflection problem for functions with discontinuities (contractive case). *Preprint*, Univ. of Texas at Austin. Reference [26] of [22].

[11] Konstantopoulos, T. and Anantharam, V. (1995) Optimal flow control schemes that regulate the burstiness of traffic. *IEEE/ACM Trans. Networking* **3**, 423-432.

[12] Marshall, A.W. and Olkin, I. (1979) *Inequalities: Theory of Majorization and Applications.* Mathematics in Science and Engineering, Vol. 143, Academic Press, New York.

[13] Oksendal, B. K. (1998) *Stochastic differential equations : an introduction with applications.* Springer, Berlin.

[14] Port, S. C., and Stone, C. J. (1978) *Brownian motion and classical potential theory.* Academic Press, New York.

[15] Royden, H.L. (1968) *Real Analysis.* Macmillan, New York.

[16] Skorokhod, A.V. (1961) Stochastic equations for diffusions in a bounded region. *Theory of Probab. and its Applic.* **6**, 264-274.

[17] Stanley, R.P. (1986) *Enumerative Combinatorics, Vol. I.* Wadsworth and Brooks/Cole, Monterey, California.

[18] Stroock, D.W. and Varadhan, S.R.S. (1971) Diffusion processes with boundary conditions. *Comm. Pure Appl. Math.* **24**, 147-225.

[19] Stroock, D. W. and Varadhan, S.R.S. (1979) *Multidimensional diffusion processes.* Springer Verlag, Berlin.

[20] Tanaka, H. (1979) Stochastic differential equations with reflecting boundary condition in convex regions. *Hiroshima Math. J.* **9**, 163-177.

[21] Varadhan, S.R.S. and Williams, R.J. (1985) Brownian morion in a wedge with oblique reflection. *Comm. Pure Appl. Math.* **37**, 405-443.

[22] Whitt, W. (2001) The reflection map with discontinuities. *Mathematics of Operations Research* **26**, No. 3, 447-484.

[23] Yosida, K. (1995). *Functional Analysis.* Springer-Verlag, Berlin.

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