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# On the Nash dynamics of congestion games with player-specific utility

Venkat Anantharam

*Abstract*— **Strategic issues in multi-agent routing in communication networks can be studied by modeling the routing problem as a congestion game. Such a game has a finite number of players and a finite set of resources; each player chooses a nonempty subset of the resources and the payoff to a player depends only on the number of players using each resource. Rosenthal proved that congestion games with symmetric utilities have pure strategy Nash equilibria. This is because the Nash dynamics give rise to an acyclic digraph. An explicit potential function on pure strategy profiles for the Nash dynamics is known for such games; this helps in understanding the Nash dynamics and verifies that the corresponding digraph is acyclic. Milchtaich considered congestion games with player-specific utilities where each player selects exactly one resource. He proved that the Nash dynamics give rise to an acyclic digraph when there are two resources. This implies the existence of a potential function on pure strategy profiles for the Nash dynamics, but no explicit potential function appears to be known as yet. In this paper we define an acyclic digraph on the integer lattice of each dimension and show that the Nash dynamics for any such congestion game can be embedded into one of these digraphs. We find an explicit potential function on each of these lattice digraphs. This potential function can then be restricted to the embedded digraph of the Nash dynamics of such a congestion game. Apart from giving an alternate proof of the acyclicity of the digraph of Nash dynamics for such games, such an explicit function helps in understanding these dynamics.**

## I. INTRODUCTION

*Congestion games* were introduced by Rosenthal [10], [11]. They are of interest to researchers in communication networking because they form a natural class of models within which to study strategic issues in multi-agent routing problems in networks, as illustrated in Example 1.1. The connection with the so called *potential games*, as described by Monderer and Shapley [7], should also be noted. We now give a formulation of congestion games, with *player-specific utilities*, combining the formulation of [10], [11] with that of Milchtaich [6].

*Definition 1.1: [Congestion games]*

There are $n$ players and $m$ resources, The set of pure strategies of player $i$ is denoted $\Sigma^i$; each element $\sigma^i \in \Sigma^i$ is a distinct nonempty subset of the resources. Given a pure strategy profile $\sigma \overset{\text{def}}{=} (\sigma^1, \dots, \sigma^n)$, the *cost* of player $i$ is

$$c^i(\sigma) \overset{\text{def}}{=} \sum_{a \in \sigma^i} s_{ia}(\nu_a(\sigma)) \ . \tag{1}$$

Here $\nu_a(\sigma) = \sum_{j=1}^{n} 1(a \in \sigma^j)$ denotes the number of players whose strategy includes resource $a$. Further, for each $1 \leq i \leq n$ and $1 \leq a \leq m$, $s_{ia} : \{1, \dots, n\} \mapsto \mathbf{R}$ is a prescribed function; $s_{ia}(\nu)$ can be thought of as the cost to player $i$ of having resource $a$ in its strategy when the total number of players who have this resource in their strategy is $\nu$. We will assume [1] that each $s_{ia}$ is a nondecreasing function. ●

The relevance of this kind of game to the study of strategic issues in multi-agent routing in communication networks may be seen by noting that the following broad class of examples falls within this framework :

*Example 1.1:* Let $G = (V, E)$ be a directed graph (digraph) with vertex set $V$ and edge set $E$. Assume that the digraph is *strongly connected* [2]. There are $n$ players. The set of edges of the digraph is the set of resources. Each player is associated with an ordered pair of vertices, called its *source* and *destination* respectively. A pure strategy of a player is a subset of edges that form a directed path from its source to its destination. The cost to a player of playing a given strategy is the sum of costs associated to the edges that the player uses in its strategy. The cost of an edge is a player-dependent function of the total number of players who use that edge in the pure strategy profile. It is natural to assume that each such cost is a nondecreasing function. ●

Before proceeding to discuss congestion games, we mention that there is a substantial literature on routing problems in communication networks using game theoretic techniques, see e.g. [1], [3], [4], [5], [8], [12], and the references in these papers. In most of these works, the flow being routed by each agent is assumed to be divisible, and it is often assumed that the flow can be split between different paths between the source and the destination, as opposed to the model considered here.

Rosenthal [10], [11] proved, in effect, that the Nash dynamics of congestion games with symmetric utilities results in an acyclic digraph [3]; a consequence is that such games have pure strategy Nash equilibria. Here equation (1) for the utility of player $i$ when the pure strategy profile is $\sigma$ is replaced by

$$c_s^i(\sigma) \overset{\text{def}}{=} \sum_{a \in \sigma^i} s_a(\nu_a(\sigma)) \ , \tag{2}$$

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V. Anantharam is with the EECS Department, University of California, Berkeley, CA 94720, U. S. A. `ananth@eecs.berkeley.edu`

[1] This is not assumed in [10], [11].

[2] See e.g. [2] for basic notions in graph theory.

[3] This was noted in [7].

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where $s_a : \{1, \ldots, n\} \mapsto \mathbf{R}$ are given nondecreasing functions, with $s_a(\nu)$ representing the cost to player $i$ of having resource $a$ in its strategy when the total number of players who have this resource in their strategy is $\nu$.

First recall what we are talking about :

*Definition 1.2: [Nash dynamics]*

*Let* $\Sigma \overset{\text{def}}{=} \Sigma^1 \times \ldots \times \Sigma^n$ *denote the set of pure strategy profiles. A* move *from* $\sigma \in \Sigma$ *to* $\tilde{\sigma} \in \Sigma$ *is part of the Nash dynamics if, for some* $1 \leq i \leq n$, $\sigma_{-i} = \tilde{\sigma}_{-i}$, *and* $c^i(\tilde{\sigma}) < c^i(\sigma)$. *The Nash dynamics defines a digraph with vertex set* $\Sigma$ *and edge set comprised of all such* $(\sigma, \tilde{\sigma})$. ●

In Defn. 1.2 we have written $\sigma_{-i}$ for the list of strategies of players other than $i$ when the pure strategy profile is $\sigma$ [4]. For the case of symmetric utilities each $c^i$ should be replaced by $c^i_s$, as given in equation (2). Understanding the Nash dynamics of congestion games that arise in multiagent routing is a problem of considerable practical interest. It is possible that the observed instabilities in the border gateway protocol (BGP) of the Internet result from strategic considerations. See [1] for an introduction to work in this area.

A real valued function $P(\cdot)$ defined on the set of pure strategy profiles $\Sigma$ is called a *potential function* [5] if for every $1 \leq i \leq n$ and every pair of pure strategy profiles $\sigma$ and $\tilde{\sigma}$ such that $\sigma_{-i} = \tilde{\sigma}_{-i}$ we have

$$P(\sigma) > P(\tilde{\sigma}) \Longleftrightarrow c^i(\sigma) > c^i(\tilde{\sigma}) \ .$$

One sees that a potential function exists if and only if the digraph of Nash dynamics is acyclic. Pure strategy Nash equilibria are precisely the pure strategy profiles with no outgoing edges in the digraph of Nash dynamics. Thus, if the game admits a potential function, the pure strategy Nash equilibria are identical to the pure strategy Nash equilibria of a game in which every player's cost function equals the potential function. Explicitly knowing a potential function can thus be quite useful in understanding the Nash dynamics of the game.

We now give the proof of the result of [10], [11]. For this symmetric case, consider the function :

$$P(\sigma) \overset{\text{def}}{=} \sum_{a \in \cup_{i=1}^n \sigma^i} \ \sum_{l=1}^{\nu_a(\sigma)} s_a(l) \ .$$

Let $\sigma, \tilde{\sigma} \in \Sigma$ and $1 \leq i \leq n$ be such that $\sigma_{-i} = \tilde{\sigma}_{-i}$. It is straightforward to check that

$$c^i_s(\sigma) - c^i_s(\tilde{\sigma}) = P(\sigma) - P(\tilde{\sigma}) \ .$$

This verifies that $P(\cdot)$ is a potential function. It follows that the digraph associated to the Nash dynamics is acyclic. Further, the availability of such an explicit potential function helps in understanding the Nash dynamics. ●

We turn next to the case of player-specific utilities, considered by Milchtaich. Following [6] [6] we restrict each player to use exactly one resource. Thus, the formulation is as in Defn. 1.1, with the restriction that, for each $1 \leq i \leq n$, each $\sigma^i$ is of cardinality 1. The Nash dynamics are defined as in Defn. 1.2. In [6] an example is given, when the number of players is 2 and the number of resources is 3, where the digraph associated to the Nash dynamics has a cycle [7]. On the other hand, it is proved that when the number of resources is 2 the digraph associated to the Nash dynamics, for any number of players, is acyclic.

We now give the usual proof of this result. Let $(\sigma(0), \sigma(1), \ldots, \sigma(M) = \sigma(0))$, with $M \geq 2$, be a directed cycle in this digraph, i.e. such that $(\sigma(k), \sigma(k+1))$ is an edge for each $0 \leq k \leq M-1$. Let $(\nu_1(k), \nu_2(k))$ denote the occupancy vector of the resources when the pure strategy profile is $\sigma(k)$. By reindexing the pure strategy profiles in the cycle, we may assume that $\nu_2(1) = \max_{0 \leq k \leq M} \nu_2(k)$, so that $\nu_1(1) = n - \nu_2(1) = \min_{0 \leq k \leq M} \nu_1(k)$. Suppose player $i$ is the one whose move in the Nash dynamics leads to the edge $(\sigma(0), \sigma(1))$. This player must have been using resource 1 in $\sigma(0)$, and moves to use resource 2 in $\sigma(1)$, so we must have $s_{i1}(\nu_1(1)+1) > s_{i2}(\nu_2(1))$. Since $\nu_2(k) \leq \nu_2(1)$ for all $0 \leq k \leq M$ and $s_{i2}$ is nondecreasing, we have $s_{i2}(\nu_2(k)) \leq s_{i2}(\nu_2(1))$. Since $\nu_1(k) \geq \nu_1(1)$ for all $0 \leq k \leq M$ and $s_{i1}$ is nondecreasing, we have $s_{i1}(\nu_1(k)+1) \geq s_{i1}(\nu_1(1)+1)$. Hence $s_{i1}(\nu_1(k)+1) > s_{i2}(\nu_2(k))$ for all $0 \leq k \leq M$. It follows that player $i$ never moves back from using resource 2 to using resource 1 throughout the cycle, which is impossible. ●

The acyclicity of the digraph associated to the Nash dynamics implies the existence of a potential function. One can define a potential function recursively by starting with vertices with no outgoing edges (these are the pure strategy Nash equilibria, of which there is at least one), assigning them the potential zero and then assigning to each vertex a potential equal to "1 + the maximum of the potentials of the vertices that it leads to".

However, this procedure does not give the potential function in a concrete form that can be used to lend insight into the digraph of Nash dynamics. In this note we examine the Nash dynamics for this class of congestion games in some depth. We find an explicit potential function which verifies the acyclicity of the Nash dynamics digraph, and thus gives more insight into the structure of the Nash dynamics.

## II. AN ALTERNATE REPRESENTATION

For the remainder of the paper we deal with a specific class of congestion games – those with player-specific utilities and two resources, with each player choosing exactly one resource. With the notation as in Defn. 1.1, $n$ is arbitrary, we have $m = 2$, and, for each $1 \leq i \leq n$, each $\sigma^i$ is of cardinality 1. We will assume that the functions $s_{i1}(\cdot)$

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[4] See e.g. [9] for basic notions in game theory.

[5] In the terminology of [7] $P(\cdot)$ would be called an *ordinal potential function*.

[6] It is assumed in [6] that each $s_{ia}$ is nondecreasing.

[7] Nevertheless, it is proved that a pure strategy Nash equilibrium exists in all congestion games with player-specific utilities where each player uses exactly one resource, because the digraph associated to the *strict* Nash dynamics is acyclic. The edges of this digraph correspond to best replies, rather than improving replies. See [6] for the details.

and $s_{i2}(\cdot)$ have been extended in some way to be defined on all of $\mathbf{Z}$ while still being nondecreasing functions, in such a way that they differ at least at one integer value.

Let $\sigma \overset{\text{def}}{=} (\sigma^1, \dots, \sigma^n)$ be a pure strategy profile. Suppose $\sigma^i = 1$. Agent $i$ will want to move to use resource 2 iff $s_{i2}(\nu_2(\sigma) + 1) < s_{i1}(\nu_1(\sigma))$. Suppose $\sigma^i = 2$. Agent $i$ will want to move to use resource 1 iff $s_{i1}(\nu_1(\sigma) + 1) < s_{i2}(\nu_2(\sigma))$. If, for $a = 1, 2$, we define $\nu_a(\sigma_{-i})$ to be the number of players other than player $i$ who are using resource $a$ when the pure strategy profile is $\sigma$, both these statements can be compressed to the following one : agent $i$ strictly prefers to use the resource $a$ iff $s_{ia}(\nu_a(\sigma_{-i}) + 1) < s_{i\bar{a}}(\nu_{\bar{a}}(\sigma_{-i}) + 1)$ and is indifferent between the two alternatives iff $s_{i1}(\nu_1(\sigma_{-i}) + 1) = s_{i2}(\nu_2(\sigma_{-i}) + 1)$. Here, $\bar{a} = 2$ if $a = 1$ and $\bar{a} = 1$ if $a = 2$.

We next note that $\nu_1(\sigma_{-i}) + \nu_2(\sigma_{-i}) = n - 1$. Since the $s_{ia}$ are nondecreasing functions, it makes sense to make the following definitions :

$$\begin{aligned} V_i &\overset{\text{def}}{=} \sup\{u \in \mathbf{Z} \ : \ s_{i1}(u+1) < s_{i2}((n-1) - u + 1)\} \\ \Lambda_i &\overset{\text{def}}{=} \inf\{u \in \mathbf{Z} \ : \ s_{i2}((n-1) - u + 1) < s_{i1}(u+1)\} \ . \end{aligned}$$

Note that it is possible to have $V_i = -\infty$ and it is also possible to have $\Lambda_i = \infty$. For every $u \in \{0, \dots, n-1\}$, in any pure strategy profile where exactly $u$ of the other players are using resource 1 : if $u \leq V_i$, player $i$ will strictly prefer to use resource 1; if $u \geq \Lambda_i$, player $i$ will strictly prefer to use resource 2; if $V_i < u < \Lambda_i$, player $i$ is indifferent between using resource 1 and resource 2.

An important consequence of the preceding is that *for strategic purposes, user $i$ is completely characterized by the pair of extended integers $(V_i, \Lambda_i)$.* Note that, for all $1 \leq i \leq n$, we must have $V_i < \Lambda_i$, because we assumed that $s_{i1}(\cdot)$ and $s_{i2}(\cdot)$ are not the same function.

## III. MAIN RESULT

Congestion games of the kind we are studying are completely determined, for strategic purposes, by the $n$-tuple of pairs $((V_1, \Lambda_1), \dots, (V_n, \Lambda_n))$. Our goal in this paper is to come to a deeper understanding of the Nash dynamics of these games, and, in particular, to understand more clearly why the digraph associated to these dynamics is acyclic. To this end, our first step is to recognize that it suffices to consider the special case where, for each $1 \leq i \leq n$, $0 \leq V_i < V_i + 1 = \Lambda_i \leq n - 1$.

Indeed, in the original game : (1) any player $i$ for which both $V_i = -\infty$ and $\Lambda_i = \infty$ is strategically uninteresting, since such a player is always indifferent between the two alternatives; (2) any player with $V_i = -\infty$ and $\Lambda_i = 0$ is strategically uninteresting since such a player will always strictly prefer resource 2; (3) any player with $V_i = n-1$ and $\Lambda_i = \infty$ is strategically uninteresting since such a player will always strictly prefer resource 1. Hence it suffices to consider only situations where, for each player, none of these three cases occurs. We may then think of the Nash dynamics of the game $((V_1, \Lambda_1), \dots, (V_n, \Lambda_n))$ as embedded in those of the game $((\max(V_1, 0), \max(V_1, 0) + 1), \dots, (\max(V_n, 0), \max(V_n, 0) + 1)$. The digraph associated to the latter game has more edges than the original game. An understanding of why such a digraph is acyclic then explains why the original digraph is acyclic. From henceforth we will therefore only consider games of the form $((W_1, W_1 + 1), \dots, (W_n, W_n + 1))$, where $0 \leq W_i \leq n - 2$ for all $1 \leq i \leq n$. Such a game will also be more compactly represented as $(W_1, \dots, W_n)$.

Consider the game $(W_1, \dots, W_n)$. Given a pure strategy profile $\sigma = (\sigma^1, \dots, \sigma^n)$, it is now convenient to consider the sequence $(\nu_1(\sigma_{-1}) - W_1, \dots, \nu_1(\sigma_{-n}) - W_n)$, which we will write as $v(\sigma) \overset{\text{def}}{=} (v_1(\sigma), \dots, v_n(\sigma))$. Note that, when the pure strategy profile is $\sigma$, player $i$ strictly prefers resource 1 (respectively resource 2) when $v_i(\sigma) \leq 0$ (respectively $v_i(\sigma) \geq 1$). Thus, when the pure strategy profile is $\sigma$, player $i$ will move to switch his strategy iff either (1) he is currently choosing resource 1 and $v_i(\sigma) \geq 1$ or (2) he is currently choosing resource 2 and $v_i(\sigma) \leq 0$. Note that in both cases, $v_i(\sigma)$ remains the same after player $i$ changes switches his strategy. However, in case (1) each $v_j(\sigma)$ for $j \neq i$ decreases by 1, and in case (2) each $v_j(\sigma)$ for $j \neq i$ increases by 1.

We note that the vector $v(\sigma) \in \mathbf{Z}^n$ completely determines $\sigma$. We may therefore think of this as an alternate representation of the pure strategy profile, which we will call the $v$-representation. The possible moves of player $i$ described above correspond, in case (1), to the addition of the vector $-\mathbf{1} + e_i$ in the $v$-representation and, in case (2), to the addition of the vector $\mathbf{1} - e_i$ in the $v$-representation. Here $\mathbf{1}$ denotes a vector all of whose entries are 1, and $e_i$ denotes the unit vector in direction $i$. The move of case (1) is valid precisely from $v$-representations whose $i$-th coordinate is strictly positive and in which player $i$ is using resource 1 in the corresponding pure strategy profile, while the move of case (2) is valid precisely from $v$-representations whose $i$-th coordinate is nonpositive and in which player $i$ is using resource 2 in the corresponding pure strategy profile.

Now, of course, not every vector in $\mathbf{Z}^n$ is a valid $v$-representation. However, we can define a digraph with vertex set $\mathbf{Z}^n$ by following the above rules for constructing edges (edges are defined for each $1 \leq i \leq n$). In applying these rules, we do not worry about attempting to associate a pure strategy profile to the vector to try to figure out which "strategy" the "player" corresponding to the edge is "playing"; rather, we simply allow all such edges to be present. Note that each vertex $v \overset{\text{def}}{=} (v_1, \dots, v_n)$ now has precisely $n$ outgoing edges : for $1 \leq i \leq n$ with $v_i \geq 1$ there is an edge from $v$ to $v - \mathbf{1} + e_i$, and for $1 \leq i \leq n$ with $v_i \leq 0$ there is an edge from $v$ to $v + \mathbf{1} - e_i$. Let us denote this digraph as $G_n$. $G_n$ contains the digraph corresponding to the Nash dynamics of the game $(W_1, \dots, W_n)$ as a subgraph. Thus, if we can understand the structure of $G_n$, we will have better understood the structure of the Nash dynamics of the congestion games we are studying. The

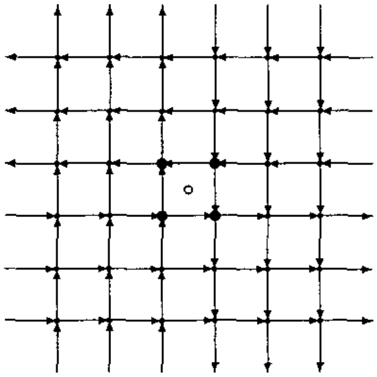

Fig. 1. The digraph $G_2$. The open circle represents $(\frac{1}{2}, \frac{1}{2})$. The lattice points $(0,0)$, $(1,0)$, $(0,1)$, and $(1,1)$ have been emphasized for clarity.

structure of $G_n$ is described in the proof of the following result, which is the main theorem of the paper.

*Theorem 3.1:* $G_n$ is an acyclic digraph.

*Proof* :

We first consider the simple case of $n = 2$ to illustrate the overall structure of the digraph. There are two outgoing edges from each $v \overset{\text{def}}{=} (v_1, v_2) \in \mathbf{Z}^2$. A sketch of the digraph, as in Figure 1, shows that the flow has a natural saddle point structure centered around $(\frac{1}{2}, \frac{1}{2})$. The digraph is plainly acyclic. An explicit potential function verifying this fact is

$$P(v) = sgn[(v_1 - \frac{1}{2})(v_2 - \frac{1}{2})][| v_1 - \frac{1}{2} | + | v_2 - \frac{1}{2} |] ,$$

where

$$sgn(x) \overset{\text{def}}{=} \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -1 & \text{if } x < 0 \end{cases} ,$$

is the sign function. This potential function can now be restricted to $v$-representations to give a potential function for the Nash dynamics. Note that, while this potential function is defined on the $v$-representations of the original problem, it can be treated as one on pure strategy profiles since the $v$-representation uniquely determines the pure strategy profile [8].

For $n \geq 3$ no such simple potential function is apparent. It is still true that for $v \in \mathbf{Z}^n$ with all coordinates strictly positive the function $P(v) = \sum_{i=1}^{n}(v_i - \frac{1}{2})$ serves as a potential function, and for $v \in \mathbf{Z}^n$ with all coordinates nonpositive the function $P(v) = \sum_{i=1}^{n} | v_i - \frac{1}{2} |$ serves as a potential function. However, for $v \in \mathbf{Z}^n$ with $\max_i v_i \geq 1 > 0 \geq \min_i v_i$, the structure of the digraph $G_n$ appears to be more involved. Let $S_n$ denote the restriction of $G_n$ to such vertices. Note that the set of such vertices is *closed* in $G_n$, in the sense that there is no directed path from any such vertex to a vertex that is not of this type. If we could find a potential function on $S_n$ taking on only nonpositive values, we could therefore combine this with the above potential functions on the other vertices to get a potential function on $G_n$. The structure of $S_n$ for $n \geq 3$ appears to be quite complicated in contrast to that of $S_2$.

For instance, in $S_n$ for $n \geq 3$ is possible to repeatedly visit the same orthant many times. Consider, for example, the path in $S_4$ given by the sequence of vertices $(-5, 0, 1, 6) \rightarrow (-5, 1, 2, 7) \rightarrow (-6, 0, 2, 6) \rightarrow (-6, 1, 3, 7) \rightarrow (-7, 0, 2, 7) \rightarrow$ etc. , which is moving back and forth between the orthant where the first two coordinates are nonpositive and the last two coordinates are strictly positive and the one where the first coordinate is nonpositive but the last three are strictly positive. The directions of the edges from a vertex of $S_n$ depend on which orthant the vertex is in, so this phenomenon makes it difficult to simply build a potential function for $G_n$ for $n \geq 3$ orthant by orthant and then cobble things together, as was done for $G_2$.

Another key difference between the digraph $S_2$ and the digraphs $S_n$ for $n \geq 3$ is that in the latter it is possible to find paths between pairs of vertices that have widely differing lengths [9]. Examining examples shows that there is a kind of "pumping" phenomenon taking place [10]. For example, let $m \geq 4$ be an arbitrary integer, and consider paths in $S_4$ from $(1, -1, -m+2, -m)$ to $(1, 1, -m, -m)$. There is a short path of length 4 given by the sequence of vertices $(1, -1, -m+2, -m) \rightarrow (2, 0, -m+2, -m+1) \rightarrow (3, 1, -m+2, -m+2) \rightarrow (2, 1, -m+1, -m+1) \rightarrow (1, 1, -m, -m)$. There is also a long path of length $2(m-1)$ given by going from $(1, -1, -m+2, -m)$ to $(m-2, -1, -1, -3)$ in $m-3$ steps, then from $(m-2, -1, -1, -3)$ to $(m, 1, -1, -1)$ in 2 steps, and then from $(m, 1, -1, -1)$ to $(1, 1, -m, -m)$ in $m-1$ steps.

It turns out that a key definition to make progress is the following :

*Definition 3.1:* Let $v \overset{\text{def}}{=} (v_1, \ldots, v_n)$ be a vertex in $S_n$. The *spread* of $v$ is defined as $s(v) \overset{\text{def}}{=} \max_i v_i - \min_i v_i$. ●

The key observation about the spread is the following :

*Lemma 3.1:* Along each edge in $S_n$ the spread is nondecreasing.

*Proof of Lemma 3.1* :

Let $(v, \tilde{v})$ be an edge in $S_n$. Let $1 \leq i \leq n$ be the coordinate defining the edge. If $v_i = \max_j v_j$ we have $s(\tilde{v}) = s(v) + 1$. If $v_i = \min_j v_j$ we again have $s(\tilde{v}) = s(v) + 1$. If $\max_j v_j > v_i > \min_j v_j$ we have $s(\tilde{v}) = s(v)$. ●

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[8] Of course, in the case $n = 2$, the full digraph $G_2$ is not of much interest for the original problem. The only situation when neither player is strategically uninteresting is the game $(W_1, W_2) = (0, 0)$. There are only 4 possible $v$-representations : $(1,1)$, $(0,0)$, $(1,0)$, and $(0,1)$, and the digraph associated to the Nash dynamics has only four edges in the $v$-representation : $(1,1) \rightarrow (1,0)$, $(1,1) \rightarrow (0,1)$, $(0,0) \rightarrow (1,0)$, and $(0,0) \rightarrow (0,1)$.

[9] Note that in $S_2$ if there is a path between a pair of vertices, then all paths between that pair of vertices have the same length.

[10] We do not attempt to make this precise; the reader will understand what we mean if he/she works out some examples.

Lemma 3.1 tells us that along any edge $(v, \tilde{v})$ with $v, \tilde{v} \in S_n$ the spread is nondecreasing. Well, what if $v$ and $\tilde{v}$ both have the same spread ? Fix a spread $s \geq 1$ and consider the restriction of $S_n$ to vertices with spread $s$. Call this digraph $S_n(s)$. Consider an edge $(v, \tilde{v})$ with $v, \tilde{v} \in S_n(s)$ [11]. The importance of the spread suggests to consider, for each $v \in S_n$, *all* differences of the form $v_i - v_j$ where $v_i \geq 1$ and $v_j \leq 0$, which we may call the set of *internal spreads* of $v$. One can now see that the proof of Lemma 3.1 gives some more information : one can conclude that whenever $(v, \tilde{v})$ is an edge with $v, \tilde{v} \in S_n(s)$, the total number of internal spreads $\tilde{v}$ that equal $s$ is at least as big as the number of internal spreads of $v$ that equal $s$.

It is now natural to hope that some kind of order exists between the collection of all internal spreads of $v$ and those of $\tilde{v}$ whenever $(v, \tilde{v})$ is an edge of $S_n$. This is indeed that case, in that one can find a weighted linear combination of internal spreads that behaves monotonically along the edges of $S_n$. By choosing this function so that it is always negative, one can combine it with the function defined earlier for vertices of $G_n$ that are not in $S_n$, in order to build a potential function for $G_n$.

We now give our proposed potential function for $G_n$. In the next lemma we prove that it works.

Define an integer valued function $P(\cdot)$ on the vertices of $G_n$ as follows :

$$\begin{aligned} P(v) &= \sum_{i=1}^{n} (v_i - \frac{1}{2}) \ , \quad \text{if } v_i \geq 1 \text{ for all } i \ , \\ P(v) &= \sum_{i=1}^{n} \mid v_i - \frac{1}{2} \mid \ , \quad \text{if } v_i \leq 0 \text{ for all } i, \text{ and } \ , \\ P(v) &= -\sum_{i=1}^{n} \sum_{j=1}^{n} n^{v_i - v_j - 1} 1(v_i \geq 1) 1(v_j \leq 0) \end{aligned}$$

if $v \in S_n$. Note that, since $\max_i v_i \geq 1 > 0 \geq \min_i v_i$ for all $v \in S_n$, we have $P(v) \leq -1$ for all $v \in S_n$. It therefore suffices to prove the following lemma to complete the proof that $P(\cdot)$ is a potential function on $G_n$.

*Lemma 3.2:* We have $P(\tilde{v}) < P(v)$ whenever $(v, \tilde{v})$ is an edge with $v, \tilde{v} \in S_n$.

*Proof of Lemma 3.2* :

Let $(v, \tilde{v})$ be an edge of $S_n$. Let $1 \leq i \leq n$ be the coordinate defining the edge.

Suppose $v_i \geq 1$. Then $\tilde{v}_i = v_i$ and $\tilde{v}_j = v_j - 1$ for $j \neq i$. We write

$$\begin{aligned} -P(\tilde{v}) &= \sum_j \sum_k n^{\tilde{v}_j - \tilde{v}_k - 1} 1(\tilde{v}_j \geq 1) 1(\tilde{v}_k \leq 0) \\ &\overset{(a)}{=} \sum_j \sum_{k \neq i} n^{\tilde{v}_j - \tilde{v}_k - 1} 1(\tilde{v}_j \geq 1) 1(\tilde{v}_k \leq 0) \\ &\overset{(b)}{=} \sum_{\substack{k \neq i \\ \tilde{v}_k \leq 0}} \left[ n^{\tilde{v}_i - \tilde{v}_k - 1} + \sum_{j \neq i} n^{\tilde{v}_j - \tilde{v}_k - 1} 1(\tilde{v}_j \geq 1) \right] \\ &\overset{(c)}{=} \sum_{\substack{k \neq i \\ \tilde{v}_k \leq 0}} \left[ n^{v_i - v_k} + \sum_{j \neq i} n^{v_j - v_k - 1} 1(\tilde{v}_j \geq 1) \right] \\ &\overset{(d)}{=} \sum_{\substack{k \neq i \\ \tilde{v}_k \leq 0}} \left[ n^{v_i - v_k} - \sum_{j \neq i} n^{v_j - v_k - 1} 1(v_j = 1) \right. \\ &\qquad\qquad \left. + \sum_{j \neq i} n^{v_j - v_k - 1} 1(v_j \geq 1) \right] \\ &\overset{(e)}{>} \sum_{\substack{k \neq i \\ \tilde{v}_k \leq 0}} \left[ n^{v_i - v_k - 1} + \sum_{j \neq i} n^{v_j - v_k - 1} 1(v_j \geq 1) \right] \\ &\overset{(f)}{\geq} \sum_{\substack{k \neq i \\ v_k \leq 0}} \left[ n^{v_i - v_k - 1} + \sum_{j \neq i} n^{v_j - v_k - 1} 1(v_j \geq 1) \right] \\ &\overset{(g)}{=} \sum_{k \neq i} \sum_j n^{v_j - v_k - 1} 1(v_j \geq 1) 1(v_k \leq 0) \\ &\overset{(h)}{=} \sum_j \sum_k n^{v_j - v_k - 1} 1(v_j \geq 1) 1(v_k \leq 0) \\ &= -P(v) \ , \end{aligned}$$

which proves the desired inequality in this case. A similar calculation handles the case when the $v_i \leq 0$. Step (a) is valid because $\tilde{v}_i = v_i \geq 1$. Step (b) is valid for the same reason. Step (c) comes from substituting for each coordinate of $\tilde{v}$ in terms of the corresponding coordinate of $v$. Step (d) comes from the observation that for $j \neq i$, if $v_j = 1$ then $\tilde{v}_j = 0$, while if $v_j > 1$ then $\tilde{v}_j \geq 1$. Step (e) is valid because, with $v_i \geq 1$, we have

$$\begin{aligned} n^{v_i - v_k} - n^{v_i - v_k - 1} &= n^{v_i - v_k - 1}(n - 1) \\ &> n^{v_i - v_k - 1}(n - 2) \\ &\overset{(e1)}{\geq} n^{v_i - v_k - 1} \sum_{j \neq i} 1(v_j = 1) \\ &\overset{(e2)}{=} \sum_{j \neq i} n^{v_j - v_k - 1} 1(v_j = 1) \ , \end{aligned}$$

where step (e1) is valid because there are at most $n - 2$ coordinates $v_j$ with $j \neq i$ and $v_j = 1$ and step (e2) is valid because $v_i \geq 1$. Step (f) is valid because $\tilde{v}_k \leq v_k$ for all $k \neq i$. Step (g) and step (h) are valid because $v_i \geq 1$. ●

We have completed the proof of Theorem 3.1. ●

We may now write down an explicit potential function on the digraph of the Nash dynamics of the original game. Suppose the original game is $((V_1, \Lambda_1), \ldots, (V_n, \Lambda_n))$. Pick any $(W_i, 1 \leq i \leq n)$ such that $\max(V_i, 0) \leq W_i < W_i + 1 \leq \min(\Lambda_i, n - 1)$. The digraph of Nash dynamics of the original game is contained in that of the game $((W_1, W_1 + 1), \ldots, (W_n, W_n + 1))$. The latter digraph is embedded in $G_n$ by the map

$$(\sigma_1, \ldots, \sigma_n) \mapsto (\nu_1(\sigma_{-1}) - W_1, \ldots, \nu_1(\sigma_{-n}) - W_n) \ .$$

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[11] $S_n(s)$ has at least one edge iff $n \geq 3$.

Let $W$ denote $\sum_i W_i$. Given strategy $\sigma_i$ of player $i$, let $\hat{\sigma}_i$ denote $1(\sigma_i = 1) + W_i$; note that this is an integer taking values in $\{0, \ldots, n-1\}$. Finally, given a strategy profile $\sigma = (\sigma_1, \ldots, \sigma_n)$, let $\Sigma(\sigma) \overset{\text{def}}{=} \sum_i 1(\sigma_i = 1)$ denote the total number of players using resource 1.

Straightforward algebraic manipulation now shows that the function $P(\sigma)$ which equals

$$\begin{array}{ll} (n-1)\Sigma(\sigma) - W - \dfrac{n}{2} & \text{if } \Sigma(\sigma) \geq \hat{\sigma}_i + 1 \text{ for all } i \;, \\ \mid (n-1)\Sigma(\sigma) - W - \dfrac{n}{2} \mid & \text{if } \Sigma(\sigma) \leq \hat{\sigma}_i \text{ for all } i, \text{ and} \end{array}$$

$-\sum_{i=1}^{n} \sum_{j=1}^{n} n^{\hat{\sigma}_j - \hat{\sigma}_i - 1} 1(\Sigma(\sigma) \geq \hat{\sigma}_i + 1) 1(\Sigma(\sigma) \leq \hat{\sigma}_j)$ otherwise, is a potential function for the digraph of Nash dynamics of the original congestion game.

## IV. CONCLUDING REMARKS

This investigation was motivated by the desire to find an explicit potential function to verify the acyclicity result of [6]. We were able to do this by embedding the Nash dynamics of the game considered there in a directed graph on the integer lattice, which we then proved is acyclic by demonstrating an explicit potential function for moves on this digraph. The structure of the integer lattice digraph into which we embedded the Nash dynamics appeared to be quite involved, as we saw through examples. It is still possible, of course, that there might exist a simpler potential function verifying the original acyclicity result. Happy hunting !

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