Optimal Sequences for CDMA Under Colored Noise: A Schur-Saddle Function Property

Pramod Viswanath, *Member, IEEE*, and Venkat Anantharam, *Fellow, IEEE*

***Abstract*—We consider direct sequence code division multiple access (DS-CDMA), modeling interference from users communicating with neighboring base stations by additive colored noise. We consider two types of receiver structures: first we consider the information-theoretically optimal receiver and use the *sum capacity* of the channel as our performance measure. Second, we consider the linear minimum mean square error (LMMSE) receiver and use the signal-to-interference ratio (SIR) of the estimate of the symbol transmitted as our performance measure. Our main result is a *constructive* characterization of the possible performance in both these scenarios. A central contribution of this characterization is the derivation of a qualitative feature of the optimal performance measure in both the scenarios studied. We show that the sum capacity is a *saddle function*: it is *convex* in the additive noise covariances and *concave* in the user received powers. In the linear receiver case, we show that the minimum average power required to meet a set of target performance requirements of the users is a *saddle* function: it is *convex* in the additive noise covariances and *concave* in the set of performance requirements.**

***Index Terms*—Code division multiple access (CDMA), colored noise, optimal sequences, saddle functions, sum capacity.**

I. Introduction

THIS paper focuses on the uplink of a single base station wireless system with direct sequence code division multiple access (DS-CDMA) modeling interference from users talking to neighboring base stations by colored additive Gaussian noise. This assumption means that the interference from outside the cell cannot be controlled but can be measured and estimated statistically. This is different from the model in [10], [11] where the authors consider joint processing at neighboring base stations. We restrict ourselves to the case when the users within a base station are symbol synchronous. This allows us to represent the signal transmitted by the users in a vector space (by modulating their *signature sequences*) with dimension equal to the spreading gain. We refer to the baseband DS-CDMA channel model as a *vector multiple access channel* (VMAC). The VMAC also models space division multiple access (SDMA), multiple access channels with multiple antennas at the receiver.

The choice of the receiver structure at the base station has an important effect on the performance of the VMAC. In this paper, we study two types of receiver structures: the information-theoretic optimal receiver and the linear receivers that are separately optimal for each user. An information-theoretic optimal receiver is a maximum-likelihood receiver that jointly estimates the users symbols and thus needs to process the signals of the users in a nonlinear fashion. In practice, linear receivers are popular and in this paper we consider the most prominent one: the optimal linear receiver. This is the linear minimum mean square (LMMSE) receiver which for each user achieves the minimum mean squared error among all linear receivers for that user. The effect of the signature sequences on the performance of these linear receivers and the role colored additive noise plays is the problem addressed in this paper. This is done by first defining appropriate performance measures, then characterizing the effect precisely, and finally conclude with some qualitative properties of the physical phenomena involved.

We assess the performance of the information-theoretically optimal receiver by its *sum capacity*. This is defined as the maximum sum of rates of users per unit degree of freedom at which the users can transmit reliably. It measures the overall spectral efficiency of the communication channel. We assess the performance of the LMMSE receiver for each user by the SIR for that user. The performance of the receiver as a whole is described in terms of its ability to simultaneously achieve given SIR requirements for the individual users. Our aim in this paper is to study the effect of colored additive noise on the performance of DS-CDMA in these scenarios. In what follows, we summarize our main findings and place them in the context of previously known results.

First, consider the information-theoretic optimal receivers at the base station and the corresponding sum capacity of DS-CDMA. Our main results are enumerated as follows.

1) We characterize the maximum sum capacity for any average received power profile of the users, the maximum being over all the choices of signature sequences of the users. Our characterization is constructive in the sense that we provide a *combinatorial* algorithm that constructs the corresponding optimal signature sequences.

Manuscript received March 27, 2001; revised December 10, 2001. The work of V. Anantharam was supported by the National Science Foundation under Grant IRI 97-12131.

P. Viswanath is with the Department of Electrical and Computer Engineering, University of Illinois, Urbana-Champaign, Urbana, IL 61801 USA (e-mail: pramodv@uiuc.edu).

V. Anantharam is with the Department of Electrical Engineering and Computer Science, University of California, Berkeley, Berkeley, CA 94720 USA (e-mail: ananth@eecs.berkeley.edu).

Communicated by S. Shamai, Guest Editor.

Publisher Item Identifier S 0018-9448(02)04030-0.

2) Our characterization allows us to derive some qualitative properties of the maximum sum capacity.
   a) Maximum sum capacity is *convex* in the noise covariance matrices and *concave* in the vector of received user powers. In particular, we have the following observations:
      i) for a given total noise power, maximum sum capacity is minimized when the additive noise is white;
      ii) for a given total received power of the users, sum capacity is maximized when the received user powers are equal.
   b) We strengthen the previous statement by working with a partial order called *Schur majorization* on vectors in $\mathbb{R}^N$ (the definition of Schur majorization can be found in Section II-B). This partial order makes precise the notion that one vector has components "more spread out" than those of another. We show that the maximum sum capacity is *Schur-convex* in the eigenvalues of the noise covariance matrix and *Schur-concave* in the received user powers.
3) We compare the sum capacity of DS-CDMA with the total capacity of parallel Gaussian channels with correlated noise. The well-known water-filling power allocation is optimal for parallel Gaussian channels. We show that in our problem, the structure of the optimal signature sequences and the resulting powers in the directions of the noise covariance eigenvectors can be viewed as a generalization of this water-filling allocation policy.

In previous work, the capacity region of the DS-CDMA channel for a fixed choice of signature sequences and received power profile of the users was characterized in [14]. The problem of characterizing the maximum sum capacity of DS-CDMA channels (maximum over all choices of signature sequences) with *white* noise, was first attempted in [7], which solved the equal user power constraint case. In [16], the general case of unequal user powers was solved, again with additive white noise, and a simple recursive algorithm was provided to construct the corresponding optimal signature sequences. In the SDMA model with white additive noise, Suard *et al.*[12] characterized the capacity region and also obtained an expression for the maximum sum capacity when user powers are symmetric. The results in [16] are also applicable to the asymmetric user power constraints case for the SDMA model (with white additive noise). The contribution of the information-theoretic portion of this work is to generalize all of the above works to the situation with *colored* additive noise.

We now turn to the linear receivers scenario. In this setting, we are interested in the *user capacity* region of DS-CDMA. We say that a set of SIR requirements of users is *admissible* if one can allocate signature sequences to the users and control their received power such that the achieved SIR of each user is at least equal to its SIR requirement. In [17], we showed the admissibility region with additive white noise and with no constraints on the power allocated to the users to be as follows: K users with SIR requirements $\beta_1, \ldots, \beta_K$ are admissible in the system with processing gain N if and only if

$$\sum_{i=1}^{K} \frac{\beta_i}{1+\beta_i} < N. \tag{1}$$

This allowed us to characterize the admissibility of users via a notion of *effective bandwidth*. If we consider $\frac{\beta}{1+\beta}$ as the effective bandwidth of a user with SIR requirement β, then users are admissible if and only if the sum of their effective bandwidths is less than the processing gain of the system. This result captures the nature of the interference limitation in CDMA systems. This is evidenced by the observation that there is no upper bound on the allocated power, and thus (1) remains true when the additive noise is colored. Thus, in this paper we are interested in characterizing admissibility of users subject to a constraint on the average transmitted power of the individual users. Our main result here is a solution to this problem for general noise covariance structures. We first characterize the optimal allocation of signature sequences and powers to the users such that the total allocated power of the users is minimized. Our characterization is constructive in that we design *combinatorial* algorithms to construct the optimal allocations. Then, using this characterization of the optimal allocations, we obtain an expression for the user capacity region when there is a constraint on the average transmitted power of the individual users. This characterization also allows us to make some qualitative observations about the user capacity region.

1) For any given SIR requirements of the users such that the users are admissible, with the optimal allocation of signature sequences and powers to the users, the LMMSE receiver of any user is simply the matched filter tuned to the background colored noise. Thus, the user capacity region with an average power constraint is unchanged if we restrict the receiver to be the *a priori* inferior matched filter tuned to the background colored noise.
2) For a given total noise power, the user capacity region increases when the eigenvalues of the noise covariance matrix become "more spread out" (the precise ordering is that of Schur majorization). In particular, the user capacity region is the smallest when the additive noise is white.
3) For a given sum of effective bandwidths of the users (and given that the sum is less than the processing gain), the minimum total power required among all valid allocations is *Schur-concave* in the effective bandwidths of the users. In particular, the minimum total power is smallest when the SIR requirements (and thus the effective bandwidths) are all equal.

In [13], the authors consider the SIR achieved by the LMMSE receiver when the signature sequences of the users are independent and randomly chosen. They show that the SIRs of the users in a large system (with large number of users and large processing gain) converges in probability to a constant. The main observation of [17] was that the user capacity region of

DS-CDMA using random signature sequences is asymptotically identical to that of the VMAC using optimal chosen signature sequences and powers. This holds when there are no average power constraints. Here, we extend the SIR analysis of [13] to the additive colored noise case and calculate the admissibility region with average power constraints of the users. We show that unlike the case when the signature sequences are specially chosen, there is no saddle property of the SIR achieved by a unit power user. The SIR of a unit power user in a large system is shown to be convex in the distribution of the colored noise and convex in the distribution of the received powers of the users. Analogous to Schur partial order in finite-dimensional vector spaces, we use the partial order of *dilation* on distributions to strengthen the saddle property.

This paper is in two parts. In Section II, we deal with the information-theoretic optimal receivers and the sum capacity of the VMAC. Section III deals with linear receivers and SIR performance of the users. Finally, a few words about our notation throughout this paper: we use lower case letters for scalars, bold face lower case letters for vectors (usually with N components), and upper case letters for matrices. We also denote the transpose by the superscript t and reserve the superscript $*$ to denote the solution of an appropriate optimization problem.

II. Sum Capacity of DS-CDMA

In this section, we characterize the maximum sum capacity of DS-CDMA, the maximum being over the signature sequences of the users. In Section II-A, we give a brief overview of the DS-CDMA baseband model as a VMAC and Section II-B provides an expression for the sum capacity of this channel along with a "sphere-packing" interpretation for the sum capacity expression. This interpretation sets the stage for our techniques which are used to find the signature sequences that achieve the maximum sum capacity. The characterization of these optimal signature sequences along with a combinatorial algorithm to construct them is given in Section II-C. Section II-D studies some properties of the optimal signature sequences and some features of the combinatorial algorithm that generates them. Section II-E brings forth the central contribution of the maximum sum capacity characterization: qualitative properties of maximum sum capacity as a function of the received user powers and colored noise variances. Section II-F summarizes our construction of the optimal signature sequences. In Section II-G, we compare the maximum sum capacity of the DS-CDMA channel with that of parallel Gaussian channels with an appropriate total transmit power constraint. This comparison allows us to interpret the construction of the optimal signature sequences as a generalization of the well-known water-pouring power allocation over parallel Gaussian channels. We relegate the proofs of the main results of this section to Appendix A.

A. DS-CDMA Model and the VMAC

There are K users in the channel and N denotes the processing gain (number of chips per symbol). K and N will be fixed throughout this paper. In DS-CDMA, each user transmits its symbols by spreading them using an assigned signature sequence. The baseband, sampled (discrete-time), received signal in the mth symbol interval at the receiver can be written as

$$\boldsymbol{y}(m) = \sum_{i=1}^{K} \boldsymbol{s}_i x_i(m) + \boldsymbol{z}(m) \tag{2}$$

where $\boldsymbol{s}_1, \ldots, \boldsymbol{s}_K$ are the signature sequences of the users, thought of as elements of $\mathbb{R}^N$. We assume that the energy of each signature sequence is unity, i.e., $\boldsymbol{s}_i^t \boldsymbol{s}_i = 1$. Here, $x_1(m), \ldots, x_K(m)$ represent the symbols transmitted by the users at the mth use of the baseband sampled channel. There is a power constraint on each user given by $\sum_{i=1}^{m} x_i^2(m) \le m p_i$ for user i. Here, $\boldsymbol{z}(m)$ is a sequence of independent and identically distributed (i.i.d.) Gaussian vectors with zero mean and covariance matrix Σ. We assume that Σ is positive definite and denote its eigenvalues by $\sigma_1^2 \le \sigma_2^2 \le \cdots \le \sigma_N^2$. We refer to this channel as a vector multiple access channel (VMAC).

B. Sum Capacity Expression

Fix the signal directions $\boldsymbol{s}_1, \ldots, \boldsymbol{s}_K$. The sum capacity of the VMAC in (2) is

$$\max I(x_1, \ldots, x_K; \boldsymbol{y})$$

where the maximum of the mutual information between the inputs and the output vector $\boldsymbol{y}$ is over independent random variables $x_1, \ldots, x_K$ with variances upper-bounded by $p_1, \ldots, p_K$, respectively. Proceeding as in [7], we see that this maximum is achieved when the distributions of all the random variables are Gaussian and thus we arrive at the following generalization of the result of [14] for colored additive noise. The sum capacity, in nats per unit degree of freedom, of the multiple access channel in (2) is given by

$$C_{\text{sum}}(S, D, \Sigma) = \frac{1}{2N} \log\det(I + \Sigma^{-1} S D S^t) \tag{3}$$

where we have written

$$S = [\boldsymbol{s}_1, \ldots, \boldsymbol{s}_K] \quad \text{and} \quad D = \operatorname{diag}\{p_1, \ldots, p_K\}.$$

Our main focus in this section is to characterize the maximum sum capacity

$$C_{\text{opt}}(D, \Sigma) \overset{\text{def}}{=} \max_{S \in \mathcal{S}} C_{\text{sum}}(S, D, \Sigma) \tag{4}$$

where $\mathcal{S}$ is the set of all $N \times K$ real matrices with each column having l_2 norm equal to 1. Observe that C_{sum} is a continuous function defined on a compact set $\mathcal{S}$ and thus the use of $\max$ in (4) above is justified. Since $\forall\, S \in \mathcal{S}$, we have $QS \in \mathcal{S}$ for every orthonormal matrix Q, it follows from the structure of C_{sum} in (3) that $C_{\text{opt}}(D, \Sigma)$ depends only on the eigenvalues of Σ. A matrix of signature sequences that achieves the maximum in (4) is called a matrix of "optimal signature sequences." This notion of optimality is specific to the user received power constraints represented by the matrix D.

Prior to attempting to characterize C_{opt} and the optimal signature sequences, we provide a "sphere-packing" interpretation of the expression for sum capacity in (3). Fix signature sequences S. Then, our baseband channel model at the nth instant can be rewritten from (2) as

$$\boldsymbol{y}(n) = SD^{\frac{1}{2}}\boldsymbol{x}(n) + \Sigma^{\frac{1}{2}}\boldsymbol{w}(n)$$

where $\boldsymbol{w}(n)$ is a sequence of vectors with components i.i.d. Gaussian zero mean unit variance vectors. Here, $x_1(n), \ldots, x_K(n)$ are the symbols transmitted by users $1, \ldots, K$ at time instant n and are subject to a unit average power constraint. In general, $\boldsymbol{x}(m)$ is chosen from an appropriately designed codebook. But the random coding argument uses a codebook that has entries generated from i.i.d. Gaussian zero mean unit variance random variables. We will choose this random codebook for our $\boldsymbol{x}(n)$. Fixing a blocklength m (number of symbols jointly decoded), we can write the total received signal as

$$\boldsymbol{y}^{(m)} = \left(SD^{\frac{1}{2}} \otimes I_m\right)\boldsymbol{x}^{(m)} + \left(\Sigma^{\frac{1}{2}} \otimes I_m\right)\boldsymbol{w}^{(m)}$$

where I_m denotes an $m \times m$ identity matrix and $\otimes$ denotes the tensor (or Kronecker) product between matrices. We have used the notation

$$\boldsymbol{y}^{(m)} \stackrel{\text{def}}{=} \begin{bmatrix} \boldsymbol{y}(1) \\ \boldsymbol{y}(2) \\ \vdots \\ \boldsymbol{y}(m) \end{bmatrix}, \quad \boldsymbol{x}^{(m)} \stackrel{\text{def}}{=} \begin{bmatrix} \boldsymbol{x}(1) \\ \boldsymbol{x}(2) \\ \vdots \\ \boldsymbol{x}(m) \end{bmatrix}, \quad \text{and } \boldsymbol{w}^{(m)} \begin{bmatrix} \boldsymbol{w}(1) \\ \boldsymbol{w}(2) \\ \vdots \\ \boldsymbol{w}(m) \end{bmatrix}.$$

Define the ellipsoids

$$E_y^{(m,\epsilon)} \stackrel{\text{def}}{=} \{\boldsymbol{y} \in \mathbb{R}^{Nm} \colon \boldsymbol{y}^t((SDS^t+\Sigma)^{-1} \otimes I_m)\boldsymbol{y} \le m(N+\epsilon)\}$$
$$E_w^{(m,\epsilon)} \stackrel{\text{def}}{=} \{\boldsymbol{w} \in \mathbb{R}^{Nm} \colon \boldsymbol{w}^t(\Sigma^{-1} \otimes I_m)\boldsymbol{w} \le m(N+\epsilon)\}$$

and consider the following claims. For every $\epsilon > 0$

$$\mathbb{P}\left(\boldsymbol{y}^{(m)} \in E_y^{(m,\epsilon)}\right) \to 1, \qquad \text{as } m \to \infty \tag{5}$$
$$\mathbb{P}\left(\left(\Sigma^{\frac{1}{2}} \otimes I_m\right)\boldsymbol{w}^{(m)} \in E_w^{(m,\epsilon)}\right) \to 1, \qquad \text{as } m \to \infty. \tag{6}$$

The claim of (5) is that the received signal over m blocks is contained within the ellipsoid $E_y^{(m)}$ and (6) claims that the colored noise is contained in the ellipsoid $E_w^{(m)}$. These claims are elementary to verify, and an argument analogous to the classical sphere-packing interpretation for the capacity of the Gaussian channel (in [2, Sec. 10.1]) shows that the sum capacity in nats per channel use is upper-bounded by

$$C = \liminf_{m\to\infty} \frac{1}{m} \log = \frac{\text{volume}\left(E_y^{(m,0)}\right)}{\text{volume}\left(E_w^{(m,0)}\right)}. \tag{7}$$

Continuing from (7), the volumes of the ellipsoids are given by

$$\text{volume}\left(E_y^{(m,0)}\right) = a(\det(SDS^t + \Sigma))^{\frac{m}{2}}$$

and

$$\text{volume}\left(E_w^{(m,0)}\right) = a(\det(\Sigma))^{\frac{m}{2}}$$

where a is a scaling constant independent of S, D, and Σ. Substituting these expressions into (7) we arrive at the expression in (3) for the sum capacity with signature sequences S. Thus, maximum sum capacity is achieved by choosing S so as to maximize the volume of the ellipsoid $E_y^{(1)}$. Now, the lengths of the axes of the ellipsoid $E_y^{(1)}$ are given by the eigenvalues of $SDS^t + \Sigma$ and thus, for any signature sequences S, the sum of the lengths of the axes is a constant equal to

$$\operatorname{tr}[SDS^t + \Sigma] = \operatorname{tr}[D] + \operatorname{tr}[\Sigma]$$

where we used the fact that the energy of each signature sequence is unity. Given this constraint, the volume is maximized when the ellipsoid $E_y^{(1)}$ is a sphere, that is, when all the eigenvalues of $SDS^t + \Sigma$ are equal. However, the condition that each of the signature sequences has to have unit energy imposes extra conditions that might rule out the possibility of making all the eigenvalues of $SDS^t + \Sigma$ equal. We make this precise by introducing the following partial order of *majorization* on the vector of eigenvalues of $SDS^t + \Sigma$. Majorization makes precise the vague notion that the components of a vector $\boldsymbol{x}$ are "less spread out" or "more nearly equal" than are the components of a vector $\boldsymbol{y}$ by the statement $\boldsymbol{x}$ is majorized by $\boldsymbol{y}$.

Definition 2.1: For $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$, say that $\boldsymbol{x}$ is majorized by $\boldsymbol{y}$ (or $\boldsymbol{y}$ majorizes $\boldsymbol{x}$) if

$$\sum_{i=1}^{k} x_{[i]} \le \sum_{i=1}^{k} y_{[i]}, \qquad k = 1 \ldots n-1$$
$$\sum_{i=1}^{n} x_{[i]} = \sum_{i=1}^{n} y_{[i]}.$$

In the above we used the following definition.

Definition 2.2: For any $\boldsymbol{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n$, let

$$x_{[1]} \ge \cdots \ge x_{[n]}$$

denote the components of $\boldsymbol{x}$ in decreasing order, called the *order statistics* of $\boldsymbol{x}$.

A comprehensive reference on majorization and its applications is [6]. A simple (trivial, but important) example of majorization between two vectors is the following.

Example 2.1: For every $\boldsymbol{a} \in \mathbb{R}^n$ such that $\sum_{i=1}^n a_i = 1$

$$(a_1, \ldots, a_n) \quad \text{majorizes} \quad \left(\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n}\right).$$

Continuing this digression, we present some definitions and facts that form a key part of the discussion ahead. It is well known that the sum of the diagonal elements of a matrix is equal to the sum of its eigenvalues. When the matrix is symmetric, the relationship between the diagonal elements and the eigenvalues is precisely characterized by majorization.

Lemma 2.1 ([6, Theorems 9.B.1 and 9.B.2]): Let H be a symmetric matrix with diagonal elements $h_1, \ldots, h_n$ and eigenvalues $\lambda_1, \ldots, \lambda_n$. We have

$$(\lambda_1, \ldots, \lambda_n) \quad \text{majorizes} \quad (h_1, \ldots, h_n).$$

Further, if $h_1 \geq \cdots \geq h_n$ and $\lambda_1 \geq \cdots \lambda_n$ are $2n$ numbers such that $\boldsymbol{\lambda}$ majorizes $\boldsymbol{h}$, then there exists a real symmetric matrix H with diagonal elements $h_1, \ldots, h_n$ and eigenvalues $\lambda_1, \ldots, \lambda_n$.

We say that a function $f: \mathbb{R}^n \mapsto \mathbb{R}$ is *Schur-convex* if $f(\boldsymbol{x}) \geq f(\boldsymbol{y})$ whenever x majorizes y. We also say that f is *Schur-concave* if $-f$ is Schur-convex. We observe that (as in [16]) $\log\det(SDS^t + \Sigma)$ is a Schur-concave function of the vector of eigenvalues of $SDS^t + \Sigma$ (which we will denote by $\mu(S)$). This allows us to conclude that if we can show that there is an $S^* \in \mathcal{S}$ such that $\mu(S^*)$ is majorized by $\mu(S)$ for every $S \in \tilde{\mathcal{S}}$ (for some set $\tilde{\mathcal{S}} \subseteq \mathcal{S}$ such that for every $S \in \mathcal{S}$ there exists $\tilde{S} \in \tilde{\mathcal{S}}$ with the property that $C_{\text{sum}}(S) \leq C_{\text{sum}}(\tilde{S})$), then we have identified S^* to be an optimal signature sequence matrix. We call this vector $\mu(S^*)$ a "Schur-minimal" vector of eigenvalues and its existence is the central focus of the next section where we also develop a combinatorial algorithm that constructs the optimal S^*. This explains our introduction of the majorization partial order into the discussion.

C. Maximum Sum Capacity Characterization

Our main result in this subsection is the solution of the optimization problem in (4) and thus the characterization of C_{opt}. Our solution completely characterizes the structure of the *optimal* signal directions (the S that achieve the maximum in (4)) and we also provide a combinatorial algorithm that explicitly constructs the optimal signature sequences. To keep the flow of our argument smooth, we relegate the proofs of all the claims in this subsection to Appendix A.

Our first observation is that in the context of the optimization problem (4), it suffices to consider only those $S \in \mathcal{S}$ with the property that SDS^t and Σ commute.

Lemma 2.2: $\forall S \in \mathcal{S}, \exists \tilde{S} \in \mathcal{S}$ such that $C_{\text{sum}}(S) \leq C_{\text{sum}}(\tilde{S})$ and $\tilde{S}D\tilde{S}^t$ commutes with Σ.

Define the set $\tilde{\mathcal{S}}$ as the subset of $\mathcal{S}$ containing $S \in \mathcal{S}$ with the property that SDS^t commutes with Σ. Then, we can restrict the optimization in (8) over $S \in \tilde{\mathcal{S}}$. Writing the vector of eigenvalues of SDS^t by $\lambda(S)$, Lemma 2.2 (along with the classical result that two matrices commute if and only if they have the same eigenvectors) allows us to write (4) in the simplified form

$$C_{\text{opt}}(D, \Sigma) = \max_{S \in \tilde{\mathcal{S}}} \frac{1}{2N} \sum_{i=1}^{N} \log\left(1 + \frac{\lambda_i(S)}{\sigma_i^2}\right). \tag{8}$$

Observe that the rank of SDS^t is upper-bounded by $\min(K, N)$ and hence only $\min(K, N)$ of the eigenvalues $\lambda_1(S), \ldots, \lambda_N(S)$ are positive. Thus, if $K < N$, we see from (8) that the optimal sequences will always have the property that the subspace they span (of dimension at most K) should not contain the eigenvectors of Σ corresponding to the largest $N - K$ eigenvalues. Hence, without loss of generality, we assume that $K \geq N$. Let us denote the vector of eigenvalues of $SDS^t + \Sigma$ by $\mu(S)$.

The following result from the proof of [16, Theorem 3.1] characterizes the map $S \mapsto \lambda(S)$.

Lemma 2.3:

$$\begin{aligned}\mathcal{L}_1' &\stackrel{\text{def}}{=} \left\{\lambda(S): S \in \tilde{\mathcal{S}}\right\} \\ &= \{(\lambda_1, \ldots, \lambda_N) \in \mathbb{R}_+^N: (\lambda_1, \ldots, \lambda_N, 0, \ldots, 0) \\ &\qquad \text{majorizes } (p_1, \ldots, p_K)\}.\end{aligned} \tag{9}$$

The proof of this result also provides an algorithm which, given $\lambda \in \mathcal{L}_1'$, constructs S such that $\lambda(S) = \lambda$. Now, recalling that the noise variances are ordered as $\sigma_1^2 \leq \sigma_2^2 \leq \cdots \leq \sigma_N^2$, we make an elementary observation that based on this ordering.

Lemma 2.4: For any vector $\lambda = (\lambda_1, \ldots, \lambda_N)$ in the positive orthant of $\mathbb{R}^N$

$$(\lambda_1 + \sigma_1^2, \ldots, \lambda_N + \sigma_N^2) \quad \text{majorizes} \quad \left(\lambda_{[1]} + \sigma_1^2, \ldots, \lambda_{[N]} + \sigma_N^2\right).$$

We summarize Lemmas 2.3 and 2.4 as the following formal result.

Lemma 2.5: Define the polyhedron $\mathcal{L}$ in the positive orthant of $\mathbb{R}^N$ as shown by the expression at the bottom of the page.

1) For every $\mu \in \mathcal{L}$ there exists $S \in \tilde{\mathcal{S}}$ such that $\mu(S) = \mu$.
2) For every $S \in \tilde{\mathcal{S}}$, there exists $\mu \in \mathcal{L}$ such that $\mu(S)$ majorizes μ.

Consider the following combinatorial algorithm (denoted by $\mathcal{A}$) that has its output a vector $\mu^* \in \mathcal{L}$. Our main result is that μ^* is a Schur-minimal element of $\mathcal{L}$ thereby completing our identification of the optimal signature sequences.

Theorem 2.6: Output μ^* of the combinatorial algorithm $\mathcal{A}$ is a Schur-minimal element of $\mathcal{L}$, i.e., μ majorizes μ^* for every $\mu \in \mathcal{L}$.

Algorithm $\mathcal{A}$:
Input $K, N, (p_1, \ldots, p_K)$ and $\sigma_1^2 \leq \sigma_2^2 \cdots \leq \sigma_N^2$.
Output $\mu^* \stackrel{\text{def}}{=} (\mu_1^*, \ldots, \mu_N^*) \in \mathcal{L}$.
Update

1. Initialization: $i = 1$, $j = N$ and $\mu_k^* = 0, \forall k = 0 \ldots N$.
2. Termination: If $i > j$ stop and output the vector $(\mu_1^*, \ldots, \mu_N^*)$. Else, go to Step 3.

$$\mathcal{L} = \left\{ (\mu_1, \ldots, \mu_N): \begin{array}{c} \mu_i \geq \sigma_i^2, \qquad \forall i = 1 \ldots N \\ \sum_{j=1}^{i} \mu_j \geq \sum_{j=1}^{i} p_{[j]} + \sigma_j^2, \ \forall i = 1 \ldots N-1 \\ \sum_{j=1}^{N} \mu_j = \sum_{j=1}^{K} p_j + \sum_{j=1}^{N} \sigma_j^2 \end{array} \right\}.$$

3. Let

$$\eta = \max\left\{\sigma_j^2, \frac{\sum_{k=i}^{K} p_{[k]} + \sum_{m=i}^{j} \sigma_m^2}{j-i+1}, \frac{1}{l-i+1}\sum_{k=i}^{l}\left(p_{[k]}+\sigma_k^2\right), i \le l < j\right\}. \quad (10)$$

   a) If $\eta = \sigma_j^2$ then set $\mu_j^* := \sigma_j^2$ and $j := j-1$. Go to Step 2.
   b) If

$$\eta = \frac{\sum_{k=1}^{K} p_{[k]} + \sum_{m=1}^{N} \sigma_m^2 - \sum_{m \notin \{i, \ldots, j\}} \mu_m^*}{j-i+1}$$

   then set $\mu_m^* := \eta, \forall m = i \ldots, j$ and $i := j+1$. Go to Step 2.
   c) If $\eta = \frac{1}{l-i+1}\sum_{k=i}^{l}(p_{[k]} + \sigma_k^2)$ for some $l \in \{i, \ldots, j\}$ then set $\mu_m^* := \eta, \forall m = i, \ldots, l$ and $i := l+1$. Go to Step 2.

D. Properties of Algorithm $\mathcal{A}$

1) The optimization problem in (8) can be rewritten, in view of Lemma 2.5, as follows:

$$C_{\text{opt}}(D, \Sigma) = \max_{\mu \in \mathcal{L}} \frac{1}{2N}\sum_{i=1}^{N}\log\left(\frac{\mu_i}{\sigma_i^2}\right).$$

The function

$$(\mu_1, \ldots, \mu_N) \mapsto \sum_{i=1}^{N}\log\frac{\mu_i}{\sigma_i^2}$$

is concave and the optimization is over $\mathcal{L}$ a convex set that has a Schur-minimal element μ^* (Theorem 2.6). Thus, the solution to the optimization problem (8) is μ^*.

2) The algorithm stops after at most N steps.
3) The updates of the components of μ^* by $\mathcal{A}$ are in nonincreasing order. Hence, $\mathcal{A}$ is a *greedy* algorithm in the sense that the algorithm first sets the largest component of μ to the smallest value it can attain and then reduces the problem to one lesser dimension. This claim is proved in Appendix A.5 where it is used in the proof of Theorem 2.6.
4) The special case of $\Sigma = \sigma^2 I$ was addressed in [16] and Algorithm $\mathcal{A}$ reduces to the following simple form [16, Sec. 3]. Define the set $\mathcal{K}_1 \subset \{1, \ldots, N\}$ to be

$$\left\{k: p_k > \frac{\sum_{j=1}^{K} p_j 1_{\{p_k > p_j\}}}{N - \sum_{j=1}^{K} 1_{\{p_j \ge p_k\}}}\right\}.$$

It follows that if $k \in \mathcal{K}_1$ then every user l with power constraint $p_l \ge p_k$ also belongs to $\mathcal{K}_1$. The optimal solution μ^* is simply

$$\mu_l^* = p_l + \sigma^2, \qquad l \in \mathcal{K}_1$$

and

$$\mu_j^* = \frac{\sum_{i=1}^{K-|\mathcal{K}_1|} p_i}{N - |\mathcal{K}_1|} + \sigma^2, \qquad j \notin \mathcal{K}_1.$$

The physical intuition is that for every $k \in \mathcal{K}_1$, the user k is *oversized*, i.e., its power is large *relative* to the power constraints of the other users and the degrees of freedom. Every oversized user is given an independent channel. (In the DS-CDMA context, this is done by allocating oversized users signature sequences that are orthogonal to all the other signature sequences.)

5) In the special case when $D = pI$, again, Algorithm $\mathcal{A}$ has a simple structure. Observe that in this case, Case 3c) of the algorithm will never be reached and this makes the algorithm have the following simple form. Define the set $\mathcal{K}_2$ to be

$$\left\{k: \sigma_k^2 > \frac{Kp + \sum_{j=1}^{N} \sigma_j^2 1_{\{\sigma_k^2 > \sigma_j^2\}}}{N - \sum_{j=1}^{N} 1_{\{\sigma_j^2 \ge \sigma_k^2\}}}\right\}.$$

Observe that if $k \in \mathcal{K}_2$ then every l such that $\sigma_l^2 \ge \sigma_k^2$ also belongs to $\mathcal{K}_2$. Thus, $\mathcal{K}_2$ is of the form $\{k, \ldots, N\}$ for some $1 < k \le N+1$ (by convention $k = N+1$ indicates $\mathcal{K}_2$ is empty). Algorithm $\mathcal{A}$ simply outputs

$$\mu_l = \sigma_l^2, \qquad k \le l \le N$$

and

$$\mu_j = \frac{Kp + \sum_{i=1}^{k-1} \sigma_i^2}{k-1}, \qquad 1 \le j < k. \quad (11)$$

The physical intuition is that for every $k \in \mathcal{K}_2$, the "channel" (the direction specified by the eigenvector of Σ corresponding to the eigenvalue σ_k^2) k is *oversized* and has noise variance σ_k^2 large *relative* to the other noise variances and the number of users and the processing gain. Hence, the transmit signals do not have any energy in the direction of these oversized "channels."

E. Properties of $C_{\text{opt}}(D, \Sigma)$

Consider a VMAC with additive white noise and variance σ^2. Suppose we make one of the noise variances, say σ_N^2, much larger than the rest while keeping the average of the variances equal to σ^2. The users can avoid using signals in the direction of the eigenvector of Σ corresponding to the large eigenvalue σ_N^2 and benefit from a reduced average noise variance (since the overall average noise variance is still σ^2). Thus, we expect that the maximal sum capacity of the latter channel will be more than the maximal sum capacity of the additive white noise channel. We make this intuitive idea precise in the following proposition using the notion of majorization to characterize when a vector has components more spread out than others.

Proposition 2.1: Fix D, the diagonal matrix of user powers. Then $C_{\rm opt}(D, \Sigma)$ is convex in Σ. Also,

$$C_{\rm opt}(D, \Sigma) \geq C_{\rm opt}(D, \tilde{\Sigma})$$

for every Σ and $\tilde{\Sigma}$ such that $(\sigma_1^2, \ldots, \sigma_N^2)$ majorizes $(\tilde{\sigma}_1^2, \ldots, \tilde{\sigma}_N^2)$.

Thus, Proposition 2.1 says that for fixed user power constraints, $C_{\rm opt}$ increases if the noise variance becomes "more colored" while keeping the total noise variance $({\rm tr}\Sigma)$ constant. On the other hand, keeping the additive noise variances fixed, if the user power constraints are asymmetric, keeping the total user power fixed, it is intuitive that there is lesser flexibility in choosing μ. We make this precise in the following.

Proposition 2.2: For fixed Σ, $C_{\rm opt}(D, \Sigma)$ is concave in D and, furthermore, for every $D \neq \tilde{D}$ such that $(p_1, \ldots, p_K)$ majorizes $(\tilde{p}_1, \ldots, \tilde{p}_K)$ we have

$$C_{\rm opt}(\tilde{D}, \Sigma) > C_{\rm opt}(D, \Sigma).$$

We conclude that $C_{\rm opt}(\Sigma, D)$ is a concave (and Schur-concave) function in D and convex (and Schur-convex) in Σ. Thus, $C_{\rm opt}$ is a *saddle function* in D and Σ (in fact, $C_{\rm opt}$ is also a "Schur-saddle function" in the sense of the results of Propositions 2.1 and 2.2). This saddle function property is reminiscent of the famous Shannon saddle function property of mutual information

$$\begin{aligned} I\left(\overline{X}_g;\, S\overline{X}_g + Z\right) &\geq I\left(\overline{X}_g;\, S\overline{X}_g + W\right) \\ &\geq I\left(\overline{X};\, S\overline{X} + W\right) \end{aligned}$$

where $\overline{X}$ is a K-dimensional random vector with covariance matrix D and $\overline{X}_g$ is a K-dimensional Gaussian random vector with the same covariance matrix D. Also, Z is an N-dimensional noise vector with covariance matrix Σ and W is an N-dimensional Gaussian noise vector with the same covariance matrix Σ. Our result says that $C_{\rm opt}$, the maximum value of $I(\overline{X};\; S\overline{X} + W)$ (maximum over $S \in \mathcal{S}$ and independent distributions on $\overline{X}$ subject to a variance constraint), is a saddle function in D and Σ. The formal proofs of Propositions 2.1 and 2.2 are in Appendix A.6.

F. Construction of Optimal Signature Sequences

The general scheme to construct the optimal signature sequences is contained in the proofs of our main results: Lemma 2.5 and Theorem 2.6. In what follows, we summarize this construction. Let $Q \stackrel{\rm def}{=} [\boldsymbol{q}_1, \ldots, \boldsymbol{q}_N]$ be an orthonormal matrix that has the property that $Q^t \Sigma Q$ is a diagonal matrix with diagonal entries $\{\sigma_1^2, \ldots, \sigma_N^2\}$. Furthermore, we assume that we first use algorithm $\mathcal{A}$ to generate μ^* and construct the vector $\lambda^* = (\mu_1^* - \sigma_1^2, \ldots, \mu_N^* - \sigma_N^2)$. We then use the recursive algorithm in [16, Sec. 4] to construct $S \in \mathcal{S}$ such that the matrix SDS^t is the diagonal matrix ${\rm diag}\{\lambda_1^*, \ldots, \lambda_N^*\}$. Then the optimal signal directions S^* are given by QS. However, the structure of $\mathcal{A}$ yields more insight into the nature of the optimal signal directions S^* and this allows the following more succinct characterization and construction of the optimal signal directions.

1) We begin with the first iteration of $\mathcal{A}$. If Step 3a) is reached, then we set $\lambda_N^* = 0$. This means that the optimal signal directions are orthogonal to $\boldsymbol{q}_N$. Thus, this allows us to recursively reduce the problem to one with only $N-1$ degrees of freedom.
2) If Step 3b) is reached, then we set

$$\lambda_j^* \frac{{\rm tr}D + {\rm tr}\Sigma}{N} - \sigma_j^2, \qquad 1 \leq j \leq N.$$

We use the recursive algorithm of [16, Sec. 4] to construct $S \in \mathcal{S}$ such that $SDS^t = {\rm diag}\{\lambda_1^*, \ldots, \lambda_N^*\}$. Then, the optimal signal directions are $S^* = QS$. This step terminates the algorithm and completes the construction.
3) If Step 3c) is reached for $1 \leq l < N$, then we have

$$\lambda_{[j]}^* = \lambda_j^* = \frac{\sum_{i=1}^{l} (p_{[i]} + \sigma_i^2)}{l} - \sigma_j^2, \qquad 1 \leq j \leq l.$$

For expository ease, we assume that the users are ordered according to their power constraints, i.e., $p_1 \geq p_2 \geq \cdots \geq p_K$. By the hypothesis that Step 3c) is reached for l, and by construction, $(\lambda_1^*, \ldots, \lambda_l^*)$ majorizes the vector $(p_1, \ldots, p_l)$. Thus, we can use the procedure in [16, Sec. 4] to construct an $l \times l$ matrix S_l such that

$$S_l {\rm diag}\{p_1, \ldots, p_l\} S_l^t = {\rm diag}\{\lambda_1^*, \ldots \lambda_l^*\}.$$

We construct the optimal signal directions for the first l users (these have the largest power constraints) as

$$[\boldsymbol{s}_1^*, \ldots, \boldsymbol{s}_l^*] = [\boldsymbol{q}_1, \ldots, \boldsymbol{q}_l] S_l.$$

The following is a key observation: recall that the output μ^* of $\mathcal{A}$ is in $\mathcal{L}$ and hence $(\lambda_1^*, \ldots, \lambda_N^*, 0, \ldots, 0)$ must majorize $(p_1, \ldots, p_K)$. By construction of $\lambda_{[i]}^* = \lambda_i^*$, $1 \leq i \leq l$ above, we must have

$$(\lambda_{l+1}^*, \ldots, \lambda_N^*, 0 \ldots, 0) \;\text{ majorizes }\; (p_{l+1}, \ldots, p_K). \tag{12}$$

Recalling the construction of $S^* \in \mathcal{S}$ from λ^* from the proof of Lemma 2.3, we see from (12) that we can construct the $N-l \times K-l$ matrix $S_{\bar{l}}$ such that

$$S_{\bar{l}} {\rm diag}\{p_{l+1}, \ldots, p_K\} S_{\bar{l}}^t = {\rm diag}\{\lambda_{l+1}^*, \ldots, \lambda_N^*\}.$$

We then let the optimal signal directions for the remaining $K-l$ users be

$$[\boldsymbol{s}_{l+1}^*, \ldots, \boldsymbol{s}_K^*] = [\boldsymbol{q}_{l+1}, \ldots, \boldsymbol{q}_N] S_{\bar{l}}.$$

We emphasize the point that each of the optimal signal directions of the first l users is orthogonal to each one of the optimal signal directions of the remaining $K-l$ users. Furthermore, the first l user signal directions span the l-dimensional subspace ${\rm span}\{\boldsymbol{q}_1, \ldots, \boldsymbol{q}_l\}$ while the signal directions of the remaining $K-l$ users span the orthogonal complement of this subspace. Thus, if Step 3c) is reached in the first iteration of $\mathcal{A}$, this observation allows us to identify the user signal directions for the first l users and recursively reduce the problem to one of l fewer users and l fewer degrees of freedom.

4) We summarize below some physical insights gained from the observations made in Section II-D and from the proof of Theorem 2.6.
   a) Consider the first iteration of $\mathcal{A}$. Step 3a) is reached if the largest noise variance σ_N^2 is "much larger" than the other noise variances (in a sense made precise in the algorithm). Our optimal signal directions are chosen in this case to be orthogonal to $\boldsymbol{q}_N$ and thus they avoid "directions" of high noise variance. We emphasize that this step is never reached if all the noise variances are equal.
   b) Suppose Step 3c) is reached for some $1 \le l < N$. This means that the average of the largest l user power constraints is "much larger" than other averages of user powers in a sense that depends on the noise variances as well (made precise in the algorithm) and the optimal signal directions are assigned to these l users so that they span a subspace (of dimension l) given by $\operatorname{span}\{\boldsymbol{q}_1, \dots, \boldsymbol{q}_l\}$. Thus, these signal directions lie in the subspace with least noise and, furthermore, the other user signal directions are orthogonal to this subspace. We emphasize that this step is never reached if all the user powers are equal.

G. Parallel Gaussian Channels

Consider the following system of parallel Gaussian channels (our notation is from [2, Sec. 10.4]):

$$y_j = x_j + z_j, \quad z_j \sim \boldsymbol{N}(0, \sigma_j^2), \qquad j = 1 \dots N$$

where the Gaussian noise is independent from channel to channel. The total power constraint on the input is $\mathbb{E}[\sum_{j=1}^N x_j^2] \le P$. Denoting the sum capacity (the maximum sum of rates per unit channel at which all information can be transmitted in each of the channels reliably) of this channel by $C_p(P, \Sigma)$ we have

$$C_p(P, \Sigma) = \max_{\{(\eta_1, \dots, \eta_N) \in \mathbb{R}_+^N:\ \sum_{j=1}^N \eta_j = P\}} \frac{1}{2N} \cdot \sum_{j=1}^{N} \log\left(1 + \frac{\eta_j}{\sigma_j^2}\right). \tag{13}$$

It is very well known that the optimal allocation of powers follows the water-filling policy $\mathbb{E}[x_j^2] = \eta_j^* = (\beta - \sigma_j^2)^+$ for some $\beta > 0$ such that $\sum_{j=1}^N \eta_j^* = P$. A further explicit expression for the water-filling policy is as follows. Define the set $\mathcal{K}_{\rm wf}$ to be

$$\left\{ k:\ \sigma_k^2 > \frac{P + \sum_{j=1}^{N} \sigma_j^2 1_{\{\sigma_k^2 > \sigma_j^2\}}}{N - \sum_{j=1}^{N} 1_{\{\sigma_j^2 \ge \sigma_k^2\}}} \right\}.$$

Observe that if $k \in \mathcal{K}_{\rm wf}$ then every l such that $\sigma_l^2 \ge \sigma_k^2$ also belongs to $\mathcal{K}_{\rm wf}$. Since we have ordered the variances $\sigma_1^2 \le \sigma_2^2 \le \cdots \le \sigma_N^2$, the set $\mathcal{K}_{\rm wf}$ is of the form $\{k, \dots N\}$ for some $1 < k \le N+1$ (if $k = N+1$, then by convention we take $\mathcal{K}_{\rm wf}$ to be empty). The water-filling policy is simply

$$\eta_l^* = 0, \qquad k \le l \le N$$

and

$$\eta_j^* = \frac{P + \sum_{i=1}^{k-1} \sigma_i^2}{k-1} - \sigma_j^2, \qquad 1 \le j < k. \tag{14}$$

Then (13) becomes

$$C_p(P, \Sigma) = \frac{1}{2N} \sum_{j=1}^{k-1} \log\left(\frac{P + \sum_{i=1}^{k-1} \sigma_i^2}{(k-1)\sigma_j^2} \right).$$

On the other hand, for every $S \in \mathcal{S}$, the sum capacity of the multiaccess vector channel (2) is

$$\begin{aligned} &C_{\rm sum}(S, D, \Sigma) \\ &\quad = \frac{1}{2N} \log\det(I + \Sigma^{-1} S D S^t) \\ &\quad = \frac{1}{2N} \log\det(I + \operatorname{diag}\{\sigma_1^{-2}, \dots, \sigma_N^{-2}\} Q S D S^t Q^t), \\ &\qquad\qquad\qquad \text{for some orthogonal } Q \\ &\quad \le \frac{1}{2N} \sum_{i=1}^{N} \log\left(1 + \frac{d_i}{\sigma_i^2}\right) \end{aligned}$$

where we have denoted the diagonal entries of $QSDS^tQ^t$ by $d_1, \dots, d_n$ and used the Hadamard inequality [4, Sec. 4.3, Problem 11] in the derivation of the last step. Since $\sum_{j=1}^N d_i = \operatorname{tr} SDS^t \operatorname{tr} D$, comparing with (13) we arrive at the following simple upper bound to $C_{\rm opt}(D, \Sigma)$:

$$C_{\rm opt}(D, \Sigma) \le C_p(\operatorname{tr} D, \Sigma).$$

Though this upper bound is to be expected, our characterization of the sum capacity $C_{\rm opt}(D, \Sigma)$ shows that if the user powers are not too "spread apart" (in a sense that depends on the size of the problem and Σ) the upper bound can actually be attained. In fact, we recognize that the simple form of $\mathcal{A}$ in (11) in Section II-D coincides with the water-filling policy in (14). Thus, the case of symmetric user powers $(D = pI)$ is a sufficient condition for the sum capacity $C_{\rm opt}$ to be equal to the corresponding parallel Gaussian channels sum capacity C_p. Our claim is that a necessary and sufficient for equality of $C_{\rm opt}$ and C_p is Step 3c) of Algorithm $\mathcal{A}$ being never reached. This follows the fact that when Step 3c) is never reached, $\mathcal{A}$ simply reduces to the water-filling allocation of (14).

III. Linear Receivers and User Capacity

Linear receivers are popular and practical choices in wireless communication systems. For instance, IS-95 uses RAKE receivers which are matched filters tuned to the multiple paths of the received signal [18], and the LMMSE receiver is considered a promising candidate in the 3G implementation. It is important to characterize the performance of these receivers in a *network-centric* way. In this section, we consider a set of users trying to communicate reliably with a single base station. The users have SIR requirements to be met for reliable communication and we are focusing on DS-CDMA as the multiple-access scheme. The problem of determining when the requirements of the users can be satisfied (i.e., they are admissible) and characterizing the *optimal* allocations of signature sequences and powers (optimal in the sense that the average powers of the users are minimized among all allocations that let the users meet their SIR requirements) is the central focus of [17]. In this section, we are interested in the effect of additive colored noise on the admissibility region with average power constraints on the users. Our main result is a precise characterization that quantifies the effect of additive colored noise and also allows us to derive important qualitative properties of the solution.

We have organized the material in this section as follows. In Section III-A, we briefly describe the model and precisely state the problem of interest. Section III-B contains our main result: a complete characterization of the optimal allocation of signature sequences and powers that minimize the average transmit powers of the users. Section III-C uses the characterization of optimal allocations to derive some important qualitative properties of the solution. In Section III-D, we use the characterization of the optimal allocations to determine the precise region of admissible SIRs given an average power constraint. Section III-E studies the effect of additive colored noise when the signature sequences are random, extends the results of [13], and allows us to compare the penalty paid on the average power transmitted by choosing the signature sequences randomly. Section III-F concludes the exposition with some discussion.

A. Model and Problem Statement

We begin by recalling the baseband model of DS-CDMA as a VMAC in (2)

$$\boldsymbol{y}(m) = \sum_{i=1}^{K} \boldsymbol{s}_i x_i(m) + \boldsymbol{z}(m). \tag{15}$$

Here, $\boldsymbol{s}_i$ is the signature sequence of user i and is an element of $\mathbb{R}^N$ with unit energy (i.e., $\boldsymbol{s}_i^t\boldsymbol{s}_i = 1$). The symbol transmitted by user i in the mth symbol interval is denoted by $x_i(m)$ and the *received* power of user i is denoted by p_i. Suppose the symbol of user i is decoded using a linear receiver, characterized by $\boldsymbol{c}_i$ (a vector in $\mathbb{R}^N$), then the SIR of user i (SIR_i) is

$$\text{SIR}_i = \frac{(\boldsymbol{c}_i, \boldsymbol{s}_i)^2 p_i}{\sigma^2(\boldsymbol{c}_i, \boldsymbol{c}_i) + \sum_{j\neq i} (\boldsymbol{c}_i, \boldsymbol{s}_j)^2 p_j}. \tag{16}$$

We say that K users are *admissible* in the system if there is an allocation of positive powers $p_1, \dots, p_K$, signature sequences $\boldsymbol{s}_1, \dots, \boldsymbol{s}_K$ (vectors $\in \mathbb{R}^N$ with unit l_2 norm), and linear receiver structures $\boldsymbol{c}_1, \dots, \boldsymbol{c}_K$ such that

$$\text{SIR}_i \geq \beta_i, \qquad \forall i = 1 \dots K.$$

Here $\beta_i > 0$ is some fixed SIR requirement of user i that has to be met for each user for satisfactory performance. Such a choice of powers and signature sequences is called a *valid* allocation. Our focus here is on the LMMSE receiver. This is the optimal linear receiver, optimal in the sense of maximizing the SIR among all linear receivers. A computation analogous to that in [17, Sec. 2.2] shows that the (unnormalized) LMMSE receiver for user i is given by

$$\boldsymbol{c}_i = (SDS^t + \Sigma)^{-1}\boldsymbol{s}_i \tag{17}$$

where we have written S for $[\boldsymbol{s}_1, \dots, \boldsymbol{s}_K]$ and D for $\text{diag}\{p_1, \dots, p_K\}$ as in Section II-B. The corresponding SIR achieved by user i is

$$\text{SIR}_i = \frac{\boldsymbol{s}_i^t(SDS^t + \Sigma)^{-1}\boldsymbol{s}_i p_i}{1 - \boldsymbol{s}_i^t(SDS^t + \Sigma)^{-1}\boldsymbol{s}_i p_i}. \tag{18}$$

Suppose the K users have SIR requirements $\beta_1, \dots, \beta_K$. The user capacity region is defined to be the set of *admissible* SIR requirements, i.e., those for which there exist an allocation of signature sequences and powers such that the SIR of the users with LMMSE receivers are at least equal to the requirements. In [17], we showed that the user capacity region is precisely given by

$$\left\{(\beta_1, \dots, \beta_K)\colon \sum_{i=1}^{K} e(\beta_i) < N\right\}, \quad \text{where } e(x) = \frac{x}{1+x}.$$

Motivated by this admissibility result, we call $e(\beta_i)$ the effective bandwidth of user i. This is because the user capacity region can then be described as the region where the sum of the effective bandwidths of the users does not exceed the processing gain.

This result assumes that there is no constraint on the user power allocated and thus the scenario of colored additive noise has no affect on this user capacity region. Thus, to sharpen our understanding of the effect of colored noise and the corresponding optimal sequences, we impose a power constraint on the users. Since it is clear that there cannot be a componentwise minimal power solution to achieve a set of target SIRs, we choose, as in [17, Sects. 4 and 5], to impose a sum received power constraint. One way to justify this constraint on the received power of the users is considering an *average transmit power* constraint on the users. If one is able to adopt a model of fading for the users that is ergodic (with the same mean fading) and independent, an average transmit power constraint on the users translates into a received power constraint. Another justification is the following. The total transmit power decides the total interference caused to neighboring cells and the total received power captures this quantity as a rough estimate. For more remarks regarding our focus on the sum received power constraint, see the discussion section, Section III-F.

B. Main Result: Optimal Allocation

Fix a set of SIR requirements $\beta_1, \dots, \beta_K$ that are admissible with no average received power constraint (thus, the sum of the effective bandwidths is less than N). In this subsection, we will address the user capacity problem section in a "dual" sense: *minimize the sum of allocated powers among all valid allocations*. The main result of this section is a complete solution to this "dual" problem. Mathematically, the dual problem can be stated as below (using the expression (18) for the SIR achieved by the LMMSE receiver).

$$\begin{aligned}\text{Dual Problem } \mathcal{P}: \quad & \text{Minimize } \mathrm{tr}[D] \text{ subject to the condition} \\ & D^{\frac{1}{2}} S^t (SDS^t + \Sigma)^{-1} S D^{\frac{1}{2}} \text{ has diagonal entries} \\ & \text{greater than or equal to } e(\beta), \dots, e(\beta_K).\end{aligned} \tag{19}$$

Here D varies over the positive diagonal matrices and $S \in \mathcal{S}$ and the condition in (19) represents the constraint that the users's SIR targets are met. Any pair (D, S) that solves Problem $\mathcal{P}$ is an *optimal allocation*.

We begin with a preliminary lemma that characterizes an important property of all optimal allocations.

Lemma 3.1: If the pair (D, S) is an optimal allocation, then SDS^t and Σ commute.

Proof: As an aid to prove this result, we continue with the following series of lemmas.

Lemma 3.2: Consider the optimization problem $\tilde{\mathcal{P}}$ as follows.

Problem $\tilde{\mathcal{P}}$: Maximize

$$\mathrm{tr}\left[WD^{\frac{1}{2}}S^t(SDS^t+\Sigma)^{-1}SD^{\frac{1}{2}}\right] \text{ subject to } \mathrm{tr}[D] \le P.$$

Here, W is a fixed nonnegative diagonal matrix and the optimization is as D varies over diagonal matrices with nonnegative entries and S ranges over $\mathcal{S}$. Then, any pair (D, S) that achieves the maximum above has the property that SDS^t and Σ commute. □

With no user received power constraint, we know that the user capacity region is the region of the positive orthant bounded above by $\sum_{i=1}^K e(\beta_i) = N$. This is a hyperplane, when the coordinates of the user capacity region are measured in terms of the effective bandwidths rather than the SIR requirements. Under this simple coordinate transformation, the user capacity region is thus convex. Our next lemma shows that the user capacity region with any given sum received power constraint P continues to be convex with these new coordinates.

Lemma 3.3: Fix a positive sum received power constraint P. Then the set (denoted by $\mathcal{F}$) of

$$\{(e_1, \dots, e_K): 0 \le e_i < 1, \ \forall i\}$$

such that users with SIRs $\frac{e_1}{1-e_1}, \dots, \frac{e_K}{1-e_K}$ are admissible when sum received power constraint P is convex.

Since any convex set is precisely the intersection of all closed half-spaces that contain it [9, Theorem 11.5], Lemmas 3.3 and 3.2 complete the proof of Lemma 3.1. We provide the proofs of Lemmas 3.3 and 3.2 in Appendixes B.2 and B.1, respectively. □

Suppose the pair (D, S) is a solution to the dual problem $\mathcal{P}$ in (19) and let a singular value decomposition of $SD^{\frac{1}{2}}$ be $U\Lambda^{\frac{1}{2}}V^t$ where U is an $N \times N$ orthonormal matrix, Λ is an $N \times N$ diagonal matrix with diagonal entries equal to the eigenvalues of SDS^t, and V^t is an $N \times K$ matrix with orthonormal rows. Since $\mathrm{tr}[D] = \mathrm{tr}[SDS^t]$, we can rewrite the dual problem $\mathcal{P}$ as below. We also observe that the optimal allocation has equality in the diagonal elements of (19).

Dual Problem $\mathcal{P}$: Minimize $\mathrm{tr}[\Lambda]$ subject to the condition

$$V\Lambda^{\frac{1}{2}}U^t(U\Lambda U^t+\Sigma)^{-1}U\Lambda^{\frac{1}{2}}V^t \text{ has diagonal entries equal to } e(\beta_1), \dots, e(\beta_K). \tag{20}$$

Here U is any $N \times N$ orthogonal matrix, Λ is any nonnegative diagonal matrix, and V is any $N \times K$ matrix with orthonormal rows.

From Lemma 3.1, we have equation

$$[SDS^t, \Sigma] \stackrel{\text{def}}{=} SDS^t\Sigma - \Sigma SDS^t = \mathbf{0}.$$

Since two symmetric matrices commute if and only if they are jointly diagonalizable, $U\Lambda U^t + \Sigma$ can be written as

$$U(\Lambda + \mathrm{diag}\{\sigma_1^2, \dots, \sigma_N^2\})U^t.$$

This result allows us to rewrite the condition (20) as

$$V\Lambda(\Lambda + \mathrm{diag}\{\sigma_1^2, \dots, \sigma_N^2\})^{-1}V^t \text{ has diagonal entries equal to } e(\beta_1), \dots, e(\beta_K). \tag{21}$$

Appealing to Lemma 2.1, we can rewrite (21) using the notation $\Lambda = \mathrm{diag}\{\lambda_1, \dots, \lambda_K\}$ as

$$\left(\frac{\lambda_1}{\lambda_1+\sigma_1^2}, \dots, \frac{\lambda_N}{\lambda_N+\sigma_N^2}, 0, \dots, 0\right) \text{ majorizes } (e(\beta_1), \dots, e(\beta_K)).$$

This allows us to rewrite the dual problem $\mathcal{P}$ using (21) as follows:

Dual Problem $\mathcal{P}$: Minimize $\sum_{i=1}^N \lambda_i$ subject to the condition

$$\left(\frac{\lambda_1}{\lambda_1+\sigma_1^2}, \dots, \frac{\lambda_N}{\lambda_N+\sigma_N^2}, 0, \dots, 0\right) \text{ majorizes } (e(\beta_1), \dots, e(\beta_K)) \text{ and } \lambda \ge 0, \ \forall i = 1 \dots N. \tag{22}$$

Suppose $\boldsymbol{\lambda}^* \stackrel{\text{def}}{=} (\lambda_1^*, \dots, \lambda_N^*)$ is a solution to the problem above. The optimal allocations are formed as follows. We use the combinatorial algorithm of [16, Sec. 4] to generate a $K \times K$ symmetric matrix H with eigenvalues $\lambda_1^*, \dots, \lambda_N^*$ and the eigenvalue 0 with multiplicity $K - N$ and diagonal entries $e(\beta_1), \dots, e(\beta_K)$. Then there exists an $N \times K$ matrix V^* with orthonormal rows such that

$$V \mathrm{diag}\left\{\frac{\lambda_1^*}{\lambda_1^*+\sigma_1^2}, \dots, \frac{\lambda_N^*}{\lambda_N^*+\sigma_N^2}\right\} V^t$$

has diagonal entries $e(\beta_1), \dots, e(\beta_K)$. We now allocate the user powers to be $p_1, \dots, p_K$ as the diagonal entries of the matrix $V\Lambda^*V^t$. Writing the matrix $D = \mathrm{diag}\{p_1, \dots, p_K\}$, we allocate the signature sequences to the users (denoted by the matrix S) to be $U^*\Lambda^{*\frac{1}{2}}V^tD^{-\frac{1}{2}}$. Here, we have taken U^* to be any orthonormal matrix that diagonalizes Σ. It is straightforward to verify that S has unit norm columns and that the pair (D, S) is a valid allocation. Thus, it suffices to solve the optimization problem (22) above.

Consider the following *combinatorial* Algorithm $\mathcal{B}$. It terminates in at most N steps.

Algorithm $\mathcal{B}$:

Input $K, N, \beta_1 \ge \dots \ge \beta_K > 0$ and $0 < \sigma_1^2 \le \sigma_2^2 \dots \le \sigma_N^2$.

Output $\boldsymbol{\lambda}^* = (\lambda_1^*, \dots, \lambda_N^*)$.

Update

1. Initialization: $i = 1$, $j = N$ and $\lambda_k^* = 0, \ \forall k = 1 \dots N$.
2. Termination: If $i > j$ stop and output the vector $\boldsymbol{\lambda}^*$. Else, go to Step 3.

3. Let

$$\eta = \max\left\{\sigma_j, \frac{\sum_{m=i}^{j} \sigma_m}{j-i+1-\sum_{m=i}^{K} e(\beta_m)}, \frac{\sum_{m=i}^{t} \sigma_m}{t-i+1-\sum_{m=i}^{t} e(\beta_m)}, t = i \cdots j-1\right\}.$$

   a) If $\eta = \sigma_j$ then set $\lambda_j^* := 0$ and $j := j-1$. Go to Step 2.
   b) If

$$\eta = \frac{\sum_{m=i}^{j} \sigma_m}{j-i+1-\sum_{m=i}^{K} e(\beta_m)}$$

   then set $\lambda_m^* := \sigma_m(\eta - \sigma_m), \forall m = i \ldots, j$ and $i := N+1$. Go to Step 2.
   c) If

$$\eta = \frac{\sum_{m=i}^{t} \sigma_m}{t-i+1-\sum_{m=i}^{l} e(\beta_m)}$$

   for some $i \le t < j$, then set $\lambda_m^* = \sigma_m(\eta - \sigma_m)$, $\forall m = i, \ldots, l$ and $i := t+1$. Go to Step 2.

Our next main result is the following.

Theorem 3.4: Output $\boldsymbol{\lambda}^*$ of the combinatorial algorithm $\mathcal{B}$ solves the dual problem $\mathcal{P}$ in (22).

The proof is relegated to Appendix B.3. Further, for both the special cases of $\sigma_i = \sigma, i = 1 \cdots N$ and $\beta_j = \beta, j = 1 \cdots K$, the algorithm simplifies substantially. This is analogous to the simplification mentioned in Section II-D. We now derive some properties of the optimal allocation of signature sequences and powers, optimal in the sense of the dual problem $\mathcal{P}$. Our main result is the following.

Proposition 3.1: Let S^* and D^* be a solution to the dual problem $\mathcal{P}$ (defined in (19)). Then, the LMMSE receiver for each user i (as in (17)) is given by

$$\boldsymbol{c}_i^* \stackrel{\text{def}}{=} (S^* D^* S^{*t} + \Sigma)^{-1} \boldsymbol{s}_i^* = a_i \Sigma^{-\frac{1}{2}} \boldsymbol{s}_i^*$$

for some constant a_i.

We conclude that the LMMSE receiver with the optimal choice of signature sequences and powers simplifies to the *matched filter*, matched to the background noise. The proof of Proposition 3.1 is in Appendix B.4.

C. Qualitative Properties of Admissibility

We now study properties of the optimal allocation identified in the previous section and derive some qualitative features of the solution to the user admissibility problem. Let $P^*((\beta_1, \ldots, \beta_K), (\sigma_1^2, \ldots, \sigma_N^2))$ denote the minimum sum power required to achieve the SIR requirements $(\beta_1, \ldots, \beta_K)$ when the colored additive noise variances are $(\sigma_1^2 \ldots, \sigma_N^2)$. We also assume that the SIR requirements are such that the users are admissible, i.e., $\sum_{i=1}^{K} e(\beta_i) < N$. The proofs of the assertions of this section are in Appendix B.5.

We would like to study the effect of colored noise on the minimal sum power required to meet a given set of SIR requirements. Suppose the largest of the noise variances, σ_N^2, was very large, then the allocation could potentially avoid the direction with this noise component and thereby communicate on the "cleaner" directions. We make this observation precise in what follows.

Proposition 3.2: Fix a set of SIR requirements $(\beta_1, \ldots, \beta_K)$. Then, P^* is a concave function of $(\sigma_1^2, \ldots, \sigma_N^2)$ and furthermore, P^* is a Schur-concave function of $(\sigma_1^2, \ldots, \sigma_N^2)$, i.e.,

$$P^*((\beta_1, \ldots, \beta_K), (\sigma_1^2, \ldots, \sigma_N^2)) \ge P^*((\beta_1, \ldots, \beta_K), (\tilde{\sigma}_1^2, \ldots, \tilde{\sigma}_N^2))$$

whenever $(\tilde{\sigma}_1^2, \ldots, \tilde{\sigma}_N^2)$ majorizes $(\sigma_1^2, \ldots, \sigma_N^2)$.

In particular, Proposition 3.2 says that for a fixed set of SIR requirements, additive white noise requires the most sum power allocation among all additive colored noises with the same total power. On the other hand, keeping the additive colored noise covariances fixed, we have the following behavior of P^*.

Proposition 3.3: Fix the additive noise variances $(\sigma_1^2, \ldots, \sigma_N^2)$. Then, P^* is a convex function of $(e(\beta_1), \ldots, e(\beta_K))$ and furthermore,

$$P^*((\beta_1, \ldots, \beta_K), (\sigma_1^2, \ldots, \sigma_N^2)) \ge P^*\left(\left(\tilde{\beta}_1, \ldots, \tilde{\beta}_K\right), (\sigma_1^2, \ldots, \sigma_N^2)\right)$$

whenever $(e(\beta_1), \ldots, e(\beta_K))$ majorizes $(e(\tilde{\beta}_1), \ldots, e(\tilde{\beta}_K))$.

Thus, Proposition 3.3 says that P^* is a Schur-convex function of $(e(\beta_1), \ldots, e(\beta_K))$ for fixed additive noise variances. In particular, this means that among all SIR requirements with the same sum of effective bandwidths, the minimal sum power required to meet them is the one corresponding to all SIR requirements being equal.

D. Admissibility Region

In Section III-B, we characterized the allocations that achieve a given set of SIR requirements while minimizing the sum of received power among such allocations. Now, we will fix P, an upper bound on sum received power that can be allocated and characterize the region of SIRs admissible with this constraint. In Lemma 3.3, we have already shown that the region of admissible $\frac{\text{SIR}}{1+\text{SIR}}$ with the power constraint is convex. We now characterize this convex set. The proof is in Appendix B.6.

Proposition 3.4: SIRs $(\beta_1, \ldots, \beta_K)$ are achievable with a sum received power constraint P if and only if $(e(\beta_1), \ldots, e(\beta_K))$ is contained in the convex hull of $\mathcal{C}(P, \Sigma)$ as shown at the top of the following page. Here π ranges over all permutations on the set $\{1, \ldots, K\}$ and $\eta_{\boldsymbol{w}}$ is a positive number chosen such that

$$\sum_{i=1}^{N} \sigma_i (\eta_{\boldsymbol{w}} \sqrt{w_{[i]}} - \sigma_i)^+ = P.$$

$$\mathcal{C}(P,\,\Sigma) \stackrel{\text{def}}{=} \bigcup_{\pi} \bigcup_{w_1,\,\ldots,\,w_N>0} \left\{ \left(e_{\pi(1)},\,\ldots,\,e_{\pi(K)}\right) : \begin{array}{ll} 0 \le e_i \le \left(1 - \dfrac{\sigma_i}{\eta_{\boldsymbol{w}}\sqrt{w_{[i]}}}\right)^+, & 1 \le i \le N \\ e_j = 0, & N+1 \le j \le K \end{array} \right\}.$$

Observe that Section III-B contains the recipe to construct allocations $(S,\,D)$ that achieve any set of $\frac{\text{SIR}}{1+\text{SIR}}$ requirements contained in $\mathcal{C}(P,\,\Sigma)$ earlier. Since $\mathcal{C}(P,\,\Sigma)$ is convex, any point on its boundary is characterized by a vector $\boldsymbol{w}$ with nonnegative entries. We can then use the constructive approach of Section III-B to construct the corresponding allocation pair $(S,\,D)$.

In the special case when all the SIR requirements are identical there is a simple characterization of achievability (akin to [17, Theorem 4.1]). First, we find a positive constant η such that

$$\sum_{i=1}^{N} \sigma_i(\eta - \sigma_i)^+ = P.$$

Then, it follows that the largest achievable common SIR, denoted by β, satisfies

$$e(\beta) < \frac{\sum_{i=1}^{N} (\eta - \sigma_i)^+}{K\eta}.$$

E. Random Signature Sequences and Admissibility

In many communication systems employing DS-CDMA, signature sequences cannot be chosen optimally as a function of the loading (number of users in the system). In such situations, it is reasonable to model the signature sequences as random, but fixed once chosen. This model is used in [13] where the authors derived substantial insight into the performance of linear receivers (in terms of the SIR of the estimate) using results on the limiting distribution of the eigenvalues of large random matrices. One of the important results was the derivation of the limiting behavior of the SIR of the LMMSE estimate of a unit power user in a large system (large K and large N keeping the ratio of K to N fixed, which we denote to be α). We now present this result extended to our scenario of interest: colored additive noise. Let us recall the channel model from (15)

$$\boldsymbol{y}(m) = \sum_{i=1}^{K} \boldsymbol{s}_i x_i(m) + \boldsymbol{z}(m).$$

We assume that the entries of the signature sequences $\boldsymbol{s}_i$ are independent with zero mean and variance $\frac{1}{N}$. This normalization ensures that the signature sequences have unit expected energy. We assume that the empirical distribution of the eigenvalues of the covariance matrix of $\boldsymbol{z}$ (denoted by Σ) converges weakly (as $N \to \infty$) to a distribution which we denote F_σ. We also assume that the support of F_σ is strictly positive and thus bounded away from zero. We denote the received power of user i by p_i and assume that the empirical distribution of user received powers converges weakly to a distribution (as $K \to \infty$) which we denote F_p. We are now ready to present the extension of the main result of [13]: limiting behavior of the SIR of the LMMSE estimate.

Proposition 3.5: Let $\beta_i^{(N)}$ denote the (random) SIR of the LMMSE estimate of user i. Then

$$\beta_1^{(N)} \xrightarrow{\text{prob}} \beta^* p_1, \qquad \text{as } N \to \infty \text{ and } K = \lfloor \alpha N \rfloor$$

where β^* is the unique positive solution for β in the equation

$$\beta = \int_0^\infty \frac{dF_\sigma(z)}{z + \alpha \int \frac{p\,dF_p(p)}{1+p\beta}}. \tag{23}$$

The proof in [13, Sec. 4] extends directly to prove the proposition above with the use of the general result in [1]. We omit a discussion of this extension and focus our attention on qualitative features of β^*, the limiting SIR of the LMMSE estimate of a unit power user.

We would like to study the behavior of β^* as a function of F_p and F_σ and to this end we introduce the following partial order on such distributions. This partial order, known as *dilation*, is one way of generalizing the partial order of majorization from finite vector spaces to infinite dimensional spaces (with a locally convex topology). In our context, the distributions F_p and F_σ are probability distributions on the nonnegative reals, so we focus on the space of such distributions. For F_1 and F_2, two probability distributions on the nonnegative reals, we say that "F_1 is a *dilation* of F_2" if for every integrable nonnegative convex function ϕ on the positive reals we have

$$\int_0^\infty \phi(x)\,dF_1(x) \ge \int_0^\infty \phi(x)\,dF_2(x).$$

Analogous to Schur convexity, we say that a function f that maps probability distributions on nonnegative reals to the reals *dilatory-convex* if for every pair of probability distributions $(F_1,\,F_2)$ such that F_1 is a dilation of F_2, we have $f(F_2) \le f(F_1)$. We say f is *dilatory-concave* if $-f$ is dilatory-convex. We refer the interested reader to [8] for further details on the partial order of dilation and for its properties. We are now ready to state one of the main results of this section.

Proposition 3.6:

1) $\beta^*(F_p,\,F_\sigma)$ is dilatory-convex in F_p for fixed F_σ.
2) $\beta^*(F_p,\,F_\sigma)$ is dilatory-convex in F_σ for fixed F_p.

The proof is in Appendix B.7. The result says that for a fixed received power profile, as the noise becomes more colored (keeping the same average noise level constant), the SIR of the LMMSE estimate of a unit power user increases. Also, for fixed noise covariance, the SIR of the LMMSE estimate of a unit power user increases when the received power profile becomes "more spread out" (all users received at the same power is "least spread") while keeping the average received power constant. In the context of specific design of signature sequences, we saw in Section III-C that performance is better when the noise gets more colored keeping the average noise power constant. The optimal signature sequences utilized the particular structure of the noise covariance and we attributed the gain in performance to this. However, in the current context, the signature sequences are chosen randomly and independent of the noise covariance and still the performance improves when the noise gets more colored. The reason is that even though the signature sequences were chosen independently, the LMMSE receiver uses the information about the color of the noise to obtain the estimate of the symbol transmitted and thus the SIR improves.

Having characterized the performance of the LMMSE receiver, we now turn to the dual question of *user capacity*. Given a target requirement β, under what conditions on the loading α will there exist positive power allocations to the users and random signature sequences such that the SIR of the LMMSE estimate is at least equal to the target β? Tse and Hanly [13, Sec. 5] show that, in the case of white additive noise (with variance σ^2), the requirement is $\alpha \frac{\beta}{1+\beta} < 1$ and the minimum received power required is

$$\frac{\sigma^2 \beta}{1-\alpha \frac{\beta}{1+\beta}}.$$

Colored noise does not change the admissibility requirement $\alpha \frac{\beta}{1+\beta} < 1$ since there is no upper limit on the power allocated. However, the minimum power that needs to be allocated is now a function of the colored noise covariance. We characterize this quantity below.

Proposition 3.7: Fix β such that $\alpha \frac{\beta}{1+\beta} < 1$ and let $P_{\min}(\beta, F_\sigma)$ denote the minimum received power of every user such that with random signature sequences the SIR of the LMMSE estimate is at least β. Then, $P_{\min}(\beta, F_\sigma)$ is the unique positive solution to the equation (for p)

$$\int \frac{dF_\sigma(z)}{\frac{z}{p} + \frac{\alpha}{1+\beta}} = \beta. \tag{24}$$

Furthermore, $P_{\min}(\beta, F_\sigma)$ is dilatory-concave in F_σ.

The proof is in Appendix B.7. Our result shows that the minimum power (for fixed target SIR of β) required decreases as the noise becomes more colored.

F. Discussion

In this subsection, we have characterized the user capacity region—the tuples of SIRs of the users that can be jointly attained—when using the LMMSE receiver. This problem was addressed in both the case when the signature sequences are chosen optimally and in the case when they are randomly chosen. Our main result is a complete characterization of this region and this allowed us to derive some qualitative properties of the user capacity region. In particular, we showed that the minimum average transmit power required to attain a given set of SIRs is a saddle function: it is convex in the tuple of SIRs and concave in the eigenvalues of the covariance matrix of the colored noise. One context in which to place our results in this section is as an extension of the results of [17] and [13] to the case of colored additive noise.

In our study of the user capacity region, we have focused on the constraint of sum of allocated powers of the users. A more general formulation is to address the *dual* problem: given a tuple of SIR requirements, characterize the region of *admissible* power allocations of the users. A tuple of power allocations to the users is admissible if there exists a choice of signature sequences such that with these powers and signature sequences the SIR of the LMMSE estimate is at least equal to the requirement. We leave this characterization as an interesting open problem.

We studied the impact of optimal signature sequence design in the presence of colored noise for two different types of receiver structures. Though the two receiver structures are very different, the mathematical techniques used to analyze both the scenarios are very similar. In particular, we found the partial order of majorization a very appropriate mathematical tool for both the problems. Further, the signature sequence design problem was posed as an optimization problem in both scenarios where we were minimizing a convex function with majorization constraints. We showed that there exist combinatorial algorithms, such as those in $\mathcal{A}$ and $\mathcal{B}$, which solve such optimization problems. This could be of independent interest in the optimization literature.

We motivated the setting of colored additive noise as interference from mobiles communicating with other base stations. It is interesting to study the behavior of distributed signature sequence adaptation, independently by each base station. The convergence properties of such asynchronous distributed adaptation will, in general, depend on the propagation model across base stations. A study of this dependence and the corresponding convergence properties is a natural step motivated by this paper and the qualitative properties derived from the saddle function property of the optimal capacity.

APPENDIX A
PROOFS FROM SECTION II

A.1 Proof of Lemma 2.2

This result follows directly from [6, Lemma 9.G.4], which says the following. For any positive definite G and H

$$\det(G+H) \le \Pi_{i=1}^{n} \left(\lambda_{[i]}(G) + \lambda_{[n+1-i]}(H)\right). \tag{25}$$

We substitute SDS^t for G and Σ for H and define $\tilde{S} \stackrel{\text{def}}{=} QS$ where Q is any orthonormal matrix such that $QSDS^tQ^t$ and Σ commute and, furthermore, has the following property. The eigenvalues of $QSDS^tQ^t + \Sigma$ are given by $\lambda_{[i]}(SDS^t) + \sigma_i^2$, $i = 1 \ldots N$. The claim now directly follows from (25).

A.2 Proof of Lemma 2.3

Recall Lemma 2.1 which states that for any symmetric matrix, the *precise* relationship between the diagonal elements and the eigenvalues is that of majorization. Thus, if $(\lambda_1, \ldots, \lambda_N, 0, \ldots, 0)$ majorizes $(p_1, \ldots, p_K)$, then there exists a symmetric matrix H with eigenvalues $\lambda_1, \ldots, \lambda_N$, $0, \ldots, 0$ and diagonal elements $p_1, \ldots, p_K$. Let $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_N \in \mathbb{R}^K$ be the normalized eigenvectors of H corresponding to the eigenvalues $\lambda_1, \ldots, \lambda_N$. Let $V^t = [\boldsymbol{v}_1 \boldsymbol{v}_2 \ldots \boldsymbol{v}_N]$. If we let Λ be the diagonal matrix with entries $\lambda_1, \ldots, \lambda_N$, then $H = V^t \Lambda V$. Now define $S = \Lambda^{\frac{1}{2}} V D^{-\frac{1}{2}}$. Then, we verify that $S^t S = D^{-\frac{1}{2}} H D^{-\frac{1}{2}}$ has unit diagonal entries, concluding that $S \in \mathcal{S}$. Further, $SDS^t = \Lambda^{\frac{1}{2}} V V^t \Lambda^{\frac{1}{2}}$ has eigenvalues $(\lambda_1, \ldots, \lambda_N)$. Now, defining $\tilde{S} = QS$ where Q is a orthonormal matrix chosen such that $QSDS^tQ^t$ commutes with Σ we have found $\tilde{S} \in \tilde{\mathcal{S}}$ with eigenvalues in $\mathcal{L}_1'$. Conversely, if $S \in \mathcal{S}$ is such that SDS^t has eigenvalues $(\lambda_1, \ldots, \lambda_N)$ then $D^{\frac{1}{2}}S^tSD^{\frac{1}{2}}$ has eigenvalues $(\lambda_1, \ldots, \lambda_N, 0, \ldots, 0)$. But it also has diagonal entries $(p_1, \ldots, p_K)$ and so, by Lemma 2.1, we must have $(\lambda_1, \ldots, \lambda_N, 0, \ldots, 0)$ majorizing $(p_1, \ldots, p_K)$.

A.3 Proof of Lemma 2.4

If $\lambda_i = \lambda_{[i]}$ for all $i = 1 \cdots N$, then the assertion is trivial. Suppose there exits at least one pair of indexes (i, j) such that $i > j$, and $\lambda_i > \lambda_j$. Define the vector $\tilde{\lambda}$ that differs from λ only in the components indexed by i and j as $\tilde{\lambda}_i = \lambda_j$ and $\tilde{\lambda}_j = \lambda_i$. It is seen that

$$\tilde{\lambda}_i + \sigma_i^2 = \alpha(\lambda_i + \sigma_i^2) + (1-\alpha)(\lambda_j + \sigma_j^2)$$

and

$$\tilde{\lambda}_j + \sigma_j^2 = (1-\alpha)(\lambda_i + \sigma_i^2) + \alpha(\lambda_j + \sigma_j^2)$$

where

$$\alpha = \frac{\sigma_i^2 - \sigma_j^2}{\lambda_i + \sigma_i^2 - \lambda_j - \sigma_j^2} \in [0, 1).$$

By definition of majorization, it follows that

$$(\lambda_1 + \sigma_1^2, \ldots, \lambda_N + \sigma_N^2) \quad \text{majorizes } \left(\tilde{\lambda}_1 + \sigma_1^2, \ldots, \tilde{\lambda}_N + \sigma_N^2\right).$$

By repeatedly interchanging every pair (i, j) with the property that $i > j$ and $\lambda_i > \lambda_j$ and using the associative property of majorization, the proof is complete. $\square$

A.4 Proof of Lemma 2.5

Fix $\mu \in \mathcal{L}$. We first show that there exists $S \in \tilde{\mathcal{S}}$ such that $\mu(S) = \mu$. Define $\lambda = (\lambda_1, \ldots, \lambda_N)$ by $\lambda_i \stackrel{\text{def}}{=} \mu_i - \sigma_i^2$. By definition of $\mu \in \mathcal{L}$ we see that $\lambda \in \mathcal{L}_1'$. An appeal to Lemma 2.3 confirms the existence of the $S \in \tilde{\mathcal{S}}$ with the property that $\mu(S) = \mu$.

Fix $\tilde{S} \in \tilde{\mathcal{S}}$. Thus $\mu(\tilde{S})$ can be written as

$$\lambda(\tilde{S}D\tilde{S}^t) + (\sigma_1^2, \ldots, \sigma_N^2)$$

for a particular ordering of $\lambda(\tilde{S}D\tilde{S}^t)$. Appealing to Lemma 2.4, the vector $\tilde{\mu} \stackrel{\text{def}}{=} (\tilde{\mu}_1, \ldots, \tilde{\mu}_N)$ defined by $\tilde{\mu}_i = \lambda_{[i]}(\tilde{S}D\tilde{S}^t) + \sigma_i^2$ is majorized by $\mu(\tilde{S})$. Appealing to Lemma 2.3, we verify that $\tilde{\mu} \in \mathcal{L}$. This completes the proof. $\square$

A.5 Proof of Theorem 2.6

Consider the following optimization problem:

$$\max_{(\mu_1, \ldots, \mu_N) \in \mathcal{L}} \left\{ H_g^{(N)}(\mu_1, \ldots, \mu_N) \stackrel{\text{def}}{=} \frac{1}{2N} \sum_{i=1}^{N} g\left(\frac{\mu_i}{\sigma_i^2}\right) \right\} \tag{26}$$

and denote it by

$$\mathcal{P}_g = (g, K, N, (p_1, \ldots, p_K), (\sigma_1^2, \ldots, \sigma_N^2)).$$

Here, g is any real continuous, increasing, strictly concave function. We show below that the output of Algorithm $\mathcal{A}$ achieves the maximum in (26) for every real, continuous, increasing, strictly concave function g. Appealing to [6, Proposition 4.B.2], and observing that the sums of the components of every vector in $\mathcal{L}$ are equal, we conclude that the output of Algorithm $\mathcal{A}$ is majorized by every element of $\mathcal{L}$ and is thus the Schur-minimal element of $\mathcal{L}$.

We begin with some preliminary observations about Algorithm $\mathcal{A}$.

1) If

$$\frac{\sum_{i=1}^{K} p_i + \sum_{i=1}^{N} \sigma_i^2}{N} \geq \max\left\{ \sigma_N^2, \ \frac{\sum_{i=1}^{l}(p_{[i]} + \sigma_i^2)}{l}, \ l = 1 \ldots N-1 \right\}$$

then Algorithm $\mathcal{A}$ output μ^* has all equal components. Hence, we have that μ^* is majorized by μ for any $\mu \in \mathcal{L}$ (see Example 2.1). This will complete the claim that μ^* is indeed the optimizing argument. We henceforth assume that this case does not occur.

2) We claimed in Section II-D that the updates of μ^* by algorithm $\mathcal{A}$ are in nonincreasing order without a proof. We develop some notation and give a formal proof of this statement. In Algorithm $\mathcal{A}$, the termination condition is $i > j$ and since either i is incremented (at least by 1) or j is decremented by 1 at every iteration, the algorithm has to stop in $n \leq N$ iterations. Denote the pairs (i, j) as the algorithm runs through the n iterations by $(i_1, j_1), \ldots, (i_n, j_n)$ and the value of η in Step 3 by $\eta_1, \ldots, \eta_{n-1}$. Observe that the algorithm always terminates in Step 2 (and, by definition, terminates at the nth iteration). Let us define $\eta_n \stackrel{\text{def}}{=} 0$. It suffices to show that

$$\eta_1 \geq \eta_2 \cdots \geq \eta_{n-1} > 0.$$

Fix $1 \leq l \leq n-2$. In the lth iteration, either j_l gets decremented or i_l gets incremented and we consider each case separately.

a) *Case 1*: $j_{l+1} = j_l - 1$. By hypothesis, $\eta_l = \sigma_{j_l}^2$ and

$$\eta_l \geq \sigma_{j_l - 1}^2 = \sigma_{j_{l+1}}^2 \tag{27}$$

$$\geq \frac{\sum_{k=i_l}^{K} p_{[k]} + \sum_{m=i_l}^{j_l} \sigma_m^2}{j_l - i_l + 1} \geq \frac{\sum_{k=i_l}^{K} p_{[k]} + \sum_{m=i_l}^{j_l - 1} \sigma_m^2}{j_l - i_l} \tag{28}$$

$$\geq \frac{\sum_{k=i_l}^{t}(p_{[k]} + \sigma_k^2)}{t - i_l + 1}, \qquad i_l \leq t < j_l. \tag{29}$$

We used the fact that

$$\sigma_{j_l}^2 \geq \frac{\sum_{k=i_l}^{K} p_{[k]} + \sum_{m=i_l}^{j_l} \sigma_m^2}{j_l - i_l + 1}$$

in the derivation of (28). Combining (27)–(29) we have shown that $\eta_l \geq \eta_{l+1}$.

b) *Case 2*: $i_{l+1} > i_l$. By hypothesis we have

$$\eta_l = \frac{\sum_{k=i_l}^{i_{l+1}-1}\left(p_{[k]} + \sigma_k^2\right)}{i_{l+1} - i_l}. \tag{30}$$

By hypothesis, we have

$$\eta_l \geq \sigma_{j_l}^2 = \sigma_{j_{l+1}}^2. \tag{31}$$

We also have for every $i_{l+1} \leq t < j_l$, by substituting for η_l from (30)

$$\eta_l \geq \frac{\sum_{k=i_l}^{t} \left(p_{[k]} + \sigma_k^2\right)}{t - i_l + 1}$$

$$(t - i_l + 1)\left(\sum_{k=i_l}^{i_{l+1}-1} \left(p_{[k]} + \sigma_k^2\right)\right) \geq (i_{l+1} - i_l)\left(\sum_{k=i_l}^{t} \left(p_{[k]} + \sigma_k^2\right)\right).$$

Rearranging the terms above leads us to conclude that

$$\eta_l \geq \frac{\sum_{k=i_{l+1}}^{t} \left(p_{[k]} + \sigma_k^2\right)}{t - i_{l+1} + 1}. \tag{32}$$

We also have

$$\eta_l \geq \frac{\sum_{k=i_l}^{K} p_{[k]} + \sum_{m=i_l}^{j_l} \sigma_m^2}{j_l - i_l + 1}$$

$$(j_l - i_l + 1)\left(\sum_{k=i_l}^{i_{l+1}-1} \left(p_{[k]} + \sigma_k^2\right)\right) \geq (i_{l+1} - i_l)\left(\sum_{k=i_l}^{K} p_{[k]} + \sum_{m=i_l}^{j_l} \sigma_m^2\right).$$

Rearrangement of the terms above leads us to conclude that

$$\eta_l \geq \frac{\sum_{k=i_{l+1}}^{K} p_{[k]} + \sum_{m=i_{l+1}}^{j_l} \sigma_m^2}{j_l - i_{l+1} + 1}. \tag{33}$$

Combining (31)–(33) we have shown that $\eta_l \geq \eta_{l+1}$.

3) For any $\mu \in \mathcal{L}$ we have $\mu_{[1]}^* \leq \mu_{[1]}$. This observation follows from the fact that $\mu_{[1]}^*$ is updated in the first step of $\mathcal{A}$ and η in Step 3 of $\mathcal{A}$ is always less than or equal to $\mu_{[1]}$ for every $\mu \in \mathcal{L}$.

Our proof that the output of Algorithm $\mathcal{A}$ is optimal is by induction. First consider the case $N = 2$ and arbitrary $K \geq 2$. Since for every $\mu \in \mathcal{L}$ we have $\mu_{[1]}^* \leq \mu_{[1]}$ and $\mu_1^* + \mu_2^* = \mu_1 + \mu_2$, we conclude that μ^* is majorized by μ and thus $H_g^{(N)}(\mu^*) \geq H_g^{(N)}(\mu)$. This completes the proof. We now make the induction hypothesis that the output of $\mathcal{A}$ is optimal for all $N \leq n$ and all $K \geq N$. We show that the output of $\mathcal{A}$ is optimal for $N = n+1$ and any $K \geq n+1$. Suppose $\mu \in \mathcal{L}$ is the optimal argument to the optimization problem in (26) and the output μ^* of $\mathcal{A}$ is such that

$$(\mu_{[1]}^*, \ldots, \mu_{[N]}^*) \neq (\mu_{[1]}, \ldots, \mu_{[N]}).$$

We now proceed to get a contradiction to the hypothesis that μ is the optimal solution to (26).

1) Suppose $\mu_{[1]} = \mu_{[1]}^*$. By the earlier observation that the updates of μ^* by Algorithm $\mathcal{A}$ are in nonincreasing order, we see that

$$\mu_{[1]}^* = \max\left\{\sigma_N^2, \frac{\sum_{k=1}^{K} p_{[k]} + \sum_{m=1}^{N} \sigma_m^2}{N}, \frac{\sum_{k=1}^{l} \left(p_{[k]} + \sigma_k^2\right)}{l}, l = 1 \ldots, N-1\right\}.$$

a) If $\mu_{[1]}^* = \sigma_N^2$, then $(\mu_{[2]}^*, \ldots, \mu_{[N]}^*)$ is the output of $\mathcal{A}$ with parameters K, $N-1$, $(p_1, \ldots, p_K)$, $(\sigma_1^2, \ldots, \sigma_{N-1}^2)$. By hypothesis, $\mu_{[1]} = \mu_{[1]}^* = \sigma_N^2$ and thus $(\mu_1, \ldots, \mu_{N-1}) \in \mathcal{L}$ for these parameters. By the induction hypothesis, we have

$$H_g^{(N-1)}(\mu_{[2]}^*, \ldots, \mu_{[N]}^*) \geq H_g^{(N-1)}(\mu_1, \ldots, \mu_{N-1}).$$

Since $\mu_{[j]} \neq \mu_{[j]}^*$ for some $j \in \{2, \ldots, N\}$ we have by the strict concavity of $H_g^{(N)}$ that

$$H_g^{(N)}(\mu^*) > H_g^{(N)}(\mu).$$

Thus, we arrive at a contradiction to the hypothesis that μ is the optimal argument in (26) completing the proof.

b) If

$$\mu_{[1]}^* = \frac{\sum_{i=1}^{l} (p_{[i]} + \sigma_i^2)}{l}$$

for some $l \in \{1, \ldots, N-1\}$ then from $\mathcal{A}$ we have $\mu_{[1]}^* = \mu_1^* = \cdots = \mu_l^*$. Using the fact that $\mu \in \mathcal{L}$ we arrive at $\mu_{[1]}^* = \mu_{[1]} = \mu_1 = \cdots = \mu_l$. Thus, $(\mu_{l+1}^*, \ldots, \mu_N^*)$ and $(\mu_{l+1}, \ldots, \mu_N)$ belong to $\mathcal{L}$ with parameters $K-l$, $N-l$, $(p_{l+1}, \ldots, p_K)$, $(\sigma_{l+1}^2, \ldots, \sigma_N^2)$. By the induction hypothesis, $(\mu_{l+1}^*, \ldots, \mu_N^*)$ is the optimal argument of $H_g^{(N-l)}$ in $\mathcal{L}$ with these reduced number of parameters and hence

$$H_g^{(N-l)}(\mu_{l+1}^*, \ldots, \mu_N^*) > H_g^{(N-l)}(\mu_{l+1}, \ldots, \mu_N)$$

contradicts the hypothesis that μ is the optimal argument of $H_g^{(N)}$ in $\mathcal{L}$ with parameters K, N, $(p_1, \ldots, p_K)$, $(\sigma_1^2, \ldots, \sigma_N^2)$. This completes the proof.

c) As observed earlier, we do not need to consider the case when

$$\mu_{[1]}^* = \frac{\sum_{i=1}^{K} p_i + \sum_{i=1}^{N} \sigma_i^2}{N}$$

since in this case μ^* is the optimal argument.

2) Henceforth, we take $\mu_{[1]} > \mu_{[1]}^*$. Let $1 \leq j \leq N$ be the largest index such that $\mu_{[1]} = \mu_j$. Observe that we have

$$\mu_j = \mu_{[1]} > \mu_{[1]}^* \geq \sigma_j^2. \tag{34}$$

a) Suppose $j = N$. We have

$$\mu_N = \mu_{[1]} > \mu^*_{[1]} \geq \frac{\sum_{i=1}^{K} p_i + \sum_{i=1}^{N} \sigma_i^2}{N}$$

and thus there exists $1 \leq l < N$ such that $\mu_l < \mu_N$. We can now define a vector $\tilde{\mu}$ differing from μ only in components indexed by N and l as follows:

$$\tilde{\mu}_N = \mu_N - \epsilon, \qquad \tilde{\mu}_l = \mu_l + \epsilon$$

where

$$\epsilon = \min\left\{ \frac{\mu_N - \sigma_N^2}{2}, \frac{\mu_N - \mu_l}{2} \right\}.$$

Using (34) we observe that $\epsilon > 0$. It is clear that $\tilde{\mu} \in \mathcal{L}(\mathcal{P})$ and that $\tilde{\mu}$ is majorized by μ. Thus, $H_g^{(N)}(\tilde{\mu}) < H_g^{(N)}(\mu)$ and we arrive at a contradiction to our hypothesis that μ was optimal on $\mathcal{L}(\mathcal{P})$.

b) Suppose $j \neq N$. By definition we have $\mu_{j+1} < \mu_j$ and by hypothesis that

$$j\mu_j = j\mu_{[1]} > j\mu^*_{[1]} \geq \sum_{i=1}^{j} (p_{[i]} + \sigma_i^2).$$

We define a vector $\tilde{\mu}$ (strictly) differing from μ only in components j and $j+1$ by

$$\tilde{\mu}_j = \mu_j - \epsilon, \qquad \tilde{\mu}_{j+1} = \mu_{j+1} + \epsilon,$$

where

$$\epsilon = \min\left\{ \frac{\mu_j - \mu_{j+1}}{2}, \frac{\mu_j - \sigma_j^2}{2}, \frac{\mu_j - \sum_{i=1}^{j}(p_{[i]} + \sigma_i^2)}{2} \right\} > 0.$$

Using (34), observe that $\epsilon > 0$. It is clear that $\tilde{\mu} \in \mathcal{L}(\mathcal{P})$ and that $\tilde{\mu}$ is majorized by μ and we arrive at a contradiction as before.

This exhausts all the cases and completes the proof of Theorem 2.6. $\square$

A.6 Proof of Propositions 2.1 and 2.2

Fix Σ and consider $\tilde{D}$ and D such that $(\tilde{p}_1, \ldots, \tilde{p}_K)$ majorizes $(p_1, \ldots, p_K)$. Using the transitivity of the partial order of majorization we have

$$\mathcal{L}_1'\left(K, N, \tilde{D}\right) \subseteq \mathcal{L}_1'(K, N, D). \tag{35}$$

Now, observe the following one to one relationship between $\mathcal{L}_1'(K, N, D)$ and $\mathcal{L}(K, N, D, \Sigma)$: For every $\lambda \in \mathcal{L}_1'(K, N, D)$, we have $\mu \in \mathcal{L}(K, N, D, \Sigma)$ where $\mu_i = \lambda_i + \sigma_i^2$. Conversely, for every $\mu \in \mathcal{L}(K, N, D, \Sigma)$, we have $\lambda \in \mathcal{L}_1'(K, N, D)$ where $\lambda_i = \mu - \sigma_i^2$. This allows us to conclude from (35) that

$$\mathcal{L}(K, N, \tilde{D}, \Sigma) \subseteq \mathcal{L}(K, N, D, \Sigma)$$

and thus that C_{opt} is Schur-concave in D for fixed Σ. To see concavity, fix D_1 and D_2. From (8), (9), and Lemma 2.4, we can write for $j = 1, 2$

$$C_{\text{opt}}(D_j, \Sigma) = \frac{1}{2N} \max_{\lambda \in \mathcal{L}'(K,N,D_j)} \sum_{i=1}^{N} \log\left(1 + \frac{\lambda_i}{\sigma_i^2}\right) \tag{36}$$

where

$$\mathcal{L}'(K, N, D) \stackrel{\text{def}}{=} \mathcal{L}_1'(K, N, D) \bigcap \{\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_N \geq 0\}.$$

Observe now that if $\lambda^{(j)} \in \mathcal{L}'(K, N, D_j)$ for $j = 1, 2$, then for every $\alpha \in (0, 1)$

$$\alpha\lambda^{(1)} + (1-\alpha)\lambda^{(2)} \in \mathcal{L}'(K, N, \alpha D_1 + (1-\alpha)D_2). \tag{37}$$

Using the concavity of the logarithm, we have for every $\alpha \in (0, 1)$, from (36)

$$\begin{aligned}
&\alpha C_{\text{opt}}(D_1, \Sigma) + (1-\alpha)C_{\text{opt}}(D_2, \Sigma)\\
&\quad\leq \max_{\{\lambda^{(j)} \in \mathcal{L}'(K,N,D_j),\, j=1,2\}} \frac{1}{2N}\\
&\qquad\cdot \sum_{i=1}^{N} \log\left(1 + \frac{\alpha\lambda_i^{(1)} + (1-\alpha)\lambda_i^{(2)}}{\sigma_i^2}\right)\\
&\quad\leq \max_{\{\lambda \in \mathcal{L}'(K,\, N,\, \alpha D_1 + (1-\alpha)D_2)\}} \sum_{i=1}^{N} \log\left(1 + \frac{\lambda_i}{\sigma_i^2}\right)\\
&\quad= C_{\text{opt}}(\alpha D_1 + (1-\alpha)D_2, \Sigma)
\end{aligned}$$

where we used (37) in the second step. This shows Proposition 2.2. Now fix D, Σ and $\tilde{\Sigma}$. Here Σ and $\tilde{\Sigma}$ are such that the vector of their eigenvalues (arranged in nondecreasing order)

$$\text{eig}(\Sigma) \stackrel{\text{def}}{=} (\sigma_1^2, \ldots, \sigma_N^2) \text{ majorizes } \text{eig}(\tilde{\Sigma}) \stackrel{\text{def}}{=} (\tilde{\sigma}_1^2, \ldots, \tilde{\sigma}_N^2).$$

We will show that

$$\begin{aligned}
C_{\text{opt}}(D, \Sigma) &= C_{\text{opt}}(D, \text{eig}(\Sigma)) \geq C_{\text{opt}}\left(D, \text{eig}\left(\tilde{\Sigma}\right)\right)\\
&= C_{\text{opt}}\left(D, \tilde{\Sigma}\right).
\end{aligned}$$

Recall the characterization of C_{opt} as in (36). Using the convexity of the map $x \mapsto \log\left(1 + \frac{a}{x}\right)$ with $a > 0$, for every fixed $\lambda \in \mathcal{L}'(K, N, D)$, we have that the function

$$(\sigma_1^2, \ldots, \sigma_N^2) \mapsto \frac{1}{2N} \sum_{i=1}^{N} \log\left(1 + \frac{\lambda_i}{\sigma_i^2}\right)$$

is convex. Since a pointwise supremum of convex functions is also convex we have shown that C_{opt} is convex in $\text{eig}(\Sigma)$. Since C_{opt} is also symmetric in $\text{eig}(\Sigma)$ (that is, it is invariant to permutations of the elements of $\text{eig}(\Sigma)$), we have shown that C_{opt} is Schur-convex in $\text{eig}(\Sigma)$. In our notation, we have shown that $C_{\text{opt}}(D, \Sigma) \geq C_{\text{opt}}(D, \tilde{\Sigma})$.

To see the convexity of C_{opt} in the covariance matrix Σ, fix any two noise covariance matrices Σ and $\tilde{\Sigma}$. We use the following result [6, Theorem 9.G.1].

Lemma A.1: For any two symmetric matrices A and B, with vectors of eigenvalues $\lambda(A)$ and $\lambda(B)$, respectively,

$$(\lambda_1(A+B), \ldots, \lambda_n A + B)^t \text{ is majorized by } \left(\lambda_{[1]}(A) + \lambda_{[2]}(B), \ldots, \lambda_{[n]}(A) + \lambda_{[n]}(B)\right)^t.$$

Continuing from (36) we have, using the convexity of the map $x \mapsto \log(1 + \frac{a}{x})$; $a > 0$

$$\begin{aligned}
&\alpha C_{\text{opt}}(D, \Sigma) + (1-\alpha) C_{\text{opt}}\left(D, \tilde{\Sigma}\right) \\
&\quad \geq \frac{1}{2N} \max_{\lambda \in \mathcal{L}'(K,N,D)} \sum_{i=1}^{N} \log\left(1 + \frac{\lambda_i}{\alpha\sigma_i^2 + (1-\alpha)\tilde{\sigma}_i^2}\right) \\
&\quad = C_{\text{opt}}\left(D, \alpha \text{eig}(\Sigma) + (1-\alpha)\text{eig}\left(\tilde{\Sigma}\right)\right) \\
&\quad \geq C_{\text{opt}}\left(D, \text{eig}\left(\alpha\Sigma + (1-\alpha)\tilde{\Sigma}\right)\right) \\
&\quad = C_{\text{opt}}\left(D, \alpha\Sigma + (1-\alpha)\tilde{\Sigma}\right)
\end{aligned}$$

where we used Lemma A.1 in conjunction with the earlier proof of the Schur-convexity of C_{opt} for fixed D in arriving at the last but one step. This shows the convexity of C_{opt} in Σ and completes the proof of Proposition 2.1. □

Appendix B
Proofs From Section III

B.1 Proof of Lemma 3.2

We begin with some relabeling. For any D and S, let a singular value decomposition of $SD^{\frac{1}{2}}$ be $U\Lambda^{\frac{1}{2}}V^t$ where U is an $N \times N$ orthonormal matrix, Λ is an $N \times N$ diagonal matrix with diagonal entries equal to the eigenvalues of SDS^t, and V^t is an $N \times K$ matrix with orthonormal rows. Then, the problem $\tilde{\mathcal{P}}$ can be written as follows.

$$\text{Maximize } \operatorname{tr}\left[V^t W V \Lambda^{\frac{1}{2}} (\Lambda + U^t \Sigma U)^{-1} \Lambda^{\frac{1}{2}}\right] \quad \text{subject to } \operatorname{tr}[\Lambda] \leq P.$$

Here U is any $N \times N$ orthonormal matrix and V^t is any $N \times K$ matrix with orthonormal rows. Suppose (U_o, V_o, Λ_o) is a solution to $\tilde{\mathcal{P}}$. Then we will show that

$$[U_o^t \Sigma U_o, \Lambda_o] = \mathbf{0} \tag{38}$$

and

$$[V_o^t W V_o, \Lambda_o] = \mathbf{0}. \tag{39}$$

We will show these two claims by a perturbation argument and this will complete the proof of the lemma.

Let H be an $N \times N$ skew-symmetric matrix (i.e., $H^t + H = \mathbf{0}$.) Then, $\exp(sH)$ is an orthonormal matrix, for any real s. By hypothesis, we have

$$\begin{aligned}
&\operatorname{tr}\left[V_o^t W V_o \Lambda_o^{\frac{1}{2}} (\Lambda_o + U_o^t \Sigma U_o)^{-1} \Lambda_o^{\frac{1}{2}}\right] \\
&\quad \geq \operatorname{tr}\left[V_o^t W V_o \Lambda_o^{\frac{1}{2}} \right. \\
&\qquad \left. \cdot (\Lambda_o + \exp(-sH) U_o^t \Sigma U_o \exp(sH))^{-1} \Lambda_o^{\frac{1}{2}}\right].
\end{aligned} \tag{40}$$

Now $\exp(sH) = I + sH + o(s)$ and thus

$$\Lambda_o + \exp(-sH) U_o^t \Sigma U_o \exp(sH) = A_o + B_s \tag{41}$$

where

$$A_o \stackrel{\text{def}}{=} U_o^t \Sigma U_o + \Lambda_o,$$

and

$$B_s \stackrel{\text{def}}{=} s(U_o^t \Sigma U_o H - H U_o^t \Sigma U_o) + o(s).$$

Now, by the matrix inversion lemma

$$(A_o + B_s)^{-1} = A_o^{-1} - A_o^{-1}(I + B_s A_o^{-1})^{-1} B_s A_o^{-1}. \tag{42}$$

Substituting (41) in (40) and using (42), we obtain

$$\operatorname{tr}\left[V_o^t W V_o \Lambda_o^{\frac{1}{2}} A_o^{-1} (I + B_s A_o^{-1})^{-1} B_s A_o^{-1} \Lambda_o^{\frac{1}{2}}\right] \geq 0. \tag{43}$$

Also, we have

$$B_s \to \mathbf{0}, \qquad \text{as } s \to 0 \tag{44}$$

$$\frac{B_s}{s} \to U_o^t \Sigma U_o H - H U_o^t \Sigma U_o, \qquad \text{as } s \to 0. \tag{45}$$

Dividing throughout by $s > 0$ in (43), and letting $s \downarrow 0$, we obtain (using (44) and (45))

$$\operatorname{tr}\left[V_o^t W V_o \Lambda_o^{\frac{1}{2}} A_o^{-1} (U_o^t \Sigma U_o H - H U_o^t \Sigma U_o) A_o^{-1} \Lambda_o^{\frac{1}{2}}\right] \geq 0. \tag{46}$$

Similarly, dividing by $s < 0$ in (43), and letting $s \uparrow 0$, we obtain

$$\operatorname{tr}\left[V_o^t W V_o \Lambda_o^{\frac{1}{2}} A_o^{-1} (U_o^t \Sigma U_o H - H U_o^t \Sigma U_o) A_o^{-1} \Lambda_o^{\frac{1}{2}}\right] \leq 0. \tag{47}$$

From (46) and (47), we conclude that for every skew-symmetric matrix H we have

$$\operatorname{tr}\left[V_o^t W V_o \Lambda_o^{\frac{1}{2}} A_o^{-1} (U_o^t \Sigma U_o H - H U_o^t \Sigma U_o) A_o^{-1} \Lambda_o^{\frac{1}{2}}\right] = 0$$

which can be rewritten as

$$\operatorname{tr}\left[H [A_o^{-1} \Lambda_o^{\frac{1}{2}} V_o^t W V_o \Lambda_o^{\frac{1}{2}} A_o^{-1}, U_o^t \Sigma U_o]\right] = 0 \tag{48}$$

where $[A, B] \stackrel{\text{def}}{=} AB - BA$ is the Lie bracket. Choosing

$$H = H_o \stackrel{\text{def}}{=} [A_o^{-1} \Lambda_o^{\frac{1}{2}} V_o^t W V_o \Lambda_o^{\frac{1}{2}} A_o^{-1}, U_o^t \Sigma U_o]$$

we conclude from (48) that $\operatorname{tr}[H_o^2] = 0$ and thus

$$H_o = \left[A_o^{-1} \Lambda_o^{\frac{1}{2}} V_o^t W V_o \Lambda_o^{\frac{1}{2}} A_o^{-1}, U_o^t \Sigma U_o\right] = \mathbf{0}. \tag{49}$$

We can perturb V_o to $V_o \exp(sH)$ and by analogous arguments, conclude that

$$\left[V_o^t W V_o, \Lambda_o^{\frac{1}{2}} A_o^{-1} \Lambda_o^{\frac{1}{2}}\right] = \mathbf{0}. \tag{50}$$

Let us define $C_o \stackrel{\text{def}}{=} \Sigma^{-\frac{1}{2}} U_o \Lambda_o^{\frac{1}{2}}$. Then

$$\begin{aligned}
\Lambda_o^{\frac{1}{2}} A_o^{-1} \Lambda_o^{\frac{1}{2}} &= C_o^t (I + C_o C_o^t)^{-1} C_o, \\
&= C_o^t (I - C_o (I + C_o^t C_o)^{-1} C_o^t) C_o, \\
&= C_o^t C_o - C_0^t C_o (I + C_o^t C_o)^{-1} C_o^t C_o.
\end{aligned} \tag{51}$$

From (50) and (51) we conclude that $V_o^t W V_o$ and $C_o^t C_o$ have a common set of eigenvectors and thus

$$[V_o^t W V_o, C_o^t C_o] = \mathbf{0}. \tag{52}$$

Now, continuing from (49), we have

$$\begin{aligned}
\left[U_o A_o^{-1} \Lambda_o^{\frac{1}{2}} V_o^t W V_o \Lambda_o^{\frac{1}{2}} A_o^{-1} U_o^t, \Sigma\right] &= \mathbf{0} \\
\left[\Sigma^{\frac{1}{2}} U_o A_o^{-1} \Lambda_o^{\frac{1}{2}} V_o^t W V_o \Lambda_o^{\frac{1}{2}} A_o^{-1} U_o^t \Sigma^{\frac{1}{2}}, \Sigma\right] &= \mathbf{0} \\
[(C_o C_o^t + I)^{-1} (C_o V_o^t W V_o C_o^t)(C_o C_o^t + I)^{-1}, \Sigma] &= \mathbf{0} \qquad (53) \\
[C_o C_o^t, \Sigma] &= \mathbf{0}. \qquad (54)
\end{aligned}$$

Here we used (52) and (53) in the derivation of (54): $C_o V_o^t W V_o C_o^t$ and $C_o C_o^t$ have a common set of eigenvectors.

Recalling the definition of C_o as $\Sigma^{-\frac{1}{2}}U_o\Lambda^{\frac{1}{2}}$, we see that (52) and (54) are the same as (38) and (39). This completes the proof. $\square$

B.2 Proof of Lemma 3.3

Define $\mathcal{F}_1$ to be the set of $(e_1, \ldots, e_K)$ such that $0 \le e_i < 1$, $\forall\, i = 1 \cdots K$ and such that $(\frac{e_1}{1-e_1}, \ldots, \frac{e_K}{1-e_K})$ SIRs are achievable with the following constraints on the allocation of signature sequences S and powers D.

1) The sum of powers meets the average sum constraint, i.e., $\mathrm{tr}[D] \le P$.
2) S and D satisfy the condition $[SDS^t, \Sigma] = \mathbf{0}$.

From Lemma 3.2, we know that

$$\text{convex hull of } \mathcal{F}_1 = \text{convex hull of } \mathcal{F}.$$

Since $\mathcal{F}_1 \subseteq \mathcal{F}$, it follows that $\mathcal{F}$ is convex if we can show that $\mathcal{F}_1$ itself is convex. The rest of this proof shows that $\mathcal{F}_1$ is indeed convex.

Consider a pair $\{\boldsymbol{e}^{(i)} \stackrel{\text{def}}{=} (e_1^{(i)}, \ldots e_K^{(i)}), i = 1, 2\}$. Then there exist a two pairs of allocations $\{(S_{(i)}, D_{(i)}), i = 1, 2\}$ such that $[S_{(i)}D_{(i)}S_{(i)}^t, \Sigma] = \mathbf{0}$ and $\mathrm{tr}[D_{(i)}] \le P$ for $i = 1, 2$ with the property that the SIRs achieved by allocation $(S_{(i)}, D_{(i)})$ are

$$\left(\frac{e_1^{(i)}}{1-e_1^{(i)}}, \ldots, \frac{e_K^{(i)}}{1-e_K^{(i)}}\right), \qquad \text{for } i = 1, 2.$$

There exists a singular value decomposition $S_{(i)}D_{(i)}^{\frac{1}{2}}U_{(i)}\Lambda_{(i)}^{\frac{1}{2}}V_{(i)}^t$ with the property that

$$U_{(i)}^t\Sigma U_{(i)} = \mathrm{diag}\{\sigma_1^2, \ldots, \sigma_N^2\}, \qquad \text{for } i = 1, 2. \tag{55}$$

From the expression for SIR for the LMMSE receiver (18), we also have, for $i = 1, 2$

$$V_{(i)}\mathrm{diag}\left\{\frac{\lambda_1^{(i)}}{\lambda_1^{(i)}+\sigma_1^2}, \ldots, \frac{\lambda_N^{(i)}}{\lambda_N^{(i)}+\sigma_N^2}\right\}V_{(i)}^t \quad \text{has diagonal entries } e_1^{(i)}, \ldots, e_K^{(i)}.$$

In arriving at this expression, we have used (55) and have written $\Lambda_{(i)} = \mathrm{diag}\{\lambda_1^{(i)}, \ldots, \lambda_N^{(i)}\}$ for $i = 1, 2$. Appealing to Lemma 2.1, we also have, for $i = 1, 2$

$$\left(\frac{\lambda_1^{(i)}}{\lambda_1^{(i)}+\sigma_1^2}, \ldots, \frac{\lambda_N^{(i)}}{\lambda_N^{(i)}+\sigma_N^2}, 0, \ldots, 0\right) \quad \text{majorizes } (e_1, \ldots, e_K). \tag{56}$$

Let us define, for $i = 1, 2$

$$y_j^{(i)} \stackrel{\text{def}}{=} \frac{\lambda_j^{(i)}}{\lambda_j^{(i)}+\sigma_j^2}, \qquad j = 1, \ldots, N \tag{57}$$

$$\tilde{\lambda}_j^{(i)} \stackrel{\text{def}}{=} \sigma_j^2\frac{y_{[j]}^{(i)}}{1-y_{[j]}^{(i)}}, \qquad j = 1, \ldots, N \tag{58}$$

where we used the notation of order statistics from Definition 2.2. Consider the following claims, for $i = 1, 2$:

$$\left(\frac{\tilde{\lambda}_1^{(i)}}{\tilde{\lambda}_1^{(i)}+\sigma_1^2}, \ldots, \frac{\tilde{\lambda}_N^{(i)}}{\tilde{\lambda}_N^{(i)}+\sigma_N^2}\right)$$

is a reordering of

$$\left(\frac{\lambda_1^{(i)}}{\lambda_1^{(i)}+\sigma_1^2}, \ldots, \frac{\lambda_N^{(i)}}{\lambda_N^{(i)}+\sigma_N^2}\right). \tag{59}$$

$$\frac{\tilde{\lambda}_1^{(i)}}{\tilde{\lambda}_1^{(i)}+\sigma_1^2} \ge \cdots \ge \frac{\tilde{\lambda}_N^{(i)}}{\tilde{\lambda}_N^{(i)}+\sigma_N^2}. \tag{60}$$

$$\sum_{j=1}^{N}\tilde{\lambda}_j^{(i)} \le P. \tag{61}$$

Equations (59) and (60) follow directly from (57) and (58). To see (61), it suffices to see that, for $i = 1, 2$

$$\begin{aligned} P &\ge \sum_{j=1}^{N}\lambda_j^{(i)} \\ &= \sum_{j=1}^{N}\sigma_j^2\frac{y_j^{(i)}}{1-y_j^{(i)}} \\ &\ge \sum_{j=1}^{N}\sigma_j^2\frac{y_{[j]}^{(i)}}{1-y_{[j]}^{(i)}} \\ &= \sum_{j=1}^{N}\tilde{\lambda}_j^{(i)}. \end{aligned}$$

The first step used the hypothesis that $\mathrm{tr}[\Lambda_{(i)}] \le P$, $i = 1, 2$, the second step follows from (57), the third step used the convexity of the map $x \mapsto \frac{x}{1-x}$, $x \in [0, 1)$, and the last step used (58).

Fix $\alpha \in [0, 1]$ and define

$$\lambda_j^{(3)} \stackrel{\text{def}}{=} \alpha\lambda_j^{(1)} + (1-\alpha)\lambda_j^{(2)}, \qquad j = 1, \ldots, N.$$

Define $\tilde{\lambda}_j^{(3)}$, $j = 1 \ldots N$ in terms of $\lambda_j^{(3)}$, $j = 1 \ldots, N$ as in (57) and (58). From (59), (60), and (56), it now follows that

$$\left(\frac{\tilde{\lambda}_1^{(3)}}{\tilde{\lambda}_1^{(3)}+\sigma_1^2}, \ldots, \frac{\tilde{\lambda}_N^{(3)}}{\tilde{\lambda}_N^{(3)}+\sigma_N^2}, 0, \ldots, 0\right) \text{ majorizes } \left(\alpha e_1^{(1)} + (1-\alpha)e_1^{(2)}, \ldots, \alpha e_K^{(1)} + (1-\alpha)e_K^{(2)}\right). \tag{62}$$

Appealing to Lemma 2.1, (62) allows us to conclude that there exists a $K \times N$ matrix $V_{(3)}$ with orthonormal columns such that

$$V_{(3)}\mathrm{diag}\left\{\frac{\tilde{\lambda}_1^{(3)}}{\tilde{\lambda}_1^{(3)}+\sigma_1^2}, \ldots, \frac{\tilde{\lambda}_N^{(3)}}{\tilde{\lambda}_N^{(3)}+\sigma_N^2}\right\}V_{(3)}^t \text{has diagonal entries } \alpha e_1^{(1)} + (1-\alpha)e_1^{(2)}, \ldots, \alpha e_K^{(1)} + (1-\alpha)e_K^{(2)}. \tag{63}$$

Define $U_{(3)} \stackrel{\text{def}}{=} U_{(1)}$ and the diagonal matrix

$$\Lambda_{(3)} \stackrel{\text{def}}{=} \mathrm{diag}\{\lambda_1^{(3)}, \ldots, \lambda_N^{(3)}\}.$$

Now consider the following allocation of signature sequences $S_{(3)}$ and powers $D_{(3)}$: $D_{(3)}$ is defined to be the diagonal matrix with diagonal entries equal to the diagonal entries of $V_{(3)}\Lambda_{(3)}V_{(3)}^t$ and

$$S_{(3)} \stackrel{\text{def}}{=} U_{(3)}\Lambda_{(3)}^{\frac{1}{2}}V_{(3)}^tD_{(3)}^{-\frac{1}{2}}.$$

We let the reader verify, using (63) and (61), the following.

1) The allocation pair $(S_{(3)}, D_{(3)})$ is valid, i.e., $S_{(3)} \in \mathcal{S}$ and $\mathrm{tr}[D_{(3)}] \le P$.

2) The allocation pair $(S_{(3)}, D_{(3)})$ achieves SIRs equal to

$$\left(\frac{\alpha e_1^{(1)} + (1-\alpha)e_1^{(2)}}{1-\alpha e_1^{(1)} - (1-\alpha)e_1^{(2)}}, \ldots, \frac{\alpha e_K^{(1)} + (1-\alpha)e_K^{(2)}}{1-\alpha e_K^{(1)} - (1-\alpha)e_{(2)}^{K}}\right).$$

A similar verification is done in [17, Sec. 5]. This shows that $\alpha \boldsymbol{e}^{(1)} + (1-\alpha)\boldsymbol{e}^{(2)} \in \mathcal{F}_1$ for $\alpha \in [0, 1]$. Since $\boldsymbol{e}^{(i)}$, $i = 1, 2$ are arbitrary points in $\mathcal{F}_1$, we have shown that $\mathcal{F}_1$ is convex thus completing the proof. □

B.3 Proof of Theorem 3.4

We begin with some relabeling. Writing $y_i = \frac{\lambda_i}{\lambda_i + \sigma_i^2}$, the optimization problem $\mathcal{P}$ in (22) becomes

$$\mathcal{P}\text{: Minimize } f(\boldsymbol{y}) \stackrel{\text{def}}{=} \sum_{i=1}^{N} \sigma_i^2 \frac{y_i}{1-y_i} \text{ subject to}$$
$$(y_1, \ldots, y_N, 0, \ldots, 0) \text{ majorizes } (e(\beta_1,), \ldots, e(\beta_K)). \qquad (64)$$

Since we have the ordering $\sigma_1^2 \le \cdots \le \sigma_N^2$, we get the simple inequality

$$f(y_1, \ldots, y_N) \ge f\left(y_{[1]}, \ldots, y_{[N]}\right).$$

Above, we used the definition and notation of order statistics from Definition 2.2. We thus conclude that the optimal $\boldsymbol{y}^*$ that solves the problem $\tilde{\mathcal{P}}$ above has the structure $y_1^* \ge \cdots \ge y_N^*$. This observation allows us (from the definition of majorization, Definition 2.1) to rewrite $\mathcal{P}$ as follows:

$$\mathcal{P}: \text{ Minimize } f(\boldsymbol{y}) \stackrel{\text{def}}{=} \sum_{i=1}^{N} \sigma_i^2 \frac{y_i}{1-y_i} \text{ subject to}$$
$$1 > y_1 \ge y_2 \ge \cdots \ge y_N \ge 0$$

and

$$\sum_{m=1}^{l} y_m \ge \sum_{m=1}^{l} e(\beta_l), \qquad l = 1, \ldots, N-1 \qquad (65)$$
$$\sum_{m=1}^{N} y_m = \sum_{m=1}^{K} e(\beta_l). \qquad (66)$$

We now rewrite the combinatorial algorithm $\mathcal{B}$ in this new notation as follows.

Algorithm $\mathcal{B}$:
Input $K, N, (\beta_1 \ge \cdots \ge \beta_K)$ and $\sigma_1^2 \le \sigma_2^2 \cdots \le \sigma_N^2$.
Output $\boldsymbol{y}^* = (y_1^*, \ldots, y_N^*)$
Update

1. Initialization: $i = 1$, $j = N$ and $y_k^* = 0, \ \forall k = 1 \ldots N$.
2. Termination: If $i > j$ stop and output the vector $\boldsymbol{y}^*$. Else, go to Step 3.
3. Let

$$\eta = \max\left\{\sigma_j, \frac{\sum_{m=i}^{j} \sigma_m}{j-i+1-\sum_{m=i}^{K} e(\beta_m)}, \frac{\sum_{m=i}^{t} \sigma_m}{t-i+1-\sum_{m=i}^{t} e(\beta_m)}, t = i \ldots j-1\right\}. \qquad (67)$$

a) If $\eta = \sigma_j$ then set $y_j^* := 1 - \frac{\sigma_j}{\eta} = 0$ and $j := j-1$. Go to Step 2.

b) If

$$\eta = \frac{\sum_{m=i}^{j} \sigma_m}{j-i+1-\sum_{m=i}^{K} e(\beta_m)},$$

then set $y_m^* := 1 - \frac{\sigma_m}{\eta}$, $\forall m = i \ldots, j$ and $i := N+1$. Go to Step 2.

c) If

$$\eta = \frac{\sum_{m=i}^{t} \sigma_m}{t-i+1-\sum_{m=i}^{l} e(\beta_m)},$$

for some $i \le t < j$ then set $y_m^* = 1 - \frac{\sigma_m}{\eta}$, $\forall m = i, \ldots, l$ and $i := t+1$. Go to Step 2.

We now show that the output $\boldsymbol{y}^*$ of the combinatorial algorithm $\mathcal{B}$ solves the problem $\mathcal{P}$.

Since the termination condition is $i > j$ and either i is incremented (at least by 1) or j is decremented by 1 at every iteration, the algorithm has to stop in $n \le N$ iterations. Denote the pairs (i, j) as the algorithm runs through the n iterations by $(i_1, j_1), \ldots, (i_n, j_n)$ and the value of η in Step 3 by $\eta_1, \ldots, \eta_{n-1}$. Observe that the algorithm always terminates in Step 2 (and, by definition, terminates at the nth iteration). Let us define $\eta_n \stackrel{\text{def}}{=} 0$. We begin with some simple observations.

1) $i_1 = 1$, $j_1 = N$, and

$$j_l \ge j_{l+1} \ge i_{l+1} \ge i_l, \qquad 1 \le l \le n-1. \qquad (68)$$

Furthermore, we claim that in iteration $(n-1)$ (this is the final iteration in which Step 3 is reached), the algorithm must have visited Step 3b). Since the condition for termination $i_n > j_n$ is satisfied, this rules out Step 3c) being visited in the $(n-1)$th iteration. Suppose Step 3a) was visited in the $(n-1)$th iteration. Then it must be the case that $i_{n-1} = j_{n-1}$ (since $i_n > j_n$). Thus, we have

$$\eta_{n-1} = \max\left\{\sigma_{j_{n-1}}, \frac{\sigma_{j_{n-1}}}{1-e(\beta_{j_{n-1}})}\right\} > \sigma_{j_{n-1}}.$$

Thus, a contradiction to the hypothesis that Step 3a) is reached in the $(n-1)$th iteration is derived. We conclude that Step 3b) must have been reached in the $(n-1)$th iteration and thus that $i_n = N+1$ and $j_{n-1} = j_n$.

2) For $1 \le l < n-1$

$$\eta_l = \begin{cases} \sigma_{j_l}, & j_l > j_{l+1} \\ \dfrac{\sum_{m=i_l}^{i_{l+1}-1} \sigma_m}{i_{l+1} - i_l - \sum_{m=i_l}^{i_{l+1}-1} e(\beta_m)}, & \text{else} \end{cases} \qquad (69)$$

and in the final iteration

$$\eta_{n-1} = \frac{\sum_{m=i_{n-1}}^{j_{n-1}} \sigma_m}{j_{n-1} - i_{n-1} + 1 - \sum_{m=i_{n-1}}^{K} e(\beta_m)}. \quad (70)$$

3) Our next observation is that the value of η decreases. Formally

$$\eta_1 \geq \eta_2 \geq \cdots \geq \eta_{n-1} > 0. \quad (71)$$

This observation is a bit more involved and we provide a detailed proof.

Proof: Fix $1 \leq l \leq n-2$. In the lth iteration either j_l gets decremented or i_l gets incremented and we consider each case separately.

a) *Case 1*: $j_{l+1} = j_l - 1$. By hypothesis $\eta_l = \sigma_{j_l}$ and

$$\eta_l \geq \sigma_{j_l - 1} = \sigma_{j_{l+1}} \quad (72)$$

$$\geq \frac{\sum_{m=i_l}^{j_l} \sigma_m}{j_l - i_l + 1 - \sum_{m=i_l}^{K} e(\beta_m)}$$

$$\geq \frac{\sum_{m=i_l}^{j_l - 1} \sigma_m}{j_l - i_l - \sum_{m=i_l}^{K} e(\beta_m)} \quad (73)$$

$$\geq \frac{\sum_{m=i}^{t} \sigma_m}{t - i + 1 - \sum_{m=i}^{t} e(\beta_m)}, \qquad \forall\, i_l \leq t \leq j_l - 1. \quad (74)$$

We used the fact that

$$\sigma_{j_l} \geq \frac{\sum_{m=i_l}^{j_l - 1} \sigma_m}{j_l - i_l + 1 - \sum_{m=i_l}^{K} e(\beta_m)}$$

in arriving at (73). Combining (72)–(74) and using the definition of η in (67) we have shown that $\eta_l \geq \eta_{l+1}$.

b) *Case 2*: $i_{l+1} > i_l$. In this case we have, from (69)

$$\eta_l = \frac{\sum_{m=i_l}^{i_{l+1}-1} \sigma_m}{i_{l+1} - i_l - \sum_{m=i_l}^{i_{l+1}-1} e(\beta_m)}.$$

By hypothesis, we have

$$\eta_l \geq \sigma_{j_l} = \sigma_{j_{l+1}}. \quad (75)$$

Also by hypothesis, for every $i_{l+1} \leq t \leq j_l - 1$

$$\eta_l \geq \frac{\sum_{m=i_l}^{t} \sigma_m}{t - i_l + 1 - \sum_{m=i_l}^{t} e(\beta_m)}$$

which implies

$$\left(t - i_{l+1} + 1 - \sum_{m=i_{l+1}}^{t} e(\beta_m)\right) \sum_{m=i_l}^{i_{l+1}-1} \sigma_m \geq \left(i_{l+1} - i_l - \sum_{m=i_l}^{i_{l+1}-1} e(\beta_m)\right) \sum_{m=i_{l+1}}^{t} \sigma_m$$

$$\eta_l \geq \frac{\sum_{m=i_{l+1}}^{t} \sigma_m}{\left(t - i_{l+1} + 1 - \sum_{m=i_{l+1}}^{t} e(\beta_m)\right)}. \quad (76)$$

Again by hypothesis, we have

$$\eta_l \geq \frac{\sum_{m=i_l}^{t} \sigma_m}{j_l - i_l + 1 - \sum_{m=i_l}^{K} e(\beta_m)}$$

which implies

$$\left(j_l - i_{l+1} + 1 - \sum_{m=i_{l+1}}^{K} e(\beta_m)\right) \sum_{m=i_l}^{i_{l+1}-1} \sigma_m \geq \left(i_{l+1} - i_l - \sum_{m=i_l}^{i_{l+1}-1} e(\beta_m)\right) \sum_{m=i_{l+1}}^{t} \sigma_m$$

$$\eta_l \geq \frac{\sum_{m=i_{l+1}}^{j_l} \sigma_m}{\left(j_l - i_{l+1} + 1 - \sum_{m=i_{l+1}}^{K} e(\beta_m)\right)}. \quad (77)$$

Combining (75)–(77) with the hypothesis of $i_{l+1} > i_l$, we have from the definition of η in (67) that $\eta_l \geq \eta_{l+1}$.

This completes the proof of the observation that $\eta_1 \geq \cdots \geq \eta_{n-1}$. □

4) We can also express the output $\boldsymbol{y}^*$ in terms of the pairs $(i_1, j_1), \ldots, (i_n, j_n)$. Fix $1 \leq k \leq N$. Define

$$l^{(k)} \overset{\text{def}}{=} \begin{cases} \{1 \leq t < n \colon i_t \leq k < i_{t+1}\}. & \text{if } k \leq j_n \\ \{1 \leq t < n-1 \colon j_t \geq k > j_{t+1}\}, & \text{if } k > j_n. \end{cases} \quad (78)$$

Observe that $l^{(k)}$ is well defined and has the interpretation that y_k^* is updated in iteration $l^{(k)}$.

- Suppose $k > j_n$. In this case $\boldsymbol{y}_k^* = 0$.
- Suppose $k \leq j_n$. In this case

$$y_k^* = 1 - \frac{\sigma_k}{\eta_{l^{(k)}}}. \quad (79)$$

We have $k \leq j_n \leq j_{l^{(k)}}$ and thus $\eta_{l^{(k)}} \geq \sigma_{j_{l^{(k)}}} \geq \sigma_k$. This shows that, for all $1 \leq k \leq N$

$$1 > y_k^* \geq 0. \quad (80)$$

It also follows from the definition of $l^{(k)}$ (in (78)) and (68) that whenever $\tilde{k} \leq k \leq j_n$, we have $l^{(k)} > l^{(\tilde{k})}$.

Combining this with the ordering $\sigma_k \geq \sigma_{\tilde{k}}$ and (71), we have, for all $1 \leq \tilde{k} \leq k \leq N$

$$y_k^* = 1 - \frac{\sigma_k}{\eta_{l^{(k)}}} \geq 1 - \frac{\sigma_{\tilde{k}}}{\eta_{l^{(\tilde{k})}}} = y_{\tilde{k}}^*. \tag{81}$$

5) We now show that $\boldsymbol{y}^*$ satisfies (65) and (66). Fix any $1 \leq m < N$ and consider two cases. First, suppose $k \leq j_n$. Then

$$\sum_{q=1}^{m} y_q^* = \sum_{q=i_{l^{(m)}}}^{m} y_q^* + \sum_{t=1}^{l^{(m)}-1} \sum_{q=i_t}^{i_{t+1}-1} y_q^* \tag{82}$$

$$= \sum_{q=i_{l^{(m)}}}^{m} \left(1 - \frac{\sigma_q}{\eta_{l^{(m)}}}\right) + \sum_{t=1}^{l^{(m)}-1} \sum_{q=i_t}^{i_{t+1}-1} \left(1 - \frac{\sigma_q}{\eta_t}\right) \tag{83}$$

$$= (m - i_{l^{(m)}} + 1) - \frac{\sum_{q=i_{l^{(m)}}}^{m} \sigma_q}{\eta_{l^{(m)}}} + \sum_{t=1}^{l^{(m)}-1} \left(i_{t+1} - i_t - \frac{\sum_{q=i_t}^{i_{t+1}-1} \sigma_q}{\eta_t} \right) \tag{84}$$

$$= (m - i_{l^{(m)}} + 1) - \frac{\sum_{q=i_{l^{(m)}}}^{m} \sigma_q}{\eta_{l^{(m)}}} + \sum_{t=1}^{l^{(m)}-1} \sum_{q=i_t}^{i_{t+1}-1} e(\beta_q) \tag{85}$$

$$\geq \sum_{q=1}^{m} e(\beta_m). \tag{86}$$

Here, we used the definition of $l^{(m)}$ as the iteration number in arriving at (82) while (83) used the update in Step 3c) of Algorithm $\mathcal{B}$. Equations (84) and (85) follow from (69) and (70). In (86), we used the property of $\eta_{l^{(m)}}$ that

$$\eta_{l^{(m)}} \geq \frac{\sum_{q=i_{l^{(m)}}}^{t} \sigma_q}{t - i_{l^{(m)}} + 1 - \sum_{q=i_{l^{(m)}}}^{t} e(\beta_q)}, \qquad \forall\, i_{l^{(m)}} \leq t \leq i_{l^{(m)}+1} - 1$$

with equality when $t = i_{l^{(m)}+1} - 1$. In the derivation of (86), we also used the fact that

$$\sum_{q=i_{l^{(m)}}}^{m} e(\beta_q) + \sum_{t=1}^{l^{(m)}-1} \sum_{q=i_t}^{i_{t+1}-1} e(\beta_q) = \sum_{q=1}^{m} e(\beta_q).$$

This allows us to conclude that we have equality in (86) when $m = i_{l^{(m)}+1} - 1$ or when $m = N$. Thus, we have shown that $\boldsymbol{y}^*$ satisfies (66) and (65) whenever $m \leq j_n$. Since we have $y_m^* = 0$, $\forall j_n + 1 \leq m \leq N$, we have shown that $\boldsymbol{y}^*$ satisfies both (65) and (66). We conclude that the output $\boldsymbol{y}^*$ satisfies all the constraints in $\mathcal{P}$ and is thus a possible solution.

We are now ready to prove that $\boldsymbol{y}^*$ solves the problem $\tilde{\mathcal{P}}$. Consider the relaxed version of the problem $\tilde{\mathcal{P}}$ below where the constraint $y_1 \geq y_2 \geq \cdots \geq y_N$ is dropped.

$$\hat{\mathcal{P}}: \text{Minimize } f(\boldsymbol{y}) \stackrel{\text{def}}{=} \sum_{i=1}^{N} \sigma_i^2 \frac{y_i}{1 - y_i} \text{ subject to}$$

$$1 > y_m \geq 0,\; 1 \leq m \leq N \text{ and}$$

$$\sum_{m=1}^{l} y_m \geq \sum_{m=1}^{l} e(\beta_l), \qquad l = 1, \ldots, N-1$$

$$\sum_{m=1}^{N} y_m = \sum_{m=1}^{K} e(\beta_l).$$

We will first show that $\boldsymbol{y}^*$ is a solution to $\hat{\mathcal{P}}$. Since the constraint set in problem $\mathcal{P}$ is larger than that in $\hat{\mathcal{P}}$ and since $\boldsymbol{y}^*$ satisfies the constraints of $\mathcal{P}$, we have completed the proof of the theorem.

It remains to show that $\boldsymbol{y}^*$ solves the relaxed problem $\hat{\mathcal{P}}$. Our first step is the observation that the function f being minimized in $\mathcal{P}$ is concave in $\boldsymbol{y} \stackrel{\text{def}}{=} (y_1, \ldots, y_N)$ and, furthermore, $\boldsymbol{y}$ is constrained to be in a convex polytope (defined by linear inequalities). Such optimization problems are classical and a complete characterization of the solution is given by the Kuhn–Tucker conditions [9, Theorem 28.3]. Define (the Lagrangian)

$$\begin{aligned} L\colon &\left(\boldsymbol{y}, \gamma^{(1)}, \gamma^{(2)}, \gamma^{(3)}\right) \\ &\mapsto f(\boldsymbol{y}) - \sum_{i=1}^{N} \gamma_i^{(1)} y_i - \sum_{i=1}^{N} \gamma_i^{(2)} (1 - y_i) \\ &\quad - \sum_{i=1}^{N-1} \gamma_i^{(3)} \left(\sum_{m=1}^{i} y_m - e(\beta_m)\right) \\ &\quad - \gamma_N^{(3)} \left(\sum_{m=1}^{N} y_m - \sum_{m=1}^{K} e(\beta_m)\right). \end{aligned}$$

Consider the following choice of $\gamma^{(1)}$, $\gamma^{(2)}$, and $\gamma^{(3)}$. Define $\gamma_j^{(2)*} \stackrel{\text{def}}{=} 0$ for all $1 \leq j \leq N$. For $1 \leq t < n-1$, define

$$\gamma_{j_t}^{(1)*} \stackrel{\text{def}}{=} \eta_t^2 - \eta_{n-1}^2, \qquad \text{if } j_t > j_{t+1} \tag{87}$$

$$\gamma_{i_{t+1}-1}^{(3)*} \stackrel{\text{def}}{=} \eta_t^2 - \eta_{\min\{m>t:\, i_{m+1} > i_{t+1}\}}^2 \tag{88}$$

$$\gamma_N^{(3)*} = \eta_{n-1}^2. \tag{89}$$

We set the remaining components of $\gamma^{(1)*}$ and $\gamma^{(3)*}$ to be equal to zero. Observe from this definition that we always have

$$\gamma_m^{(1)*} = 0, \qquad 1 \leq m \leq j_n \tag{90}$$

$$\gamma_m^{(3)*} = 0, \qquad j_n < m < N. \tag{91}$$

We claim that

$$\frac{\partial L}{\partial y_k}\left(\boldsymbol{y}^*, \gamma^{(1)*}, \gamma^{(2)*}, \gamma^{(3)*}\right) = 0, \qquad \forall\, k = 1 \ldots N. \tag{92}$$

If this is true, then appealing to [9, Theorem 28.3], we have proved that $\boldsymbol{y}^*$ solves $\hat{\mathcal{P}}$ and this completes the proof of the theorem. We will now verify (92) which can be expanded into

$$\frac{\partial L}{\partial y_k}\left(\boldsymbol{y}^*, \gamma^{(1)*}, \gamma^{(2)*}, \gamma^{(3)*}\right) = \frac{\sigma_k^2}{(1 - y_k^*)^2} - \gamma_k^{(1)*} - \sum_{t=k}^{N} \gamma_t^{(3)*}. \tag{93}$$

Fix $1 \le k \le N$. We consider two cases: $k \le j_n$ and $k > j_n$.

1) *Case 1*: $k > j_n$. Recall the definition of $l^{(k)}$ (from (78)), with the interpretation that y_k^* is updated (and set to zero) in iteration numbered $l^{(k)}$. Here $\eta_{l^{(k)}} = \sigma_k$ and $\gamma_k^{(1)*} = \eta_{l^{(k)}}^2 - \eta_{n-1}^2$. Using (91) and the hypothesis that $k > j_n \ge i_n$, we have $\gamma_t^{(3)*} = 0$, $k \le t < N$. Substituting these quantities in (93), we have

$$\frac{\partial L}{\partial y_k}\left(\boldsymbol{y}^*, \gamma^{(1)*}, \gamma^{(2)*}, \gamma^{(3)*}\right) = \eta_{l^{(k)}}^2 - \left(\eta_{l^{(k)}}^2 - \eta_{n-1}^2\right) - \eta_{n-1}^2 = 0$$

and the claim of (92) is shown.

2) *Case 2*: $k \le j_n$. Again, recall the definition of $l^{(k)}$ from (78) as the iteration number where y_k^* is updated. In this case, $y_k^* = 1 - \frac{\sigma_k}{\eta_{l^{(k)}}}$ (see (79)). From (90), we have $\gamma_k^{(1)*} = 0$. Substituting these quantities in (93), we have

$$\begin{aligned}\frac{\partial L}{\partial y_k}&\left(\boldsymbol{y}^*, \gamma^{(1)*}, \gamma^{(2)*}, \gamma^{(3)*}\right)\\ &= \eta_{l^{(k)}}^2 - 0 - \sum_{m \in \{i_{l^{(k)}+1}, \dots, i_{n-1}, i_n\}} \gamma_{m-1}^{(3)*}\\ &= \eta_{l^{(k)}}^2 - \sum_{m=0}^{n-2-l^{(k)}} \left(\eta_{l^{(k)}+m}^2 - \eta_{l^{(k)}+m+1}^2\right) - \eta_{n-1}^2.\\ &= 0.\end{aligned}$$

Thus, (92) is shown in this case as well.

This completes the proof of the claim that $\boldsymbol{y}^*$ solves $\hat{\mathcal{P}}$ and the theorem is proved. □

B.4 Proof of Proposition 3.1

We use the notation developed in the proof of Theorem 3.4 in Appendix B.3. Suppose Algorithm $\mathcal{B}$ concludes in $n \le N$ steps. We denote the pairs (i, j) as the algorithm runs through the n iterations by $(i_1, j_1), (i_2, j_2), \dots, (i_n, j_n)$ and the value of η in Step 3 by $\eta_1, \dots, \eta_{n-1}$. Fix $1 \le t < n-1$. Now, it follows that the output of Algorithm $\mathcal{B}$ has the property, for every $i_t \le r < i_{t+1}$, that

$$\lambda_r^* = \sigma_r(\eta_t - \sigma_r), \qquad \text{where } \eta_t = \frac{\sum_{k=i_t}^{i_{t+1}-1} \sigma_k}{i_{t+1} - i_t - \sum_{k=i_t}^{i_{t+1}-1} e(\beta_k)}.$$

Let $V_{(t)}$ be any $(i_{t+1} - i_t)$-dimensional orthonormal matrix such that

$$V_{(t)} \operatorname{diag}\left\{\frac{\lambda_r^*}{\lambda_r^* + \sigma_r^2}, i_t \le r < i_{t+1}\right\} V_{(t)}^t \text{ has diagonal entries } e(\beta_{i_t}), \dots, e(\beta_{i_{t+1}-1}).$$

Define $D_{(t)}$ to be a $(i_{t+1} - i_t)$-dimensional diagonal matrix with diagonal entries equal to the diagonal values of

$$V_{(t)} \operatorname{diag}\{\lambda_r^*, i_t \le r < i_{t+1}\} V_{(t)}^t.$$

Define the $N \times (i_{t+1} - i_t)$ matrix defined as

$$S_{(t)} \stackrel{\text{def}}{=} U_{(t)}^* \operatorname{diag}\{\lambda_r^*, i_t \le r < i_{t+1}\} V_{(t)}^t D_{(t)}^{-\frac{1}{2}}$$

where

$$U_{(t)}^* \stackrel{\text{def}}{=} [\boldsymbol{u}_r, \; i_t \le r < i_{t+1}]$$

and $U^* = [\boldsymbol{u}_1, \dots, \boldsymbol{u}_N]$ is an orthonormal matrix that diagonalizes Σ. Now, for every $i_{n-1} \le r < N$ we have

$$\lambda_r^* = \sigma_r(\eta_{n-1} - \sigma_r), \qquad \text{where } \eta_{n-1} = \frac{\sum_{k=i_{n-1}}^{j_{n-1}} \sigma_k}{j_{n-1} - i_{n-1} + 1 - \sum_{k=i_{n-1}}^{K} e(\beta_k)}.$$

Let $V_{(n-1)}$ be any $(K + 1 - i_{n-1})$-dimensional orthonormal matrix such that

$$V_{(n-1)} \operatorname{diag}\left\{\frac{\lambda_r^*}{\lambda_r^* + \sigma_r^2}, i_{n-1} \le r \le j_{n-1}\right\} V_{(n-1)}^t \text{ has diagonal entries } e(\beta_{i_{n-1}}), \dots, e(\beta_K).$$

Define $D_{(n-1)}$ to be a $(K + 1 - i_{n-1})$-dimensional diagonal matrix with diagonal entries equal to the diagonal values of

$$V_{(n-1)} \operatorname{diag}\{\lambda_r^*, i_{n-1} \le r \le j_{n-1}\} V_{(t)}^t.$$

Define the $N \times (K + 1 - i_{n-1})$ matrix defined as

$$S_{(n-1)} \stackrel{\text{def}}{=} U_{(n-1)}^* \operatorname{diag}\{\lambda_r^*, i_{n-1} \le r \le j_{n-1}\} V_{(n-1)}^t D_{(n-1)}^{-\frac{1}{2}}$$

where

$$U_{(n-1)}^* \stackrel{\text{def}}{=} [\boldsymbol{u}_r, i_{n-1} \le r \le j_{n-1}].$$

Now define the $N \times K$ matrix

$$S^* \stackrel{\text{def}}{=} \left[S_{(1)}, \dots, S_{(n-1)}\right]$$

and the $K \times K$ diagonal matrix D^* with diagonal entries D_{rr}^* equal to the $(r - i_t + 1)$st diagonal entry of $D_{(t)}$. Here t is either $n-1$ or the unique number between 1 and $n-2$ such that $i_t \le r < i_{t+1}$. It follows from the construction of Section III-B that the pair (S^*, D^*) is a solution to the dual problem $\mathcal{P}$. We make the observation that the signature sequences $\boldsymbol{s}_k^*$ and $\boldsymbol{s}_{\tilde{k}}^*$ are orthogonal whenever k and $\tilde{k}$ belong to different intervals of the form $[i_1, i_2), [i_2, i_3), \dots, [i_{n-1}, K]$. We are now in a position to prove the proposition. For any user k that belongs to the interval $[i_t, i_{t+1})$ for some $1 \le t < n-1$ we have

$$\begin{aligned}\boldsymbol{c}_k^* &= (S^* D^* S^{*t} + \Sigma)^{-1} \boldsymbol{s}_k^*\\ &= U_{(t)}^* \operatorname{diag}\left\{\frac{1}{\eta_t \sigma_r}, i_t \le r < i_{t+1}\right\} U_{(t)}^{*t} \boldsymbol{s}_k^*\\ &= \frac{1}{\eta_t} \Sigma^{-\frac{1}{2}} \boldsymbol{s}_k^*.\end{aligned}$$

A similar calculation for users k that lie in the interval $[i_{n-1}, K]$ shows that $\boldsymbol{c}_k^* = \frac{1}{\eta_{n-1}} \Sigma^{-\frac{1}{2}} \boldsymbol{s}_k^*$. This completes the proof of the proposition. □

B.5 Proof of Propositions 3.2 and 3.3

Fix a set of SIR requirements $(\beta_1, \ldots, \beta_K)$ such that $\sum_{i=1}^K e(\beta_i) < N$. Now, the minimum sum power P^* is the solution of the optimization problem $\mathcal{P}$ in (22) which was rewritten as (64). It is clear that P^* as a function of $(\sigma_1^2, \ldots, \sigma_N^2)$ is the minimum of a sequence (indexed by $y_1, \ldots, y_N$ which range over a convex polytope, see (65) and (66)) of linear functions (given by $\sum_{i=1}^N \sigma_i^2 \frac{y_i}{1-y_i}$). Hence, it follows that P^* is a concave function of $(\sigma_1^2, \ldots, \sigma_N^2)$. Furthermore, P^* is a symmetric function of $(\sigma_1^2, \ldots, \sigma_N^2)$. We conclude that P^* is a Schur-concave function of $(\sigma_1^2, \ldots, \sigma_N^2)$. This concludes the proof of Proposition 3.2.

Fix the additive colored noise variances $(\sigma_1^2, \ldots, \sigma_N^2)$ and consider a pair of SIR requirements $(\beta_1, \ldots, \beta_K)$ and $(\tilde{\beta}_1, \ldots, \tilde{\beta}_K)$ such that

$$(e(\beta_1), \ldots, e(\beta_K)) \quad \text{majorizes} \quad \left(e\left(\tilde{\beta}_1\right), \ldots, e\left(\tilde{\beta}_K\right)\right).$$

Using the transitivity of the majorization relation in the optimization problem (64) (whose value is P^*), we see that the set of $(y_1, \ldots, y_N)$ over which the optimization is carried corresponding to SIR requirements $(\beta_1, \ldots, \beta_K)$ out is contained in the set corresponding to the SIR requirements $(\tilde{\beta}_1, \ldots, \tilde{\beta}_K)$. Thus, we arrive at the inequality

$$P^*((\beta_1, \ldots, \beta_K), \sigma^2) \geq P^*\left(\left(\tilde{\beta}_1, \ldots, \tilde{\beta}_K\right), \sigma^2\right).$$

We conclude that P^* is a Schur-convex function of $(e(\beta_1), \ldots, e(\beta_K))$ and is thus convex as well. $\square$

B.6 Proof of Proposition 3.4

From Lemma 3.3, we know that the region of $\frac{\text{SIR}}{1+\text{SIR}}$ that are achievable with a sum received power constraint P is convex. We will now characterize this convex set (henceforth denoted by $\mathcal{C}(P, \Sigma)$) by its extreme points. Each extreme point $(e_1, \ldots, e_K)$ on this convex set is characterized by a vector $\boldsymbol{w}$ with nonnegative entries with the property that $(e_1, \ldots, e_K)$ is the argument of the optimization problem $\mathcal{P}$ in the statement of Lemma 3.2. We reformulate $\mathcal{P}$ below, using Lemma 3.2 and the convexity of the map $x \mapsto \frac{x}{x+\sigma^2}$ and the ordering $\sigma_1^2 \leq \cdots \leq \sigma_N^2$ (as in the beginning of the proof of Theorem B.3).

Problem $\tilde{\mathcal{P}}$:

$$\text{Minimize } \operatorname{tr}\left[V^t W V \operatorname{diag}\left\{\frac{\lambda_1}{\lambda_1+\sigma_1^2}, \ldots, \frac{\lambda_N}{\lambda_N+\sigma_N^2}\right\}\right]$$

subject to V, a $K \times N$ matrix with orthonormal columns and

$$\lambda_1 \geq \cdots \geq \lambda_N \geq 0 \text{ and } \sum_{i=1}^N \lambda_i \leq P.$$

Since the diagonal entries (denoted by, say, $d_1, \ldots, d_N$) of $V^t W V$ are majorized by the eigenvalues of $V^t W V$ (denoted by, say, $\mu_1, \ldots, \mu_N$) and since $\mu_1, \ldots, \mu_N$ satisfy the constraints [4, Theorem 4.3.15]

$$\mu_{[N-i]} \leq w_{[N-i]}, \qquad 1 \leq i \leq N,$$

we see that for the optimization problem $\tilde{\mathcal{P}}$ above, V should always be chosen such that $V^t W V$ is a diagonal matrix with diagonal entries $w_{[1]}, \ldots, w_{[N]}$. Thus, $\tilde{\mathcal{P}}$ is reduced to

$$\text{Problem } \tilde{\mathcal{P}}: \text{Minimize } \sum_{i=1}^N w_{[i]} \frac{\lambda_i}{\lambda_i + \sigma_i^2}$$

$$\text{subject to } \lambda_1 \geq \cdots \lambda_N \geq 0, \quad \sum_{i=1}^N \lambda_i \leq P.$$

It is easily verified that the solution to this reduced problem is $\lambda_i^* = \sigma_i(\eta_{\boldsymbol{w}}\sqrt{w_{[i]}} - \sigma_i)^+$ where $\eta_{\boldsymbol{w}}$ is chosen such that $\sum_{i=1}^N \lambda_i^* = P$. This allocation of V, Λ translates into signature sequence and powers allocation as follows. Users corresponding to the weights $w_{[1]}, \ldots, w_{[N]}$ are assigned powers $\lambda_1^*, \ldots \lambda_N^*$ and signature sequences equal to the eigenvectors of Σ corresponding to the eigenvalues $\sigma_1^2, \ldots, \sigma_N^2$ in that order. We can now verify that the SIRs attained by these N users (given by the same ordering as the users corresponding to $w_{[1]}, \ldots, w_{[N]}$) are $(\frac{\eta_{\boldsymbol{w}}\sqrt{w_{[i]}}}{\sigma_i} - 1)^+$, $i = 1 \cdots N$. This completes the proof of the proposition. $\square$

B.7 Proof of Propositions 3.6 and 3.7

We begin with Proposition 3.6. Consider the map

$$f\colon \beta \mapsto \int \frac{dF_\sigma(x)}{x + \alpha \int \frac{p\,dF_p}{1+p\beta}}.$$

We see that β^* is the unique positive fixed point of f. We claim that f is concave and monotonically increasing. Suppose this is true. Fix F_p and consider F_σ that is a dilation of $\tilde{F}_\sigma$. Now, f is seen to be dilatory-convex in F_σ, we have that

$$f(F_\sigma, F_p, \beta) \geq f\left(\tilde{F}_\sigma, F_p, \beta\right), \qquad \forall \beta \geq 0.$$

The concavity and increasing property of f (as a function of β) coupled with the relation above, shows that $\beta^*(F_\sigma, F_p) \geq \beta^*(\tilde{F}_\sigma, F_p)$. Now fix F_σ and consider F_p that is a dilation of $\tilde{F}_p$. Again, f is seen to be dilatory-convex in F_p and thus

$$f(F_\sigma, F_p, \beta) \geq f\left(F_\sigma, \tilde{F}_p, \beta\right), \qquad \forall \beta \geq 0.$$

As before, this observation coupled with the concavity and increasing property of f (as a function of β) shows that $\beta^*(F_\sigma, F_p) \geq \beta^*(F_\sigma, \tilde{F}_p)$. We only need to show that f is concave and increasing monotonically in β.

A straightforward calculation shows that

$$f'(\beta) = \alpha \int \left(\frac{p}{1+p\beta}\right)^2 dF_p(p) \cdot \int \frac{1}{\left(x + \alpha \int \frac{p}{1+p\beta}\,dF_p(p)\right)^2} dF_\sigma(x). \tag{94}$$

Since the first derivative of f is strictly positive, we have shown that f is monotonically increasing with β. Continuing from (94), we arrive at

$$f''(\beta) = -\alpha \int \frac{\alpha g(\beta) + x \int \left(\frac{p}{1+p\beta}\right)^3 dF_P(p)}{\left(x + \alpha \int \frac{p}{1+p\beta}\, dF_P(p)\right)^3}\, dF_\sigma(x) \tag{95}$$

where we have written

$$g(\beta) \stackrel{\text{def}}{=} \left(\int \frac{p}{1+p\beta}\, dF_P(p)\right)\left(\int \left(\frac{p}{1+p\beta}\right)^3 dF_P(p)\right) - \left(\int \left(\frac{p}{1+p\beta}\right)^2 dF_P(p)\right)^2. \tag{96}$$

Using Holder's inequality

$$\int \|g_1(x) g_2(x)\| \mu(dx) \le \left(\int \|g_1(x)\|^p \| \mu(dx)\right)^{\frac{1}{p}} \left(\int g_2(x)\|^q \mu(dx)\right)^{\frac{1}{q}}$$

for every μ-integrable functions g_1 and g_2 and $p,\, q > 0$ such that $\frac{1}{p} + \frac{1}{q} = 1$ in (96) (with $g_1(x) = (\frac{x}{1+x\beta})^{\frac{1}{2}}$, $g_2(x) = (\frac{x}{1+x\beta})^{\frac{3}{2}}$, $p = q = 2$, and $\mu = dF_p$) we see that $g(\beta) \ge 0$. Using this in (95), we see that the second derivative of f is negative and thus conclude that f is concave in β. This completes the proof of Proposition 3.6. □

We now turn to the proof of Proposition 3.7. Defining the map

$$f_1 \colon p \mapsto \int \frac{dF_\sigma(x)}{\frac{x}{p} + \frac{\alpha\beta}{1+\beta}}$$

we observe that $P_{\min}(\beta, F_\sigma)$ is the inverse of f_1 at 1 (f_1 is strictly monotonically increasing and thus the inverse is well defined). We also observe that $f_1(p, F_\sigma)$ is dilatory-concave as a function of F_σ. This observation coupled with the concavity (easily verified) and monotonically increasing property of f_1 as a function of p shows that $P_{\min}(\beta, F_\sigma)$ is dilatory-concave in F_σ. The proof is complete. □

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