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User Capacity of a Power Controlled CDMA System with Multiple Base Stations

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Abstract — **User capacity of the uplink of a multiple cell synchronous CDMA system is analyzed along with power and channel allocation. For the most part, attention is focussed on the situation when the signature sequences are chosen from an orthogonal sequence set. The user capacity of the system is the maximum number of users per unit processing gain admissible in the system such that each user has its quality of service (QoS) requirement (expressed in terms of its desired signal-to-interference ratio) met. The user capacity depends on the path gains of the users to the base stations and the comparison of the user capacity per unit processing gain with the maximum number of admissible users in a single channel system allows us to describe (in terms of the path gains) scenarios when the user capacity of the system can be increased by having more processing gain. We conclude by conjecturing the user capacity region when the signature sequences are not constrained and the base station estimates the users' symbols with matched filter and linear MMSE receiver structures.**

I. Introduction

We consider a multi cell spread spectrum wireless system and focus on the uplink. The users share the same bandwidth and the performance of the system depends on how efficiently the resources - bandwidth and transmit power - are utilized and on the design of the multiuser receiver structure at the base station. The users are separated from each other by their *signature sequences* or *codes*. The processing gain represents the *degrees of freedom* in the system. The users arrive with some Quality-of-Service (QoS) requirement (expressed in terms of the Signal-to-Interference Ratio (SIR) of the users' signal at the base station it is communicating with) and the problem is to allocate powers and signature sequences to the users so that their QoS requirements are satisfied. For the most part in this paper we restrict ourselves to the scenario when the codes of the users are chosen from an orthogonal sequence set. We are interested in the problem of characterizing the set of admissible users per unit degree of freedom called the *user capacity* of the system. One motivating factor for such a characterization is to understand "good" power and channel allocation strategies as a function of the fading gains. In Section 2 we formally state our model and formulate the problem and point out the connection to existing literature on this topic. In Section 3 we state our main results on the characterization of the admissibility region. A 2-cell system exemplifies the channel and power allocation strategies as a function of the path gains. We conclude in Section 4 with some conjectures and direction for future work.

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II. Model and Problem Formulation

The system consists of $M > 1$ cells each of which has a base station. We shall refer interchangeably to the cell j and base station j. The users arrive into the system and we fix the base station the user will communicate to (the base station that is nearest to the mobile user is typically the choice). We make the assumption of *perfect power control*, i.e., we assume that the channel can be estimated perfectly (implicit is the assumption that the estimation of the channel is done faster than the fading rate) and thus the user can control the received power at the base station it is communicating with. We distinguish the users by the base station they talk to and by the vector of path gains to all the base stations. To quantify this notion, let the number of *types* of users communicating with base station i be finite, say $n_i \geq 1$, denoted by $\{m_{iu}, u = 1 \dots n_i\}$ and user of *type* iu has path gain to the base station j given by $g(m_{iu}, j) > 0$. A technical assumption we make is that any two types of users communicating with the same base station are *different* in the following sense: for any two types iu and iv, we have $u \neq v$ iff there exists some base station j such that $\frac{g(m_{iu},j)}{g(m_{iu},i)} \neq \frac{g(m_{iv},j)}{g(m_{iv},i)}$, i.e., the path gains to at least one base station normalized appropriately are different. We consider an immobile system, ie., the number of types of mobiles and the path gains of mobiles of type iu to base station j are fixed. This is known as the *snapshot* analysis of the system and can be anticipated to provide intuition and information on what to expect when mobility of the users is modeled into the system. One snapshot analysis of allocation of resources, namely transmit powers and channel to transmit in, is in [2] and the authors perform simulations on a one-dimensional cellular system to study the performance of their allocation schemes. In [1], the authors propose a distributed scheme allocation of resources, namely channel and transmit powers to the mobiles and analyze the performance of their proposal through simulations.

We are interested in the scenario when the users are allocated signature sequences from an orthogonal sequence set (henceforth denoted by channels or slots) and the processing gain, say n, translates into a choice of n channels. Each user is the system is allocated transmit power and a channel and at the base station it is communicating with the user is linearly estimated from the received signal (a comprehensive reference on linear multiuser receivers is Chapters 4,5 and 6 of [6]). Corresponding to this linear estimate is the SIR that has to be above a target value. We shall assume that each user has the same target SIR value denoted by say β. Formally, admissibility of a set of users is defined as follows: a set of mobiles $\{k_{iu} : \forall i = 1 \dots M, \ \forall u = 1 \dots n_i\}$ is admissible into the system with processing gain n if for all users m of type iu (there are k_{iu} of them) there is an allocation of channels s_m^{iu} and received powers p_m^{iu} such that the SIR of the linear estimate of the user m at the base station i is above the threshold β. In our notation the SIR of the estimate of user m can be written as

$$SIR_m^{iu} = \frac{p_m^{iu}}{\sigma_i^2 - p_m^{iu} + \sum_{j=1}^{M} \sum_{v=1}^{n_j} \frac{g(m_{jv},i)}{g(m_{jv},j)} \sum_{l=1}^{N_{ju}} 1_{\{s_l^{jv}=s_m^{iu}\}} p_{ll}^{jv}} \quad (1)$$

and should be greater than the threshold β for every user $m = 1 \ldots N_{iu}$. Here σ_i^2 is the variance of the white Gaussian additive noise at the base station i. We summarize our main results on the admissibility of a set of users in the next section.

III. Main Results

Since only users in the same channel (i.e., they are allocated the same signature sequence) interfere with each other. Hence we can focus on a single channel (representing, say the first sequence from the orthogonal set). Suppose there are k_{iu} users of type iu in this channel. Admissibility of these users in this (simplified) setup amounts to the feasibility of a set of linear inequalities (given in (1) above) and can be shown to be equivalent (following well known studies in the power control literature, see [3]) to

$$r(A) < 1 + \frac{1}{\beta} \quad (2)$$

where $r(A)$ is the Perron eigenvalue of the positive matrix A by a direct application of Theorem 2.1 in [5]. A is a $\sum_{iu} k_{iu} \times \sum_{iu} k_{iu}$ matrix and let us parameterize A by the pair (l_{iu}, t_{jv}) where for every pair of types iu and jv, $l_{iu} \in \{1, \ldots, k_{iu}\}$ and $t_{jv} \in \{1, \ldots, k_{jv}\}$. Then, for every pair of types iu and jv and every pair of users l_{iu} and t_{jv} the entry $A_{l_{iu},t_{jv}}$ depends only on the base station i and type jv and is given by $\frac{g(m_{jv},i)}{g(m_{jv},j)}$. When the condition in (2) is satisfied, there is a unique component wise minimal allocation of powers to the users so that the SIR requirement of the users is exactly met. There is a simple closed form expression for the minimal power solutions and [3] provides a distributed algorithm to update the powers of the users that converges to the minimal power solution and [8] is a good reference to various schemes that update the powers of the users in a way that the updates converge to the minimal power solution. Henceforth we shall focus only on the condition (2) and assume that power allocation is the minimal power solution. Thus user capacity region of a single channel - the set of admissible mobiles - is given by

$$\tilde{\mathcal{K}} = \left\{ k = (k_{iu}) : k_{iu} \in \mathcal{Z}_+,\ r(A) \leq 1 + \frac{1}{\beta} \right\} \quad (3)$$

In the general situation of a slotted system, the user capacity region (set of admissible users per unit degree of freedom) is easily seen to be

Proposition III.1

$$\mathcal{K} = co\left\{ k = (k_{iu}) : k_{iu} \in \mathcal{Z}_+ \ \ r(A) \leq 1 + \frac{1}{\beta} \right\} = co\left(\tilde{\mathcal{K}}\right)$$

where A is defined above and $co(C)$ is the convex hull of the set C.

The only gain in admissibility when they are multiple channels comes from "channel sharing" and thus the user capacity region in the slotted system is the convex hull of the user capacity region for the single channel system. A priori the capacity region for the single channel system can be convex and then the two capacities are identical. However, if this is not the case, then $\mathcal{K}$ is strictly larger and some points in this region can be achieved only by appropriately sharing the channels, a version of "time-sharing" among channels. The path gains determine the convexity of $\tilde{\mathcal{K}}$ and towards obtaining a more tractable representation of $\tilde{\mathcal{K}}$ we have the following result:

Proposition III.2 *The spectral radius of A is equal to the spectral radius of the matrix $K\Gamma$ where $K = diag\{k_{iu}, \forall i, u\}$ and Γ is a $\sum_{i=1}^{M} n_i \times \sum_{i=1}^{M} n_i$ matrix and $\Gamma_{iu,jv} = \frac{g(m_{jv},i)}{g(m_{jv},j)}$ with the same parameterization as before.*

Observe that the dimension of the matrix Γ does not vary with the number of users. Following the notation used above, let us adopt the convention that K represents the diagonal matrix whose diagonal entries come from the vector k; the dimensions being $\sum_{i=1}^{M} n_i$. We are now ready to state our main result: A partial characterization of convexity of the single channel admissibility region as a function of the path gains:

Theorem III.3 *Recall our notation for the user capacity region of a single channel as $\tilde{\mathcal{K}}$ in (3).*

1. *The level set $\left\{k : r(K\Gamma) \leq 1 + \frac{1}{\beta}\right\}$ is convex if and only if the map $k \mapsto r(K\Gamma)$ is convex*
2. *$f(k) = r(K\Gamma)$ is non convex whenever $n_i > 1$ for some $i = 1 \ldots M$*
3. *Let $n_i = 1$. Then the function $f(k)$ is convex when*
   - *Γ is invertible and Γ^{-1} is a M-matrix or*
   - *there exist D_1, D_2 positive diagonal matrices such that $D_1 \Gamma D_2$ is positive semidefinite*
4. *If $\forall i, u, j$ $\frac{g(m_{iu},j)}{g(m_{iu},i)} \geq 1$, then $f(k)$ is concave.*
5. *If Γ is invertible and Γ is element wise non-negative then $f(k)$ is concave.*

It is obvious that if the function $f : k \mapsto r(K\Gamma)$ is convex then the level set $\left\{k : r(K\Gamma) \leq 1 + \beta^{-1}\right\}$ is also convex. Assertion 1 in the theorem says that the converse is also true. Thus we can focus on the convexity of the map f. The second assertion in the theorem says that whenever there is more than one type of user in any base station the function f is non convex. Thus there is always a gain by channel sharing when $n_i > 1$ for any base station i. In the situation when $n_i = 1$ for each base station i, the third assertion in the theorem identifies two (separate and independent in general) sufficient conditions on the path gain matrix Γ such that the map f is convex. We illustrate these conditions in the special case of $M = 2$ below:

Example III.4 *User Capacity regions for $M = 2, n_1 = 1$ and $n_2 = 1$.*

Consider the case when there are only two base stations ($M = 2$) and $n_1 = n_2 = 1$. In this case the matrix Γ is 2×2 and is given by

$$\Gamma = \begin{bmatrix} 1 & \frac{g(m_2,1)}{g(m_2,2)} \\ \frac{g(m_1,2)}{g(m_1,1)} & 1 \end{bmatrix}$$

where we have simplified the notation and m_1 and m_2 are users that talk to base stations 1 and 2 respectively. Let us represent the determinant of Γ (equal to $1 - \frac{g(m_2,1)}{g(m_2,2)} \frac{g(m_1,2)}{g(m_1,1)}$) by x. Then, it is straightforward to verify that

$$\tilde{\mathcal{K}} = \text{ co } \left\{ (k_1, k_2) \in \mathcal{Z}_+^2 : k_1 + k_2 - \frac{x\beta k_1 k_2}{1+\beta} \leq 1 + \frac{1}{\beta} \right\}$$

There are now two regimes of path gains:

$x < 0$ In this situation, $\tilde{\mathcal{K}}$ is concave and $\mathcal{K} = \text{co}\left(\tilde{\mathcal{K}}\right)$ is strictly bigger than $\tilde{\mathcal{K}}_\beta$.

$x \geq 0$ In this situation, $\tilde{\mathcal{K}}$ is convex and $\mathcal{K} = \tilde{\mathcal{K}}$. Thus the user capacity region in the slotted system is identical to that of the single channel system.

When $x < 0$ it is easy to verify that condition in assertion (5) of Theorem III.3 is met which implies the concavity of $f(k)$ and when $x \geq 0$ both the conditions in assertion (3) of Theorem III.3 are met implying the convexity of $f(k)$. A qualitative explanation is that when $x < 0$ the interference in the system is "high" (since the path gains to the other base stations is large in the appropriate sense) and $\tilde{\mathcal{K}}$ is concave and thus the optimal strategy (in the sense of achieving every point in $\mathcal{K}$) is to separate users of the two different types to different channels. When $x \geq 0$ the interference is "low" and the optimal strategy is not to separate the users into different channels but instead use the same strategy in all the channels. For notation on M-matrices, see [5] and on when the inverse of a matrix is a M-matrix see [4]. Thus in the simple setting of 2 base stations there is a simple interpretation of when there is a need to convexify by channel sharing. In the general case the corresponding generalization appears to be more complicated and we are currently investigating intuitive explanations of the conditions in assertion (3) in terms of the path gains.

Observe that when f is concave, the user capacity region of the slotted system $\mathcal{K}$ is simply the scaled unit simplex given by co $\left\{k = (k_{iu}) : k_{iu} \in \mathcal{Z}_+, \sum_{iu} k_{iu} \leq 1 + \frac{1}{\beta}\right\}$ and thus in this situation convexifying by channel sharing strictly increases the user capacity region. When *every* user is communicating to a base station that is not the closest (in terms of path gains) then the user capacity region increases by channel sharing is made precise in assertion (4) of the theorem.

IV. Discussion

For the entire part in this paper we have restricted ourselves to the scenario when the signature sequences of the users are drawn from an orthogonal sequence set (like the Welsh codes in IS-95). We could generalize this situation to allocating arbitrary signature sequences (visualized as unit norm vectors in $\mathbb{R}^n$). Two important linear receiver structures at the base station are the conventional matched filter and linear MMSE receiver. In [7], we have completely characterized the user capacity regions for the special case of $M = 1$, i.e., in the situation of a single base station. Our main result was that K users with SIR requirements $\beta_1, \ldots, \beta_K$ are admissible in the system with processing gain n if and only if

$$\sum_{i=1}^{K} \frac{\beta_i}{1+\beta_i} < n \tag{4}$$

and this condition is applicable to both the linear MMSE receiver and the (a priori inferior) matched filter receiver. The user capacity when all the users have the SIR requirement β is just $1 + \beta^{-1}$. In this simple case of a single base station, the user capacity of a single channel is easily calculated to be $\lfloor 1 + \beta^{-1} \rfloor$ and thus we notice that the gain in user capacity region per unit degree of freedom when we allow nonorthogonal signature sequences is the non integer part of $1 + \beta^{-1}$. Motivated by the symmetry in the underlying equations for the SIRs in the multiple base station scenario compared to the single base station calculation we have the following conjecture:

Conjecture IV.1 *For both the matched filter and linear MMSE receiver the user capacity region is*

$$\hat{K} = co \left\{ k = (k_{iu}) \ : \ k_{iu} \geq 0, r\left(K\Gamma\right) \leq 1 + \frac{1}{\beta} \right\}$$

and then the difference between the user capacity region per unit degree of freedom of the slotted system is in the non integer part $1 + \beta^{-1}$. A key feature of the derivation of (4) in [7] was the discovery of a conservation law among the SIRs of linear MMSE estimates and we are studying the applicability of this law to the multiple base station scenario. In the single base station solution in [7] we identified "optimal" signature sequences and corresponding (component wise minimal) optimal powers for the users. Current studies are in extending the intuition developed in this exercise.

In this work, we have focussed on a global allocation of signature sequences and powers to all the users of the system as a function of the path gains. In practice, it may only be feasible (and desirable to keep protocols simple to implement) to do this allocation only locally, say only between users communicating with the same base station. User capacities with such constraints are very interesting quantities.

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