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A FUNCTIONAL CENTRAL LIMIT THEOREM FOR THE JUMP COUNTS OF MARKOV PROCESSES WITH AN APPLICATION TO JACKSON NETWORKS

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Abstract

Each feasible transition between two distinct states i and j of a continuous-time, uniform, ergodic, countable-state Markov process gives a counting process counting the number of such transitions executed by the process. Traffic processes in Markovian queueing networks can, for instance, be represented as sums of such counting processes. We prove joint functional central limit theorems for the family of counting processes generated by all feasible transitions. We characterize which weighted sums of counts have zero covariance in the limit in terms of balance equations in the transition diagram of the process. Finally, we apply our results to traffic processes in a Jackson network. In particular, we derive simple formulas for the asymptotic covariances between the processes counting the number of customers moving between pairs of nodes in such a network.

MARKOV CHAIN; COUNTING PROCESS; COUPLING; INFINITE-DIMENSIONAL BROWNIAN MOTION

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1. Introduction

We consider a Markov process, denoted by $\{X(t), t \geqq 0\}$, in continuous time with countable state space S. The process is assumed to be uniform and ergodic with stationary measure π. Let q_{ij} be the rate of jumps from the state i to state j and $q_i = \sum_{j \neq i} q_{ij}$. Since the process is assumed uniform, we have $q_i \leqq q < \infty$ for all $i \in S$. The jump process $\{A_{ij}(t), t \geqq 0\}$ from state i to state $j \neq i$ with $q_{ij} > 0$ is defined as the right-continuous process that counts the number of jumps from i to j. Let $A(t)$ denote the vector with components $A_{ij}(t)$, ordered in one way or another. Ergodicity implies that its asymptotic mean λ exists:

$$\lambda = \lim_{t \to \infty} \frac{1}{t} A(t).$$

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Clearly λ is a vector with components λ_{ij} ordered in the same way as the components of A, given by $\lambda_{ij} = \pi_i q_{ij}$. Consider the process

$$Z_n(t) = \frac{1}{\sqrt{n}}[A(nt) - nt\lambda], \qquad t \in [0, \infty), \quad n = 1, 2, \cdots. \tag{1}$$

It is expected that the sequence $Z_n = \{Z_n(t), t \geqq 0\}$ converges weakly to a Gaussian process with zero drift. Our goal is to prove this weak convergence theorem and discuss the structure of the limiting process. For the basic definitions and results on weak convergence of stochastic processes see Billingsley [5], Ethier and Kurtz [8], Pollard [15] or Jacod and Shiryaev [12].

Here is a brief summary of the rest of the paper. We first consider finite-state Markov processes. Let d be the number of pairs (i, j) for which $q_{ij} > 0$. The vectors $A(t)$, λ, $Z_n(t)$ are vectors in $\mathbb{R}^d$. Let $D^d[0, \infty)$ be the space of $\mathbb{R}^d$-valued functions on $[0, \infty)$ which are right continuous and have left limits, equipped with the usual Skorokhod topology. Theorem 1 in Section 2 identifies the weak limit of $Z_n = \{Z_n(t), t \geqq 0\}$ as a sequence of random elements of $D^d[0, \infty)$. This theorem gives explicit formulas for the entries of the covariance matrix of the limiting Gaussian process in terms of the differences of certain sojourn times of the process. In Section 3 we discuss some ways of computing these excess sojourn times.

Let $\mathbb{R}^{\mathbb{N}}$ denote the space of real sequences with the product topology. For processes with countably infinite state space, we first view $A(t)$, λ, $Z_n(t)$ as vectors in $\mathbb{R}^{\mathbb{N}}$. Let $D^{\mathbb{N}}[0, \infty)$ denote the space of $\mathbb{R}^{\mathbb{N}}$-valued functions on $[0, \infty)$ which are right continuous and have left limits. In Theorem 2, we prove that $Z_n = \{Z_n(t), t \geqq 0\}$ converges weakly as a sequence of random elements of $D^{\mathbb{N}}[0, \infty)$. Here we need to impose the additional condition that for some state (and hence every state) the mean time to hit it, starting from the stationary distribution, is finite. The covariance of the limiting Gaussian process is explicitly identified and the formulas in Section 3 can be used to compute these covariances.

We next observe that the marginals of the limiting Gaussian process are in ℓ^2 almost surely. This suggests we consider Z_n as a sequence in $D_{\ell^2}[0, \infty)$, which is the space of mappings from $[0, \infty)$ into ℓ^2 which are right continuous and have left limits. In Theorem 5, we show that Z_n converges weakly as a sequence in $D_{\ell^2}[0, \infty)$. This is an improvement over Theorem 2, because it allows the use of the continuous mapping theorem to derive functional central limit theorems for a larger class of functionals of the path than is allowed by Theorem 2. In Theorem 6 we show that Theorem 5 is tight in some sense. For any $\epsilon > 0$, there are uniform Markov processes (in fact birth and death processes) for which the marginals of the limiting Gaussian process are almost surely *not* in $\ell^{2-\epsilon}$. Finally, Theorem 7 gives a sharpening of Theorem 2 if appropriate conditions are met.

A few simple examples are discussed in Section 4 to illustrate our results. These examples point to the fact that the range of the limiting Gaussian process is often a proper subspace of the sequence spaces in which we have proved our limit theorems. One source of this singularity is the balance equations associated to the transition diagram, which say that the total number of jumps into a state cannot differ from the total number of jumps out of the same state by more than 1. It is natural to investigate if there are any other sources of singularity in the limit. In Theorem 8 and Theorem 9 of Section 5 we characterize the subspace spanned by the limiting Gaussian process for finite-state processes and countable-state processes respectively. The characterization demonstrates that the balance equations just discussed are the only source of singularity in the limiting process.

In Section 6, we discuss an important application that served to motivate much of this work. We discuss the traditional stable Jackson network and use our results to describe the covariances between traffic processes in the network that count the number of customers jumping between pairs of nodes. Simple formulas for these covariances can be derived in terms of certain sojourn times associated to the network.

Throughout, we let $\boldsymbol{E}_i$ denote expectation when $X(0)=i$ and $\boldsymbol{E}_\pi$ when $X(0)\sim\pi$. $\boldsymbol{E}_0$ denotes $\boldsymbol{E}_{j0}$, where j_0 is a fixed state chosen as in Section 2. Statements made without specifying the initial distribution are true for all initial distributions.

2. Limit theorems

We first assume $\{X(t), t\geqq 0\}$ is a finite-state ergodic Markov process. Let m denote the number of states, and d the number of pairs (i,j) with $q_{ij}>0$. Let $Z_n=\{Z_n(t), t\geqq 0\}$ be defined as in the introduction. Let $\tau_j(t)$ be the time spent in state j, on the time interval $[0,t]$. It is easy to see by coupling that the limit

$$\alpha_{ij}=\lim_{t\to\infty}[\boldsymbol{E}_i[\tau_j(t)]-\pi_j t] \tag{2}$$

exists. The following theorem gives explicit formulas for the covariance matrix of the jump processes of the finite-state Markov process $\{X(t), t\geqq 0\}$.

Theorem 1 (finite state space). *Consider the sequence of processes Z_n as random elements of the space $D^d[0,\infty)$ equipped with the Skorokhod topology. Then Z_n converges weakly, as $n\to\infty$, to a zero-mean Brownian motion with covariance matrix C with entries $C_{ij,k\ell}$ given by the following formula:*

$$C_{ij,k\ell}=\begin{cases}q_{ij}q_{k\ell}(\pi_i\alpha_{jk}+\pi_k\alpha_{\ell i}) & \text{if } (i,j)\neq(k,\ell)\\ 2\pi_i q_{ij}^2\alpha_{ji}+\pi_i q_{ij} & \text{if } (i,j)=(k,\ell).\end{cases} \tag{3}$$

Proof. Recall that $\{X(t), t\geqq 0\}$ is piecewise constant and continuous on the right with left limits. Fix a pair of states (i_0,j_0) for which $q_{i_0j_0}>0$ and let $T_1, T_2,\cdots$ denote the increasing sequence of times t such that $X(t-)=i_0$, $X(t)=j_0$. Consider also the counting process

$$A^0(t):=A_{i_0j_0}(t)=\sum_{k=1}^{\infty}1\{T_k\leqq t\}. \tag{4}$$

In the following the superscript or subscript 0 will refer to this specific pair of states (i_0, j_0). Let $\lambda_0 = \pi_{i_0} q_{i_0 j_0}$ denote the rate of the counting process $A^0(t)$. Let

$$\psi_{ij}(n) = A_{ij}(T_n) - A_{ij}(T_{n-1}), \tag{5}$$

and $\psi(n)$ be the vector with entries $\psi_{ij}(n)$, with $q_{ij} > 0$, ordered in the same way as the entries of $A(t)$. Clearly, the random sequence $\{\psi(n), n = 1, 2, \cdots\}$ is an i.i.d. sequence in $\mathbb{R}^d$, provided that the process starts with $X(0) = j_0$. Let $v_{ij} = \boldsymbol{E}_0 \psi_{ij}(n)$. A standard regenerative argument shows that

$$v_{ij} = \frac{\lambda_{ij}}{\lambda_0} = \frac{\pi_i q_{ij}}{\pi_{i_0} q_{i_0 j_0}}. \tag{6}$$

Consider now the $\mathbb{R}^d$-valued process

$$Z_n^0(t) = \frac{1}{\sqrt{n}} \{A(T_{\lfloor nt \rfloor}) - \lambda T_{\lfloor nt \rfloor}\} = \frac{1}{\sqrt{n}} \sum_{m=1}^{\lfloor nt \rfloor} [\psi(m) - \lambda \tau_m], \tag{7}$$

where $\tau_m := T_m - T_{m-1}$. Since $E_0 \tau_n = 1/\lambda_0$, the sequence $\psi(m) - \lambda \tau_m$, $m = 1, 2, \cdots$ is a zero-mean i.i.d. sequence in $\mathbb{R}^d$. It follows from Donsker's invariance principle, see [13], Theorem 5.1.2(c), that

$$Z_n^0 \Rightarrow \sqrt{C^0} W \qquad \text{as } n \to \infty, \tag{8}$$

where W is a standard Brownian motion in $\mathbb{R}^d$ and C^0 is a positive semi-definite matrix with uniquely defined self-adjoint square root $\sqrt{C^0}$, and having entries

$$C^0_{ij,k\ell} = \boldsymbol{E}[\psi_{ij}(n) - \lambda_{ij} \tau_n][\psi_{k\ell}(n) - \lambda_{k\ell} \tau_n]. \tag{9}$$

Although (8) has been established only when $X(0) = j_0$, it is easily seen that it is true for any initial distribution. Having established (8), write $Z_n(t)$ as follows:

$$\begin{aligned} Z_n(t) &= \frac{1}{\sqrt{n}} \{A(nt) - nt\lambda\} \\ &= \frac{1}{\sqrt{n}} \{A(nt) - A(T_{A^0(nt)})\} + \frac{1}{\sqrt{n}} \{A(T_{A^0(nt)}) - \lambda T_{A^0(nt)}\} \\ &\quad + \frac{1}{\sqrt{n}} \{T_{A^0(nt)} - nt\} \lambda. \end{aligned} \tag{10}$$

As $n \to \infty$, $T_{A^0(nt)} - nt$ converges in distribution to a finite random variable distributed like the stationary age of the renewal process A^0. Hence the third term of (10) converges to zero in probability. For the first term, we have

$$0 \leqq A(nt) - A(T_{A^0(nt)}) \leqq A(T_{A^0(nt)+1}) - A(T_{A^0(nt)}) \tag{11}$$

where the inequalities are to be interpreted componentwise. The random sequence

$\{A(T_{A^0(nt)+1})-A(T_{A^0(nt)})\}_n$ is tight, because $A_{ij}(T_{A^0(nt)+1})-A_{ij}(T_{A^0(nt)})$ is the number of jumps from i to j over a typical interval $(T_{A^0(nt)}, T_{A^0(nt)+1}]$ of the renewal process A^0. It follows that the first term of (10) also converges to zero in probability. From Equation (7) we see that the second term of (10) is equal to

$$Z_n^0\left(\frac{A^0(nt)}{n}\right).$$

We now claim the following.

Lemma 1. *The sequence of processes* $\{Z_n^0(A^0(nt)/n),\ t \geqq 0\}$, $n=1,2,\cdots$, *as random elements of* $D^d[0,\infty)$, *converges weakly to a Brownian motion:*

$$\left\{Z_n^0\left(\frac{A^0(nt)}{n}\right)\right\} \Rightarrow \sqrt{\lambda_0 C^0}W.$$

Proof of Lemma 1. The proof involves a random time change argument, as in Billingsley [5], Chapter 3. We first have the fact

$$\text{for all } T>0, \qquad \sup_{t\in[0,T]}\left|\frac{A^0(nt)}{n}-\lambda_0 t\right| \xrightarrow{P} 0, \qquad \text{as } n\to\infty.$$

This implies that the family of processes $\{A^0(nt)/n, t \geqq 0\}_n$, converges weakly in $D^d[0,\infty)$ to the continuous deterministic process $\{\lambda_0 t, t\geqq 0\}$. Furthermore, from (8), we have $Z_n^0 \Rightarrow \sqrt{C^0}W$ as $n\to\infty$, where W is a Brownian motion, and hence continuous. The process $Z_n^0(A^0(nt)/n)$ is now the composition of two processes in $D^d[0,\infty)$. Since the composition operation $D^d[0,\infty)\times D^d[0,\infty)\to D^d[0,\infty)$ is continuous whenever the two arguments are continuous functions, it follows from the continuous mapping theorem that $\{Z_n^0(A^0(nt)/n)\}$ converges to $\{\sqrt{C^0}W(\lambda_0 t), t\geqq 0\}$. But the latter has the same distribution as $\sqrt{\lambda_0 C^0}W$. This proves the lemma.

From Lemma 1 and the preceding observations we conclude that $\{Z_n\}$ converges weakly to a zero-mean Brownian motion in $\mathbb{R}^d$ with covariance matrix

$$C=\lambda_0 C^0. \tag{12}$$

At first sight it seems that C depends on the rate of the chosen embedded process, but, of course, this is not the case.

Lemma 2. *The covariance* C *of the limiting Brownian motion, defined by* (9) *and* (12), *is also given by*

$$C_{ij,k\ell}=\lim_{t\to\infty}\frac{1}{t}\boldsymbol{E}[A_{ij}(t)-\lambda_{ij}t][A_{k\ell}(t)-\lambda_{k\ell}t].$$

Proof of Lemma 2. It is easily seen that the right-hand side does not depend on the initial distribution. Letting $o(t)$ be a function such that $o(t)/t \to 0$, as $t \to \infty$, we have:

$$
\begin{aligned}
&\boldsymbol{E}_0[A_{ij}(t) - \lambda_{ij}t][A_{k\ell}(t) - \lambda_{k\ell}t] \\
&\qquad = \boldsymbol{E}_0\left[\sum_{m=1}^{A^0(t)} \psi_{ij}(m) - \lambda_{ij}T_{A^0(t)}\right]\left[\sum_{m=1}^{A^0(t)} \psi_{k\ell}(m) - \lambda_{k\ell}T_{A^0(t)}\right] + o(t) \\
&\qquad = \boldsymbol{E}_0 \sum_{m=1}^{A^0(t)} \xi_m \sum_{m=1}^{A^0(t)} \eta_m + o(t),
\end{aligned}
\tag{13}
$$

where $\xi_m = \psi_{ij}(m) - \lambda_{ij}\tau_m$ and $\eta_m = \psi_{k\ell}(m) - \lambda_{k\ell}\tau_m$. Now

$$
\sum_{m=1}^{n} \xi_m \sum_{m=1}^{n} \eta_m - n\boldsymbol{E}[\xi_0\eta_0], \qquad n = 1, 2, \ldots
$$

is a martingale. From the optional sampling theorem it follows that (13) equals

$$
\boldsymbol{E}_0[A^0(t)]\boldsymbol{E}_0[\xi_0\eta_0] + o(t) = t\lambda_0 C^0_{ij,k\ell} + o(t).
$$

The conclusion follows after dividing by t and letting $t \to \infty$.

The final step is the computation of $C_{ij,k\ell}$. To this end, we use Lemma 2 and the Markov property of X. Regarding A_{ij}, $A_{k\ell}$ as random integer-valued measures, we write:

$$
\boldsymbol{E}_\pi A_{ij}(t)A_{k\ell}(t) = \boldsymbol{E}_\pi \int_0^t A_{ij}(ds) \int_0^t A_{k\ell}(du).
$$

Now split the integral into the integrals over $\{s < u\}$, $\{s > u\}$, and $\{s = u\}$. The last part is non-zero if and only if $(i, j) = (k, \ell)$, in which case it equals

$$
\boldsymbol{E}_\pi \int_0^t dA_{ij}(s) = \pi_i q_{ij} t. \tag{14}
$$

The integral over $\{s < u\}$ is

$$
\begin{aligned}
\boldsymbol{E}_\pi \iint_{0 \leqq s < u \leqq t} A_{ij}(ds)A_{k\ell}(du) &= \boldsymbol{E}_\pi \int_0^t A_{k\ell}(du) \int_0^u A_{ij}(ds) \\
&= \boldsymbol{E}_\pi \int_0^t A_{ij}(u)A_{k\ell}(du) \\
&= \boldsymbol{E}_\pi \int_0^t A_{ij}(u) 1\{X(u) = k\} q_{k\ell}\, du,
\end{aligned}
\tag{15}
$$

where the last formula follows from the fact that $A_{k\ell}(u) - \int_0^t 1\{X(u) = k\} q_{k\ell} du$ is a zero-mean martingale with respect to the filtration $\sigma\{X(u),\ u \leqq t\}$. To compute $\boldsymbol{E}_\pi[A_{ij}(u)1\{X(u) = k\}]$, fix u and let $\tilde{X}(s) = X(u - s)$, $0 \leqq s \leqq u$, be the reverse time process over the interval $[0, u]$. This has transition rates $\tilde{q}_{ji} = \pi_i q_{ij} / \pi_j$ and, of course, the same stationary distribution. So, by Bayes' formula,

$$
\begin{aligned}
\boldsymbol{E}_\pi[A_{ij}(u) 1\{X(u) = k\}] &= \pi_k \tilde{E}_k[\tilde{A}_{ji}(u)] \\
&= \pi_k \tilde{\boldsymbol{E}}_k \int_0^u \tilde{A}_{ji}(ds) \\
&= \pi_k \tilde{\boldsymbol{E}}_k \int_0^u \tilde{q}_{ji} 1\{\tilde{X}(s) = j\}\, ds \\
&= \pi_k \int_0^u \tilde{q}_{ji} \tilde{\boldsymbol{P}}_k(\tilde{X}(s) = j)\, ds \\
&= \pi_k \int_0^u \frac{\pi_i q_{ij}}{\pi_j} \frac{\pi_j \boldsymbol{P}_j(X(s) = k)}{\pi_k}\, ds.
\end{aligned}
\tag{16}
$$

Let $\boldsymbol{P}_s(j, k)$ denote $\boldsymbol{P}_j(X(s) = k)$. Substituting (16) into (15) gives the following expression for the integral over $\{s < u\}$:

$$\pi_i q_{ij} q_{k\ell} \int_0^t (t - s) \boldsymbol{P}_s(j, k)\, ds.$$

The integral over $\{s > u\}$ is similar. Hence, if $(i, j) \neq (k, \ell)$ we get

$$
\begin{aligned}
\boldsymbol{E}_\pi A_{ij}(t) A_{k\ell}(t) - \lambda_{ij} \lambda_{k\ell} t^2 &= \pi_i q_{ij} q_{k\ell} \int_0^t (t - s)(\boldsymbol{P}_s(j, k) - \pi_k)\, ds \\
&\quad + \pi_k q_{ij} q_{k\ell} \int_0^t (t - s)(\boldsymbol{P}_s(l, i) - \pi_\ell)\, ds
\end{aligned}
\tag{17}
$$

where we have used the obvious formula $t^2 = 2\int_0^t (t - s)\, ds$. Let

$$F_{jk}(t) = \int_0^t (\boldsymbol{P}_s(j, k) - \pi_k)\, ds.$$

Integration by parts gives

$$\int_0^t (t - s)(\boldsymbol{P}_s(j, k) - \pi_k)\, ds = \int_0^t F_{jk}(s)\, ds. \tag{18}$$

Substitution of (18) into (17) then gives

$$\boldsymbol{E}_\pi A_{ij}(t) A_{k\ell}(t) - \lambda_{ij} \lambda_{k\ell} t^2 = \pi_i q_{ij} q_{k\ell} \int_0^t F_{jk}(s)\, ds + \pi_k q_{ij} q_{k\ell} \int_0^t F_{\ell i}(s)\, ds.$$

Thus, if $(i, j) \neq (k, \ell)$, we have

$$\begin{aligned} C_{ij,k\ell} &= \lim_{t\to\infty} \frac{1}{t} [EA_{ij}(t)A_{k\ell}(t) - \lambda_{ij}\lambda_{k\ell}t^2] \\ &= \pi_i q_{ij} q_{k\ell} \lim_{t\to\infty} \frac{1}{t} \int_0^t F_{jk}(s)\, ds + \pi_k q_{ij} q_{k\ell} \lim_{t\to\infty} \frac{1}{t} \int_0^t F_{\ell i}(s)\, ds \\ &= \pi_i q_{ij} q_{k\ell} \alpha_{jk} + \pi_k q_{ij} q_{k\ell} \alpha_{\ell i}, \end{aligned}$$

whereas, if $(i, j) = (k, \ell)$ the additional term from (14) gives

$$C_{ij,ij} = 2\pi_i q_{ij}^2 \alpha_{ji} + \pi_i q_{ij}.$$

This completes the proof of Theorem 1.

Let now $\{X(t), t \geqq 0\}$ be a uniform and ergodic Markov process with countable state space S. We next turn to central limit theorems for jump processes counting collections of jumps of $\{X(t), t \geqq 0\}$. For convenience, let us order the states of S in some way as $S = \{1, 2, 3, \cdots\}$. Let d_K denote the number of feasible jumps between pairs of states in $\{1, \cdots, K\}$. Define a projection

$$s_K : \mathbb{R}^{\mathbb{N}} \to \mathbb{R}^{d_K}$$

by erasing all but the first d_K coordinates of a vector in $\mathbb{R}^{\mathbb{N}}$. Even though the process is uniform, it may be the case that the excess sojourn times α_{ij} are infinite. To exclude this from happening, we shall consider only Markov processes with $\alpha_{ij} < \infty$ for all i, j. From the analysis that follows in this and the next section, it will be seen that it suffices to assume that the mean hitting time to some state j is finite, given that the process starts with the stationary distribution. Then the following theorem follows easily from the finite state space Theorem 1.

Theorem 2 (countable state space). *Consider the sequence of processes Z_n as random elements of the space $D^{\mathbb{N}}[0, \infty)$. Then Z_n converges weakly, as $n \to \infty$. The limit distribution has continuous paths almost surely, and is uniquely characterized as a zero-mean $\mathbb{R}^{\mathbb{N}}$-valued Gaussian process with independent increments, and having covariance matrix C given by* (3).

Proof. Kuelbs [13] has proved that a functional central limit theorem for a sequence of i.i.d. random variables taking values in a complete separable Banach space B holds iff the usual central limit theorem holds. The result of Kuelbs is stated for the measures on the space $C_B[0, 1]$ of continuous B-valued functions on $[0, 1]$, but it is easily seen that the result extends to $D_B[0, \infty)$ with the Skorokhod topology. Further, the limiting process is continuous. Our theorem is therefore most easily proved by considering the process $(Z_n^0(t), t \geqq 0)$, which is constructed from a sum of independent and identically distributed variables when the process starts at j_0. Let $A^K(t)$, λ^K, $Z_n^{0,K}(t)$ and $Z_n^K(t)$ denote the $\mathbb{R}^{d_K}$-valued vectors $s_K(A(t))$, $s_K(\lambda)$, $s_K(Z_n^0(t))$

and $s_K(Z_n(t))$ respectively, where the definition of $Z_n^0(t)$ is made as in the proof of Theorem 1 by picking (i_0, j_0) for which $q_{i_0 j_0} > 0$. Note that

$$Z_n^{0,K}(t) = \frac{1}{\sqrt{n}}\{A^K(T_{\lfloor nt \rfloor}) - \lambda^K T_{\lfloor nt \rfloor}\} = \frac{1}{\sqrt{n}} \sum_{m=1}^{\lfloor nt \rfloor} [\psi^K(m) - \lambda^K \tau_m], \tag{19}$$

where $\tau_m := T_m - T_{m-1}$ and $\psi^K = s_K(\psi)$, with ψ defined as in (5). As before, $\psi^K(m) - \lambda^K \tau_m$, $m = 1, 2, \ldots$ is a zero-mean i.i.d. sequence in $\mathbb{R}^{d_K}$ when $X(0) = j_0$, so by Donsker's invariance principle, we have

$$Z_n^{0,K} \Rightarrow \sqrt{C^{0,K}}W \qquad \text{as } n \to \infty, \tag{20}$$

where W is a standard Brownian motion in $\mathbb{R}^{d_K}$ and C^0 has exactly the same expression in terms of the jump rates, stationary distribution and the α_{ij}. From this it follows easily that $Z_n^{0,K}(t)$ converges in $\mathbb{R}^{\mathbb{N}}$ to a limiting Gaussian distribution having covariance matrix with entries determined by the restrictions $tC^{0,K}$ of tC^0. Now the result of [13] gives the weak convergence in $D^{\mathbb{N}}[0, \infty)$ of $\{Z_n^0(t), t \geqq 0\}$ to an independent-increments zero-mean Gaussian process with continuous paths having covariance matrix C^0. The same time change argument carries over word for word to give weak convergence of $\{Z_n(t),\ t \geqq 0\}$ to a continuous-path zero-mean Gaussian process with covariance matrix $C = \lambda_0 C^0$, whose entries $C_{ij,k\ell}$ are given by (3).

The value of a functional central limit theorem is that it allows one to immediately write down central limit theorems for various functionals of the path. In applications, we are interested in taking weighted sums of the jump processes above and identifying the central limit fluctuations of such sums. In order to derive limit theorems for such sums from an underlying functional central limit theorem, we need to work with an appropriate topology on the underlying space.

For instance, let us examine Theorem 2 in more detail. Since the topology used on the marginals is the product topology on sequence space, we can only derive central limit theorems for sums of a finite number of the basic counting processes. In typical applications this is not enough. For example, in the Jackson network example discussed in more detail in Section 6, to derive functional central limit theorems for the traffic process of customers jumping from one node of the network to another, we need to be able to add up the counts from an infinite number of types of jumps. With this end in view, we now turn to refinements of Theorem 2 that allow more continuous functionals to be handled.

In the following, our main reference for discussion of Gaussian measures on sequence spaces will be [17], especially Chapters 1 and 2. For the basic facts about sequence spaces, see for example [16]. Note in particular that c_0 denotes the space of sequences converging to zero, with the topology of uniform convergence. The following result is Theorem 2.2.5 of [17] and is central to extending the scope of Theorem 2.

Theorem 3. *For a Gaussian distribution on $\mathbb{R}^{\mathbb{N}}$ with mean sequence $\{m_k\}$ and* *covariance matrix* $\{s_{jk}\}$ *to be concentrated on* ℓ^p, $1 \leqq p < \infty$, *it is necessary and sufficient that* $\{m_k\}_k \in \ell^p$ *and* $\{s_{kk}\}_k \in \ell^{p/2}$.

As an immediate consequence, we have the following.

Theorem 4 (marginals of the limit). *For any* $0 \leqq s < t$, *the marginal distribution of the weak limit of* Z_n *on* $D^{\mathbb{N}}[0,\infty)$ *is concentrated on* ℓ^2. *Hence it is also concentrated on* ℓ^p *for* $2 \leqq p \leqq \infty$ *and on* c_0.

Proof. Since the marginal distribution is Gaussian and has zero mean, by Theorem 3 it suffices to verify that the diagonal of the covariance matrix is in ℓ^1. Equivalently, by virtue of (12) we need to prove:

$$\sum_{ij} C^0_{ij,ij} < \infty, \tag{21}$$

or, using (9),

$$\sum_{ij} \boldsymbol{E}_0[\psi_{ij}(1) - \lambda_{ij}\tau_1]^2 < \infty.$$

Let $\tau_i(n)$ denote the time spent in state i on the interval $[T_{n-1}, T_n)$. We write

$$\sum_{ij} \boldsymbol{E}_0[\psi_{ij}(n) - \lambda_{ij}\tau_n]^2 \leqq 2\sum_{ij} \boldsymbol{E}_0[\psi_{ij}(n) - q_{ij}\tau_i(n)]^2 + 2\sum_{ij} q_{ij}^2 \boldsymbol{E}_0[\tau_i(n) - \pi_i\tau_n]^2. \tag{22}$$

Since

$$\begin{aligned}\sum_i \boldsymbol{E}_0[\tau_i(n) - \pi_i\tau_n]^2 &\leqq \sum_i \boldsymbol{E}_0[\tau_i(n)]^2 + \sum_i \pi_i^2 \boldsymbol{E}_0[\tau_n]^2 \\ &\leqq \boldsymbol{E}_0\Big[\sum_i \tau_i(n)\Big]^2 + \boldsymbol{E}_0[\tau_n]^2 \\ &= 2\boldsymbol{E}_0[\tau_n]^2,\end{aligned}$$

we have

$$\begin{aligned}\sum_{ij} q_{ij}^2 \boldsymbol{E}_0[\tau_i(n) - \pi_i\tau_n]^2 &\leqq q\sum_i \boldsymbol{E}_0[\tau_i(n) - \pi_i\tau_n]^2 \sum_{j\neq i} q_{ij} \\ &\leqq q^2 \sum_i \boldsymbol{E}_0[\tau_i(n) - \pi_i\tau_n]^2 \\ &\leqq 2q^2\boldsymbol{E}_0[\tau_n]^2,\end{aligned}$$

so the second term on the right of (22) is finite. As for the first term, we may realize the process on a sample space supporting independent Poisson processes N_{ij} of rates q_{ij} in the obvious way, permitting us to write

$$\begin{aligned}\boldsymbol{E}_0[\psi_{ij}(1) - q_{ij}\tau_i(1)]^2 &= \boldsymbol{E}_0\left[\int_0^{\tau_1} 1(X(s-)=i)[N_{ij}(ds) - q_{ij}\,ds]\right]^2 \\ &= \boldsymbol{E}_0\left[\int_0^{\tau_1} 1(X(s-)=i)q_{ij}\,ds\right] \\ &= q_{ij}\boldsymbol{E}_0[\tau_i(1)]\end{aligned}$$

so that

$$\begin{aligned}\sum_{ij} \boldsymbol{E}_0[\psi_{ij}(n) - q_{ij}\tau_i(n)]^2 &\leqq \sum_{ij} q_{ij}\boldsymbol{E}_0[\tau_i(1)] \\ &= \sum_i \boldsymbol{E}_0[\tau_i(1)]\sum_{j\neq i} q_{ij} \\ &\leqq q\boldsymbol{E}_0[\tau_1],\end{aligned}$$

and the first term on the right of (22) is also finite. This completes the proof.

In view of Theorem 4, we may ask if the weak convergence of Theorem 2 holds in $D_{\ell^2}[0, \infty)$. This is indeed the case.

Theorem 5 (strengthening the limit theorem). *Consider Z_n as random elements of the space $D_{\ell^2}[0, \infty)$ endowed with the Skorokhod topology. Then Z_n converges weakly, as $n \to \infty$ to the zero-mean independent increments Gaussian process described in Theorem 2. The corresponding statement is also true in $D_B[0, \infty)$, where B denotes one of the ℓ^p spaces $2 \leqq p \leqq \infty$ or c_0.*

Proof. Since ℓ^2 is complete and separable, we may appeal to the result of Kuelbs [13] for sequences of i.i.d. random variables to reduce the problem to the one of showing that the central limit theorem for the sequence $\{\psi(n) - \lambda\tau_n\}$ holds in ℓ^2. Note that this sequence is ℓ^2-valued, by virtue of (9) and Theorem 3. From Theorem 2, we know that this sequence obeys the central limit theorem in $\mathbb{R}^{\mathbb{N}}$ with a limit in ℓ^2. It follows from Theorem 2.3.2 of [17] that it obeys the central limit theorem in ℓ^2.

Finally, let B be one of the spaces ℓ^p, $2 \leqq p \leqq \infty$, or the space c_0. Note that ℓ^∞ and c_0 are not separable. Nevertheless, it suffices to observe that the inclusion $D_{\ell^2}[0, \infty) \to D_B[0, \infty)$ is continuous, and apply the continuous mapping theorem ([5], Theorem 5.1; [15]) to complete the proof of the last statement.

Using Theorem 5, we can derive a central limit theorem for weighted sums of the different types of jumps as long as the weighting sequence is in ℓ^2. This is an improvement over Theorem 2 and can lead to interesting conclusions in examples. For network examples, however, such as the Jackson network example discussed just before Theorem 3, we would like to be able to handle weighting sequences in

ℓ^∞. We therefore ask if Theorem 5 can be strengthened. It turns out this is not possible in general for the class of uniform ergodic countable-state Markov processes. The next result shows that Theorem 5 is tight in some sense. This result depends on a characterization of the excess sojourn times α_{ij} that is derived in Section 3, but is placed here because it properly belongs with the other results in this section.

Theorem 6. *For any $\epsilon > 0$, there is a birth and death process for which the limiting Gaussian process of Theorem* 2 *has marginals that do not belong to $\ell^{2-\epsilon}$.*

Proof. By Theorem 3 it suffices to show that for any $\epsilon > 0$ there is a birth and death process for which the diagonal of the covariance matrix given by (3) is not in $\ell^{1-\epsilon}$. Fix $\delta > 0$ and consider the birth and death process on $\{0, 1, 2, \cdots\}$ with up-rates λ_k, $k \geqq 0$ and down-rates μ_k, $k \geqq 1$, given by

$$\lambda_0 = 1$$

$$\lambda_k = \left(\frac{k}{k+1}\right)^{1+\delta}, \qquad k \geqq 1$$

$$\mu_k = 1, \qquad k \geqq 1.$$

It is elementary to see that the process is uniform and positive recurrent, with stationary distribution given by

$$\pi_k = \pi_0 k^{-(1+\delta)}, \qquad k \geqq 1$$

$$\pi_0 = \left[1 + \sum_{k=1}^{\infty} k^{-(1+\delta)}\right]^{-1}.$$

In Section 3 the following probabilistic characterization of the excess sojourn times is given, see Equations (29) and (30).

$$\alpha_{ij} = \pi_j(\boldsymbol{E}_\pi \sigma_j - \boldsymbol{E}_i \sigma_j), \qquad i \neq j \tag{23}$$

$$\alpha_{jj} = \pi_j \boldsymbol{E}_\pi \sigma_j, \tag{24}$$

where σ_j denotes the first hitting time of state j. The reader is referred to Section 3 for the derivation of these formulas. From (3) what we need to demonstrate is that for any $\epsilon > 0$, we can choose $\delta > 0$ so that

$$\sum_{ij} (2\pi_i q_{ij}^2 \alpha_{ji} + \pi_i q_{ij})^{1-\epsilon} = \infty,$$

and also $\boldsymbol{E}_\pi \sigma_j < \infty$. It can be seen that the latter holds for all large enough δ. So, in view of (23), and because the process is birth and death, it is enough to show that

$$\sum_{i=1}^{\infty} (2\pi_i \mu_i^2 [\pi_i(\boldsymbol{E}_\pi \sigma_i - \boldsymbol{E}_{i-1}\sigma_i)] + \pi_i \mu_i)^{1-\epsilon} = \infty.$$

Notice that $\mu_i = 1$ for all $i \geqq 1$. This leads to the observation that if we can demonstrate

$$E_\pi \sigma_i - E_{i-1}\sigma_i > 0 \qquad \text{for all sufficiently large } i, \tag{25}$$

then choosing δ so that $(1+\delta)(1-\epsilon) < 1$, the fact that $\sum_{i=1}^\infty \pi_i^{1-\epsilon} = \infty$ is itself enough to yield the claim. We proceed to demonstrate (25). First observe that for any $k \leqq i$, we have $E_i\sigma_{i+1} \leqq E_k\sigma_{i+1}$. Fix K, to be chosen later, and let $i > K$. We may write

$$\begin{aligned} E_i\sigma_{i+1} &= \frac{1}{\lambda_i+\mu_i} + \frac{\mu_i}{\lambda_i+\mu_i} E_{i-1}\sigma_{i+1} \\ &\leqq \frac{1}{\lambda_i+\mu_i} + \frac{\mu_i}{\lambda_i+\mu_i} \frac{\sum_{k=0}^K \pi_k E_k \sigma_{i+1}}{\sum_{k=0}^K \pi_k}. \end{aligned}$$

Hence

$$\begin{aligned} E_\pi\sigma_{i+1} - E_i\sigma_{i+1} &\geqq \sum_{k=0}^K \pi_k E_k \sigma_{i+1} - E_i\sigma_{i+1} \\ &\geqq \sum_{k=0}^K \pi_k E_k\sigma_{i+1} - \frac{1}{\lambda_i+\mu_i} - \frac{\mu_i}{\lambda_i+\mu_i}\frac{\sum_{k=0}^K \pi_k E_k\sigma_{i+1}}{\sum_{k=0}^K \pi_k} \\ &= \sum_{k=0}^K \pi_k\left(1 - \frac{\mu_i}{\lambda_i+\mu_i}\frac{1}{\sum_{k=0}^K \pi_k}\right) E_k\sigma_{i+1} - \frac{1}{\lambda_i+\mu_i}. \end{aligned}$$

Choose K so large that $\sum_{k=0}^K \pi_k > \frac{2}{3}$. Since $\lim_{i\to\infty} E_k\sigma_{i+1} = \infty$ for all $0 \leqq k \leqq K$, and since $\lim_{i\to\infty}\{\mu_i/(\lambda_i+\mu_i)\} = \frac{1}{2}$, it follows that the right-hand side of (26) is strictly positive for all large enough i. This concludes the proof of the theorem.

While we cannot expect a sharper result than that of Theorem 5 in full generality, we get sharper results if the covariances satisfy the appropriate condition. The result of the following theorem will be applied to Jackson networks in Section 6.

Theorem 7. Let $1 \leqq p < 2$, *and suppose* $\sum_{ij} C_{ij,ij}^{p/2} < \infty$. *Then as* $n \to \infty$, Z_n *converges weakly as random elements of* $D_{\ell^p}[0,\infty)$, *to the zero-mean independent increments Gaussian process described in Theorem* 2.

Proof. By virtue of (9), (12), and Theorem 3, the sequence $\{\psi(n) - \lambda\tau_n\}$ is ℓ^p-valued. From Theorem 2, we know that this sequence obeys the central limit theorem in $\mathbb{R}^{\mathbb{N}}$ with a limit in ℓ^p. It follows from Theorem 2.3.2 of [17] that it obeys the central limit theorem in ℓ^p. Since ℓ^p is complete and separable, we may appeal to the result of Kuelbs [13] to conclude.

3. Characterizing the excess sojourn times

In this section we give a characterization of the excess sojourn times α_{ij} defined in Equation (2) in terms of the rate matrix of the process. This characterization is useful in calculations, as will be demonstrated in some simple examples in Section 4.

It is quite straightforward to derive, and is discussed in some detail only for the convenience of the reader. Clearly we have the first step equation

$$\boldsymbol{E}_i[\tau_j(t)] = \sum_{k \neq i} \int_0^t q_{ik} \exp(-q_i s) \boldsymbol{E}_k[\tau_j(t-s)]\, ds$$

where $i \neq j$. Let $D_{ij}(w)$ denote the Laplace transform of the excess sojourn time function $\boldsymbol{E}_i[\tau_j(t)] - \pi_j t$. Then for $i \neq j$ we have

$$\begin{aligned}
D_{ij}(w) &= \int_0^\infty \exp(-wt)[\boldsymbol{E}_i[\tau_j(t)] - \pi_j t]\, dt \\
&= \sum_{k \neq i} \int_0^\infty \exp(-wt) \int_0^t q_{ik} \exp(-q_i s) \boldsymbol{E}_k[\tau_j(t-s)]\, ds\, dt - \frac{\pi_j}{w^2} \\
&= \sum_{k \neq i} \int_0^\infty q_{ik} \exp(-(q_i + w)s) \int_s^\infty \exp(-w(t-s)) \boldsymbol{E}_k[\tau_j(t-s)]\, dt\, ds - \frac{\pi_j}{w^2} \\
&= \sum_{k \neq i} \frac{q_{ik}}{q_i + w} \left[D_{kj}(w) + \frac{\pi_j}{w^2} \right] - \frac{\pi_j}{w^2} \\
&= \sum_{k \neq i} \frac{q_{ik}}{q_i + w} D_{kj}(w) - \frac{\pi_j}{w(q_i + w)}.
\end{aligned}$$

Since we assume $\alpha_{ij} = \lim_{t \to \infty} \{E_i \tau_j(t) - \pi_j t\}$ exists and is finite for all i, j, we have

$$\alpha_{ij} = \lim_{w \to 0} w D_{ij}(w) = \sum_{k \neq i} \frac{q_{ik}}{q_i} \alpha_{kj} - \frac{\pi_j}{q_i}. \tag{26}$$

For each state $j \in S$ the equations (26) may be written in matrix form for the relative sojourn times $\alpha_{ij} - \alpha_{jj}$, $i \neq j$. Let α^j and μ^j be the column vectors whose components are given by

$$\alpha_i^j = (\alpha_{ij} - \alpha_{jj}) \delta_{ij}$$

$$\mu_i^j = q_j^{-1} \delta_{ij},$$

where $\delta_{ij} = 1$ if $i = j$ and 0 otherwise, and let Γ^j denote a matrix with entries

$$\Gamma_{ik}^j = \frac{q_{ik}}{q_i}, \qquad i \neq k,\ i \neq j,\ k \neq j.$$

Thus, Γ^j is the matrix obtained by deleting the jth row and jth column from the transition matrix of the discrete time Markov chain obtained by observing the states of $X(t)$ after each jump. Then it is easily seen that (26) can be written as

$$\alpha^j = \Gamma^j \alpha^j - \pi_j \mu^j. \tag{27}$$

$I - \Gamma^j$ is invertible, where I denotes the identity matrix. Note that the $k\ell$ entry of

$(I-\Gamma^j)^{-1}$ is just the mean number of visits to state ℓ starting in state k before being absorbed in state j; this is always finite. It follows that $\alpha_{ij}-\alpha_{jj}$, $i\neq j$, are uniquely determined by (27). To complete the determination of the excess sojourn times in terms of the rate matrix we further observe that for each $j\in S$ we have

$$\sum_{i\in S}\pi_i\alpha_{ij}=0.$$

This gives

$$\alpha_{jj}+\sum_{i\neq j}\pi_i(\alpha_{ij}-\alpha_{jj})=0 \tag{28}$$

which determines α_{jj}, and hence all the α_{ij}, $i\in S$. Letting e^i denote the unit vector (row) in the direction i we can write the following explicit formula for α_{ij}:

$$\alpha_{ij}=\pi_j(\pi^j-e^i)(I-\Gamma^j)^{-1}\mu^j.$$

The excess sojourn times also admit another probabilistic interpretation. Let σ_j denote the hitting time of state j, and $\phi_i^j=\boldsymbol{E}_i\sigma_j$. Let ϕ^j be a column vector whose components are given by

$$\phi^j=\phi_i^j\delta_{ij}.$$

We then have

$$\phi^j=(I-\Gamma^j)^{-1}\mu^j.$$

Writing the inverse of $I-\Gamma^j$ as $\sum_{k=0}^{\infty}(\Gamma^j)^k$ and using (27) gives

$$\alpha_{ij}-\alpha_{jj}=-\pi_j\boldsymbol{E}_i[\sigma_j],\qquad i\neq j.$$

Substituting in (28) we also have

$$\alpha_{jj}=\pi_j\boldsymbol{E}_\pi[\sigma_j] \tag{29}$$

Hence

$$\alpha_{ij}=\pi_j(\boldsymbol{E}_\pi[\sigma_j]-\boldsymbol{E}_i[\sigma_j]),\qquad i\neq j. \tag{30}$$

The discrete-time version of (29)–(30) is further discussed in [1], Proposition 7.1.

4. Simple examples

In this section we carry out computations for some simple examples; a two-state process, a random walk on a complete graph, the countable-leafed flower, a general birth and death process, and an M/M/1 queue. These examples will also serve to motivate the discussion in Section 5.

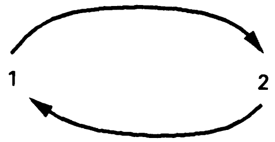

Figure 1. A two-state process

4.1. *Two-state process.* Let $S = \{1, 2\}$, and let $\{X(t), t > 0\}$ be described by the rate diagram of Figure 1. The rate matrix and the stationary distribution of this process are respectively

$$Q = \begin{bmatrix} -\lambda & \lambda \\ \mu & -\mu \end{bmatrix}$$

$$\pi = \left[\frac{\mu}{\lambda + \mu} \quad \frac{\lambda}{\lambda + \mu}\right].$$

From (27) we get

$$\alpha^1 = \alpha_{21} - \alpha_{11} = \frac{-1}{\lambda + \mu}$$

$$\alpha^2 = \alpha_{12} - \alpha_{22} = \frac{-1}{\lambda + \mu}.$$

Together with (28) this gives

$$\begin{bmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{bmatrix} = \begin{bmatrix} \dfrac{\lambda}{(\lambda + \mu)^2} & \dfrac{-\lambda}{(\lambda + \mu)^2} \\ \dfrac{-\mu}{(\lambda + \mu)^2} & \dfrac{\mu}{(\lambda + \mu)^2} \end{bmatrix}. \tag{31}$$

The counting process associated to $(X(t),\ t \geqq 0)$ is $\mathbb{R}^2$-valued, with the first and second coordinates counting the (12) and (21) jumps respectively. The covariance matrix of the limiting Brownian motion can now be identified from (31) and (3). We get

$$\begin{bmatrix} C_{12,12} & C_{12,21} \\ C_{21,12} & C_{21,21} \end{bmatrix} = \begin{bmatrix} \dfrac{\lambda\mu(\lambda^2 + \mu^2)}{(\lambda + \mu)^3} & \dfrac{\lambda\mu(\lambda^2 + \mu^2)}{(\lambda + \mu)^3} \\ \dfrac{\lambda\mu(\lambda^2 + \mu^2)}{(\lambda + \mu)^3} & \dfrac{\lambda\mu(\lambda^2 + \mu^2)}{(\lambda + \mu)^3} \end{bmatrix}.$$

Note that the rank of the limiting Brownian motion is 1 and that it lives on the subspace of $\mathbb{R}^2$ given by the linear equation $x_1 = x_2$. Also notice that in this process the count of $(1, 2)$ jumps cannot differ from that of $(2, 1)$ jumps by more than 1. More will be made of this observation in Section 5.

4.2. *Continuous-time random walk on a complete graph.* Consider a symmetric random walk on a complete graph on n nodes. Here each node has stationary distribution $\pi_i = (1/n)$. Further $\alpha_{ij} - \alpha_{jj}$ is clearly the same for all $i \neq j$, and for all j. Let this common number be denoted α. From (27) we have

$$\alpha = \frac{n-2}{n-1}\alpha - \frac{1}{n}$$

which gives $\alpha = -(n-1)/n$. From (28) we have $\alpha_{jj} = (n-1)^2/n^2$ and $\alpha_{ij} = -(n-1)/n^2$ for $i \neq j$. This allows us to compute the covariance matrix of the limiting Brownian motion, based on Theorem 1. We get

$$C_{ij,ij} = \frac{n^2-2}{n^3(n-1)}$$

$$C_{ij,ji} = \frac{2}{n^3}$$

$$C_{ij,ik} = -\frac{2}{n^3(n-1)} \qquad \text{where } k \neq j$$

$$C_{ij,kj} = -\frac{2}{n^3(n-1)} \qquad \text{where } k \neq i$$

$$C_{ij,jk} = \frac{n-2}{n^3(n-1)} \qquad \text{where } k \neq i$$

$$C_{ij,kl} = -\frac{2}{n^3(n-1)} \qquad \text{where } i, j, k, \ell \text{ are distinct.}$$

From this we learn, for example, that the excess over the mean of (ij) jumps is positively correlated with jumps into i and jumps out of j, but negatively correlated with all other kinds of jumps.

The covariance matrix can be written out as an $n(n-1) \times n(n-1)$ matrix which determines a symmetric Toeplitz form. One can easily write vectors that null the form, for example, if $w_{ij} = \delta_{ij,k\ell} - \delta_{ij,\ell k}$, then

$$\sum_{ij,\overline{ij}} w_{ij} C_{ij,\overline{ij}} w_{\overline{ij}} = 0.$$

One might ask for a complete description of all the vectors which null this form. Such a description is an immediate consequence of Theorem 8 in Section 5 below. In particular, the rank of this form is $(n-1)^2$.

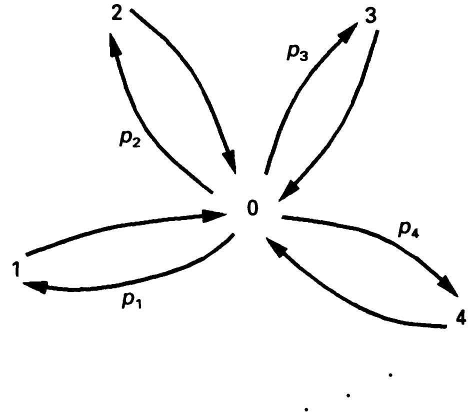

Figure 2. Countable-leafed flower

4.3. *Countable-leafed flower.* Let $S = \{0, 1, 2, \cdots\}$ and let $(X(t), t \geqq 0)$ be described by the rate diagram of Figure 2. The transition rates are given by

$$q_{0i} = p_i$$
$$q_{i0} = 1, \qquad i = 1, 2, \cdots$$

where $p_i > 0$, and $\sum_{i=1}^{\infty} p_i = 1$. The stationary distribution is given by

$$\pi_0 = \tfrac{1}{2}, \qquad \pi_i = \tfrac{1}{2} p_i, \qquad i = 1, 2, \cdots.$$

The mean time to visit 0 starting from stationarity is $\frac{1}{2}$, so Theorem 2 applies. A straightforward computation yields

$$E_\pi \sigma_0 = \tfrac{1}{2}, \qquad E_\pi \sigma_i = \frac{2}{p_i} - \frac{3}{2}, \qquad i = 1, 2, \cdots,$$

$$E_0 \sigma_i = \frac{2}{p_i} - 1, \qquad E_i \sigma_0 = 1, \qquad i = 1, 2, \cdots,$$

$$E_i \sigma_j = \frac{2}{p_j}, \qquad i, j \in \{1, 2, \cdots\}.$$

From this one calculates, using (29) and (30) that

$$\alpha_{00} = \tfrac{1}{4}, \qquad \alpha_{ii} = 1 - \frac{3p_i}{4}, \qquad i = 1, 2, \cdots,$$

$$\alpha_{0i} = -\frac{p_i}{4}, \qquad \alpha_{i0} = -\tfrac{1}{4}, \qquad i = 1, 2, \cdots$$

$$\alpha_{ij} = -\frac{3p_j}{4}, \qquad i, j \in \{1, 2, \cdots\}.$$

The basic formula (3) then yields

$$C_{0i,0i} = C_{i0,i0} = -C_{0i,i0} = -\frac{p_i^2}{4} + \frac{p_i}{2}, \qquad i = 1, 2, \cdots, \tag{32}$$

$$C_{0i,0j} = C_{i0,j0} = C_{0i,j0} = -\frac{p_i p_j}{4}, \qquad i, j \in \{1, 2, \cdots\}. \tag{33}$$

Equation (32) comes about because an excursion into a leaf must necessarily return to the root. From Equation (33) we learn in particular that the fluctuations in the number of loops made in any leaf of the flower is negatively correlated with the fluctuations in the other loops, as one would expect. Theorem 9 in Section 5 identifies which linear combinations of edge fluctuations with bounded weights are null.

4.4. *Birth and death process.* Let $\{X(t), t \geqq 0\}$ be a positive recurrent birth and death process with state space $S = \{0, 1, 2, \cdots\}$. We follow the notation in Dynkin and Yushkevitch [7], Section 4.2. Let a_n denote the rate of the exponential time the process spends in state n before jumping, p_n the probability that the jump is to state $n+1$ and $q_n = 1 - p_n$, $n = 0, 1, 2, \cdots$. Then

$$q_{n\,n+1} = a_n p_n, \qquad q_{n\,n-1} = a_n q_n, \qquad n = 1, 2, \cdots,$$

and $p_0 = 1$, $q_{01} = a_0$. Following [7], we place the process on its canonical scale by defining

$$\delta_0 = 1, \qquad \delta_n = \frac{q_1 \cdots q_n}{p_1 \cdots p_n}, \qquad n = 1, 2, \cdots,$$

$$u_0 = 0, \qquad u_1 = 1, \qquad u_n = \delta_0 + \cdots + \delta_{n-1}, \qquad n = 1, 2, \cdots.$$

We define the velocity measure (μ_n) by

$$2\mu_0 = \frac{1}{a_0}, \qquad 2\mu_n = \frac{1}{a_n}\frac{p_1 \cdots p_{n-1}}{q_1 \cdots q_n}, \qquad n = 1, 2, \cdots,$$

and let

$$v_0 = 2\mu_0, \qquad v_n = 2\mu_0 + \cdots + 2\mu_n, \qquad n = 1, 2, \cdots.$$

If $v_\infty = 2\sum_{k=0}^{\infty} \mu_k$, then $v_\infty < \infty$ for a positive recurrent process. Finally, we define the characteristic of the process by letting

$$S_0 = 0, \qquad S_n = -\sum_{m=0}^{n-1} v_m \delta_m, \qquad n = 1, 2, \cdots.$$

The mean hitting times can be expressed in terms of the characteristic, and thence the mean excess sojourn times via (29) and (30). We have, for $i < j$

$$\boldsymbol{E}_i \sigma_j = S_i - S_j = \sum_{m=i}^{j-1} v_m \delta_m$$

whereas, for $i > j$

$$E_i\sigma_j = S_i - S_j + (u_i - u_j)v_\infty = \sum_{m=j}^{i-1} (v_\infty - v_m)\delta_m.$$

From these expressions and Equations (29) and (30) all the desired covariances between jumps can be determined, provided that $E_\pi\sigma_j = \sum_i \pi_i E_i\sigma_j < \infty$ where $\pi_i \sim 2\mu_i$. A case where the computations can be carried out explicitly is given next.

4.5 *M/M/1 queue.* The M/M/1 with arrival rate λ and service rate μ is a special case of the birth and death process with

$$q(n, n+1) = \lambda, \qquad q(n+1, n) = \mu, \qquad n = 0, 1, 2, \cdots.$$

Let $\rho = \lambda/\mu$ and assume $\rho < 1$ for ergodicity. We have

$$a_0 = \lambda, \qquad a_n = \lambda + \mu, \qquad n \geqq 1,$$

$$q_0 = 0, \qquad p_0 = 1, \qquad q_n = 1, \qquad p_n = 1, \qquad n \geqq 1.$$

The quantities δ_n, μ_n, v_n are easily computed as follows:

$$\delta_n = \rho^{-n}, \qquad 2\mu_n = \lambda^{-1}\rho^n$$

$$v_n = \lambda^{-1}\frac{1-\rho^{n+1}}{1-\rho}, \qquad v_\infty - v_n = \lambda^{-1}\frac{\rho^{n+1}}{1-\rho}.$$

The mean hitting time $E_i\sigma_j$ is then given by

$$E_i\sigma_j = \begin{cases} \dfrac{\rho}{\lambda(1-\rho)}\left[\dfrac{\rho^{-j-1}-\rho^{-i-1}}{\rho^{-1}-1} - (j-i)\right], & \text{if } i < j \\ \dfrac{\rho}{\lambda(1-\rho)}(i-j) & \text{if } i > j. \end{cases}$$

The mean hitting time starting from stationarity is

$$E_\pi\sigma_j = \sum_{i=0}^{\infty} \pi_i E_i\sigma_j = \frac{\rho}{\lambda(1-\rho)}\left[\frac{\rho^{-j}}{1-\rho} - (2j+1)\right],$$

where we used the formula $\pi_i = (1-\rho)\rho^i$ for the stationary distribution. This is finite, so Theorem 2 applies. The excess sojourn times can now be computed from formulas (29), (30). We have

$$\alpha_{ij} = \begin{cases} \lambda^{-1}\rho\left[\dfrac{1}{1-\rho} - (2j+1)\rho^j\right], & \text{if } i = j \\ \lambda^{-1}\rho\left[\dfrac{1}{1-\rho}\rho^{j-i} - (i+j+1)\rho^j\right], & \text{if } i < j \\ \lambda^{-1}\rho\left[\dfrac{1}{1-\rho} - (i+j+1)\rho^j\right], & \text{if } i > j. \end{cases}$$

These formulas can be compactly written as

$$\alpha_{ij} = \lambda^{-1}\rho\left[\frac{1}{1-\rho}\rho^{(j-i)^+} - (i+j+1)\rho^j\right],$$

where $(j-i)^+ = \max\{j-i, 0\}$. Using this expression and formulas (3) we can compute the covariances $C_{ij,k\ell}$. For i, $k \geqq 0$ we find

$$C_{i,i+1;k,k+1} = C_{i+1,i;k+1,k} = C_{i,i+1;k+1,k} = \lambda(1+\rho)\rho^{i\vee k} - 2\mu(1-\rho)(i+k+2)\rho^{i+k+2}.$$

Note that, after computations, $\sum_{i\geqq 0}\sum_{k\geqq 0} C_{i+1,i,k+1,k} = \lambda$, a consequence of the fact that the arrival and departure processes are Poisson with rate λ.

This example is generalized in Section 6, where the case of a Jackson network is treated in detail.

5. Range of the limiting Brownian motion

From the examples of the preceding section we see that the range of the limiting Gaussian process is often a proper subspace of the space in which we have proved the limit theorems. One source of this singularity is clearly the balance conditions imposed on the transition counts by the requirement that the total number of entries into a state and the total number of exits from the state cannot differ in absolute value by more than 1. It turns out that this is the only source of singularity for the limiting process. The formulation of this result is straightforward for finite state space, but somewhat more tricky for countable state space.

We first discuss the finite state space result. For each $i \in S$, let $v^i \in \mathbb{R}^d$ denote the vector with entries

$$v^i_{ij} = +1 \qquad \text{if } q_{ij} > 0$$

$$v^i_{ji} = -1 \qquad \text{if } q_{ji} > 0.$$

Theorem 8 (finite state space). *The subspace of $\mathbb{R}^d$ spanned by the limiting Brownian motion is the orthogonal complement of $(v^i, i \in S)$. In particular, the rank of the covariance matrix C is $d - m + 1$.*

Proof. We first show that each v^i is orthogonal to the subspace of $\mathbb{R}^d$ spanned by the limiting Gaussian process. Consider the process

$$A^i(t) = \sum_{jk} v^i_{jk} A_{jk}(t).$$

Since this process counts the net number of jumps out of state i it is bounded in absolute value by 1 for all t. From Theorem 1 and the balance relations for the Markov process it follows that $A^i(nt)/\sqrt{n}$ converges weakly to the Gaussian process with covariance $\sum_{jk,\ell m} v^i_{jk} C_{jk,\ell m} v^i_{\ell m}$ as $n \to \infty$. But this process also clearly converges weakly to the path that is identically 0. It follows that v^i is in the null space of the self-adjoint square root of C, i.e. it is orthogonal to the subspace spanned by the limiting Gaussian process (which is the range of this self-adjoint square root).

We next observe that the subspace of $\mathbb{R}^d$ spanned by $(v^i, i \in S)$ has dimension $m-1$. Indeed, the $m \times d$ matrix formed by stacking the rows $((v^i)^{\mathrm{T}}), i \in S)$ in some order is just the incidence matrix of the directed graph $\boldsymbol{G}$ with vertex set S, which has a directed edge from i to j if and only if the transition (ij) is feasible. Irreducibility of $(X(t), t \geqq 0)$ is equivalent to strong connectedness of $\boldsymbol{G}$. It is well known that the incidence matrix of a strongly connected directed graph has rank $m-1$; for example [3], p. 38.

To complete the proof it suffices to show that the orthogonal complement of the subspace spanned by the limiting Brownian motion has dimension at most $m-1$. Let S_0^* denote the set of all finite strings of states $\boldsymbol{s} = [s_0, s_1, \cdots, s_k, s_{k+1}]$ with $k \geq 0$ and such that $s_0 = j_0$, $s_k = i_0$, $s_{k+1} = j_0$, and, if $k \geqq 1$, $(s_i, s_{i+1}) \neq (i_0, j_0)$ for all $0 \leqq i \leqq k-1$. Observe that S_0^* is a countable set. From the path of the process $\{X(t), t \geqq 0\}$ started at j_0, we define the S_0^*-valued random variable Ξ which gives the state sequence executed by the path till it completes the first transition from i_0 to j_0. Note that the path may visit i_0 and/or j_0 several times during this sojourn.

Let e_{ij} denote the unit vector in the direction (ij) in $\mathbb{R}^d$. Let $\Psi: S_0^* \to \mathbb{R}^d$ be given by

$$\Psi(\boldsymbol{s}) = \sum_{ij} N(ij \mid \boldsymbol{s}) e_{ij} \tag{34}$$

where $N(ij \mid \boldsymbol{s})$ denotes the number of (ij) transitions in $\boldsymbol{s}$ (for example $N(i_0, j_0 \mid \boldsymbol{s}) = 1$ for all $\boldsymbol{s}$). Note that

$$\Psi(\Xi) = \sum_{ij} \psi_{ij}(1) e_{ij} \tag{35}$$

where $\psi_{ij}(1)$ are defined in (5).

Since C is just a scalar multiple of C^0, see (12), the subspace spanned by the limiting Gaussian process is the range of the self-adjoint square root of C^0. Consider the null space of C^0, i.e. the space of d dimensional vectors with coordinates w_{ij} such that

$$\sum_{ij,k\ell} w_{ij} C^0_{ij,k\ell} w_{k\ell} = 0. \tag{36}$$

We will now show that this space has dimension at most $m-1$, completing the proof.

Let w denote the column vector of the w_{ij} listed in the same order as the transitions of the process, and suppose (36) holds. From (35) and (9) we have

$$\boldsymbol{E}_0[w^{\mathrm{T}}[\Psi(\Xi) - \lambda \tau_1][\Psi(\Xi) - \lambda \tau_1]^{\mathrm{T}} w] = 0.$$

In view of (34), we may rewrite this as

$$E_0\Big[\sum_{ij} w_{ij}(N(ij \mid \Xi) - \lambda_{ij}\tau_1)\Big]^2 = 0.$$

But this implies that

$$\sum_{ij} w_{ij}(N(ij \mid \Xi) - \lambda_{ij}\tau_1) = 0 \tag{37}$$

holds almost surely. Since the time spent in a state before a jump is independent of the choice of where to jump, (37) can only hold if

$$\sum_{ij} w_{ij}N(ij \mid \Xi) = 0$$

holds almost surely. Since each state sequence in S_0^* has positive probability, this can only hold if

$$\sum_{ij} w_{ij}N(ij \mid \boldsymbol{s}) = 0 \tag{38}$$

for all $\boldsymbol{s} \in S_0^*$.

Let $\sigma = [\sigma_0, \sigma_1, \cdots, \sigma_k]$ be any closed directed walk in $\boldsymbol{G}$, i.e. $\sigma_0 = \sigma_k$, and (σ_i, σ_{i+1}) is a feasible transition for all $0 \leqq i \leqq k-1$. Let $N(ij \mid \sigma)$ denote the number of (ij) transitions in σ. Then (38) implies that

$$\sum_{ij} w_{ij}N(ij \mid \sigma) = 0 \tag{39}$$

for all σ. To see this, first assume that (i_0, j_0) is a transition in σ. Fix such a transition, and consider the closed directed walk in $\boldsymbol{G}$ that starts at j_0, follows the sequence of states dictated by σ till state $\sigma_k = \sigma_0$, and then continues along states $\sigma_1, \sigma_2, \cdots$ in sequence, till it ends with the chosen transition from i_0 to j_0. This closed walk has the same weighted transition sum as σ, and it can be thought of as a concatenation of elements of S_0^*, hence the weighted transition sum of σ is 0 in this case as a consequence of (38). If the transition (i_0, j_0) does not appear in σ, we choose $\boldsymbol{s} \in S_0^*$ in which the state σ_0 appears and choose an occurrence of σ_0 in $\boldsymbol{s}$. We generate a new element of S_0^* by following $\boldsymbol{s}$ up to the chosen occurrence of σ_0, following σ once, and then finally executing the remaining transitions in $\boldsymbol{s}$. The weighted transition count of this new element of S_0^* is the same as the weighted transition count of σ, but this must be 0, by (38).

It is now a simple exercise in graph theory to show that the subspace of w satisfying (39) is at most $(m-1)$-dimensional. Note that since $\boldsymbol{G}$ is strongly connected, it admits a rooted tree, $\boldsymbol{T}$, which is directed into the root. Now, any vector of weights w_{ij} satisfying (39) is completely determined by the weights on the edges on this tree. Indeed, if $\boldsymbol{T}$ is a rooted tree in $\boldsymbol{G}$ directed into its root r, and if

$\sigma(i)$, $i \neq r$ denotes the successor of state i in $\boldsymbol{T}$, then any vector of weights $w = (w_{ij})$ satisfying (39) can be expressed as

$$w = \sum_{i \neq r} w_{i\sigma(i)} v^i.$$

This completes the proof of Theorem 8.

The formulation of the analogous theorem for countable state space is somewhat delicate. The reason is that there may be infinitely many jumps into or out of a state, so that the vectors v^i, as defined above are in general only in ℓ^∞—on the other hand, as we know from Theorem 6, the marginals of the limiting process may not be in ℓ^1, so we cannot consider them as defining functionals on the limiting process. To prove the appropriate theorem we first need the following simple lemma.

Lemma 3 *Let* $(Y_n, n \geqq 1)$ *be i.i.d. random variables such that*

$$\frac{1}{\sqrt{n}} \sum_{i=1}^{n} Y_i \Rightarrow 0 \tag{40}$$

Then each Y_n *is almost surely* 0.

Proof of Lemma 3. Note that we do not *a priori* assume anything about the existence of moments for the Y_n. Let V_n denote the left-hand side of (40). The characteristic function of V_n is related to that of the Y_n by

$$\phi_{V_n}(t) = \boldsymbol{E} \exp\left(it \frac{1}{\sqrt{n}} \sum_{i=1}^{n} Y_i\right) = \left[\phi_Y\left(\frac{t}{\sqrt{n}}\right)\right]^n$$

where $\phi_Y(t) = \boldsymbol{E} \exp(itY_1)$. By virtue of (40) we have $\phi_{V_n}(t)$ converging pointwise to 1 as $n \to \infty$. Taking logarithms gives

$$\phi_Y(\epsilon) = \exp(o(\epsilon^2)).$$

From this one concludes that ϕ_Y admits first and second derivatives at 0, and these are both 0, i.e. $\boldsymbol{E}Y$ and $\boldsymbol{E}Y^2$ both exist and equal 0. The conclusion follows immediately from an application of Chebyshev's inequality.

Now for each $i \in S$ let $v^i \in \ell^\infty$ be given by

$$v^i_{ij} = +1 \qquad \text{if } q_{ij} > 0$$

$$v^i_{ji} = -1 \qquad \text{if } q_{ji} > 0.$$

Then we have the following result.

Theorem 9 (countable state space). *For each* $i \in S$, *the sequence of processes* $(\sum_{kl} v^i_{k\ell} Z_{n,k\ell}(t), t \geqq 0)$ *converges weakly to the identically zero function in* $D[0, \infty)$. *If* $w = (w_{ij}) \in \ell^\infty$ *is such that the sequence of processes* $(\sum_{ij} w_{ij} Z_{n,ij}(t), t \geqq 0)$ *converges* *weakly to the identically zero function in* $D[0, \infty)$, *then* w *lies in the weak-* closure of the subspace of* ℓ^∞ *spanned by* $(v^i, i \in S)$. *Finally, if* $w = (w_{ij}) \in \ell^\infty$ *is in the weak-* closure of the subspace of* ℓ^∞ *spanned by* $(v^i, i \in S)$, *then the sequence of processes* $(\sum_{ij} w_{ij} Z_{n,ij}(t), t \geqq 0)$ *converges weakly to the identically zero function in* $D[0, \infty)$.

Proof. Part of the statement is that the indicated processes are well defined—since for any t the process $(X(t), t \geqq 0)$ can have only executed finitely many jumps up to time t, almost surely, and since $\sum_{ij} \pi_i q_{ij} \leqq q < \infty$, this is clear from the definition of Z_n, (1). For the first statement, it suffices to observe that the nth indicated process is almost surely bounded in absolute value by $2/\sqrt{n}$. Thus it remains to verify the latter statements. The topology mentioned here is the weak-* topology of ℓ^∞ as the dual space of ℓ^1; see [16] for background.

Let $w \in \ell^\infty$ be such that the sequence of processes $(\sum_{ij} w_{ij} Z_{n,ij}(t), t \geqq 0)$ converges weakly to the identically zero function in $D[0, \infty)$. Then we also have $(\sum_{ij} w_{ij} Z^0_{n,ij}(t), t \geqq 0)$ converges weakly to the identically zero function in $D[0, \infty)$, and in particular, that the sequence of i.i.d. variables $\{\sum_{ij} w_{ij}(\psi_{ij}(n) - \lambda_{ij}\tau_n)\}$ with central limit scaling converges weakly to 0. Note that these variables need not admit second moments *a priori*. Nevertheless, as a consequence of Lemma 3, we learn that they must be almost surely zero.

Let S_0^*, Ξ and $N(ij \mid \boldsymbol{s})$ be as defined above. The conclusion of Lemma 3 can be written as

$$\sum_{ij} w_{ij}(N(ij \mid \Xi) - \lambda_{ij}\tau_1) = 0. \tag{41}$$

As in the proof of Theorem 8, (41) can only hold if

$$\sum_{ij} w_{ij} N(ij \mid \Xi) = 0$$

holds almost surely, by virtue of the independence of the time spent in a state from the decision of where to jump on leaving that state. Since each state sequence in S_0^* has positive probability, this can only hold if

$$\sum_{ij} w_{ij} N(ij \mid \boldsymbol{s}) = 0 \tag{42}$$

for all $\boldsymbol{s} \in S_0^*$. As in the proof of Theorem 8 we can conclude that if $\sigma = [\sigma_0, \sigma_1, \cdots, \sigma_k]$ is any closed directed walk in the directed graph $\boldsymbol{G}$ associated to the process, then

$$\sum_{ij} w_{ij} N(ij \mid \sigma) = 0. \tag{43}$$

Since the underlying Markov process is irreducible, it is easily seen that one can construct an exhaustion of the state space S by a sequence of finite subsets

$$S_1 \subset S_2 \subset \cdots S_K \subset \cdots$$

such that if $\boldsymbol{G}_K$ denotes the restriction of the directed graph $\boldsymbol{G}$ associated to the process to S_K, then each $\boldsymbol{G}_K$ is strongly connected. We choose a root $r \in S_1$ and construct an infinite tree $\boldsymbol{T}$ rooted at r and directed into the root by starting with such a tree $\boldsymbol{T}_1$ for $\boldsymbol{G}_1$, and, with $\boldsymbol{T}_{K-1}$ already constructed as a tree for $\boldsymbol{G}_{K-1}$ directed into the root, constructing $\boldsymbol{T}_K$ as an extension of $\boldsymbol{T}_{K-1}$ which is also a tree for $\boldsymbol{G}_K$ directed into the root. $\boldsymbol{T}$ is the limit of $(\boldsymbol{T}_K)$. We may now define $\sigma(i)$ for $i \neq r$ as the successor of state i in $\boldsymbol{T}$.

Let $w^K \in \ell^\infty$ be given by

$$w^K = \sum_{i \in S_K, i \neq r} w_{i\sigma(i)} v^i. \tag{44}$$

Then w^K is a finite linear combination of $(v^i, i \in S)$. We claim that the sequence (w^K) converges to w in the weak-* topology of ℓ^∞, establishing the second statement of the theorem.

For $k, \ell \in S_K$, we have

$$\begin{aligned} w_{k\ell} &= \sum_{i \in S_K, i \neq r} w_{i\sigma(i)} v^i_{k\ell} \\ &= w^K_{k\ell} \end{aligned}$$

where the first equality follows from the fact that (43) holds for all loops σ in $\boldsymbol{G}_K$ (see the last paragraph in the proof of Theorem 8). Thus we may write, for any $a = (a_{ij}) \in \ell^1$,

$$\sum_{ij} w_{ij} a_{ij} - \sum_{ij} w^K_{ij} a_{ij} = \sum_{i \notin S_K \text{ or } j \notin S_K} w_{ij} a_{ij} - \sum_{i \notin S_K \text{ or } j \notin S_K} w^K_{ij} a_{ij}. \tag{45}$$

Since $a \in \ell^1$, for any $\epsilon > 0$, we can choose K large enough so that

$$\sum_{i,j \in S_K} |a_{ij}| > \|a\|_1 - \epsilon.$$

With such a choice of K, the first term on the right of (45) is bounded above by $\epsilon \, \|w\|_\infty$. For the second term on the right of (45), observe first that the summation can be taken to be over $\{i \in S_K, j \notin S_K\} \cup \{i \notin S_K, j \in S_K\}$, by virtue of (44). Further each term of the first kind is $w_{i\sigma(i)} a_{ij}$, and each term of the second kind is $w_{j\sigma(j)} a_{ij}$. Thus the second term on the right of (45) is also bounded above in absolute value by $\epsilon \, \|w\|_\infty$. Thus the absolute value of the left-hand side of (45) can be made as small as desired by choosing K sufficiently large, establishing the desired result.

Finally, it remains to show the last statement of the theorem. Let $w = (w_{ij}) \in \ell^\infty$ be in the weak-* closure of $(v^i, i \in S)$. Then (43) must hold for all closed directed walks σ, because $N(\cdot \mid \sigma) \in \ell^1$. Further, $\sum_{ij} w_{ij} \lambda_{ij} = 0$, because $\sum_{ij} v^k_{ij} \lambda_{ij} = 0$ for all $k \in S$ and $\lambda = (\lambda_{ij}) \in \ell^1$. It follows that (41) holds a.s., hence the variables $\{\sum_{ij} w_{ij}(\psi_{ij}(n) - \lambda_{ij}\tau_n)\}$ are almost surely zero. Hence the sequence of processes $(\sum_{ij} w_{ij} Z^0_{n,ij}(t), t \geqq 0)$ converges weakly to the identically zero function in $D[0, \infty)$, and so also the sequence $(\sum_{ij} w_{ij} Z_{n,ij}(t), t \geqq 0)$. This completes the proof of the theorem.

6. Fluctuations of traffic processes in a Jackson network

In this final section we illustrate the applicability of our results to the important practical example of a Jackson network. We compute the covariance matrix of the traffic processes in the network using our results and express the covariances in terms of certain simple sojourn times associated to the network.

A Jackson network of J nodes is a queueing network consisting of J first-come first-served exponential servers. The network is fed with customers by a Poisson process of rate γ. Each customer is routed independently of the others to node i with probability r_{0_i}, where $\sum_{i=1}^{J} r_{0_i} = 1$. On arriving at a node, the customer joins the tail of the queue and waits till it is his turn to be served. The service received at node i is an exponential random variable of mean μ_i^{-1}; individual service times are independent. A customer completing service at node i is routed, independently of the others to node j with probability r_{ij} and out of the network with probability r_{i0}. Here $\sum_{j=1}^{J} r_{ij} + r_{i0} = 1$. The arrival process, the routing decisions, and the service times are independent. We assume that the matrix $R = [r_{ij}]$ is irreducible and strictly substochastic. The network can clearly be described by a Markov process $\{X(t), t \geqq 0\}$ with state space $\mathbb{Z}_+^J$.

Let λ_i, $1 \leqq i \leqq J$ solve the *flow balance equations*:

$$\lambda_i = \gamma r_{0i} + \sum_{j=1}^{J} \lambda_j r_{ji}. \tag{46}$$

From the preceding conditions on the matrix R it is easily seen that these equations admit a unique solution. It is known that $\{X(t), t \geqq 0\}$ is ergodic iff $\lambda_i < \mu_i$ for all $1 \leqq i \leqq J$, in which case it admits the unique *product form* stationary distribution

$$\pi_x = \prod_{i=1}^{J} (\rho_i)^{x_i}(1 - \rho_i)$$

where $\rho_i = \lambda_i / \mu_i$. Jackson networks are a popular and widely studied model for packet-switched communication networks. For more on the subject, see for instance [18].

Certain sums of the counting processes associated to jumps of $\{X(t), t \geqq 0\}$ are of particular interest. Here we focus on the following processes: the process of arrivals to node i, $1 \leqq i \leqq J$, the process of departures from node i, $1 \leqq i \leqq J$, and the process of jumps from node i to node j, $1 \leqq i, j \leqq J$. For simplicity, we assume that $r_{ii} = 0$, $1 \leqq i \leqq J$. This results in no essential loss of generality, for similar results can easily be derived without this assumption. Our purpose in this section is to demonstrate how the results we have derived in this paper can be used to compute the covariance structure of the normalized version of all these processes. Interestingly, there is a considerable simplification in the expression, and the final expression for the covariances involves only certain simple sojourn times associated to the underlying network. This expression has a deceptive intuitive appeal, rather like the deceptive simplicity of the product form stationary distribution.

We introduce some notation specific to this example. Let $e^i \in \mathbb{Z}_+^J$ denote the unit vector in the direction i. For a state $x \in \mathbb{Z}_+^J$, let $x^i = x + e^i$. Also, let $x^0 = x$.

Note that the sums of jumps we have to consider to define each of the processes above involve weighting sequences in ℓ^∞. Thus, to apply our results, we first need to demonstrate the following.

Theorem 10 (Jackson network). *Let $\{X(t), t \geqq 0\}$ be an ergodic Jackson network, with the states ordered in some way. Then, with the sum being over feasible jumps, we have $\sum_{xy} C_{xy,xy}^{1/2} < \infty$. As a consequence, as $n \to \infty$, the normalized processes Z_n defined by Equation* (1) *converge weakly as random elements of $D_{\ell^1}[0, \infty)$.*

Proof. The jumps of $\{X(t), t \geqq 0\}$ are of three types: jumps from x to x^i (J such), jumps from x^i to x^j (J^2 such), and jumps from x^i to x (J such). From Equation (3), and using the identity $(a+b)^{1/2} \leqq a^{1/2} + b^{1/2}$ for $a,\ b \geqq 0$, $\sum_{xy} C_{xy,xy}^{1/2}$ is obviously bounded above by the sum of six terms, each of which we now proceed to bound. For each $1 \leqq i \leqq J$, we have

$$\sum_x (\pi_x q_{xx^i})^{1/2} = (\gamma r_{0i})^{1/2} \sum_x (\pi_x)^{1/2}$$

$$< \infty.$$

Similarly, for all $1 \leqq i, j \leqq J$,

$$\sum_x (\pi_{x^i} q_{x^i x^j})^{1/2} = (\rho_i \mu_i r_{ij})^{1/2} \sum_x (\pi_x)^{1/2}$$

$$< \infty,$$

and for all $1 \leqq i \leqq J$,

$$\sum_x (\pi_{x^i} q_{x^i x})^{1/2} = (\rho_i \mu_i r_{i0})^{1/2} \sum_x (\pi_x)^{1/2}$$

$$< \infty.$$

To bound the other three terms we need the following lemma.

Lemma 4. *Let $x,\ y \in \mathbb{Z}_+^J$, and let α_{xy} be defined as in Equation* (2). *There are absolute constants $C_1,\ C_2 < \infty$ depending only on the network parameters, such that*

$$|\alpha_{xy}| \leqq C_1 |x| + C_2$$

where $|x|$ denotes $\sum_{i=1}^J x_i$.

Proof of Lemma 4. Corollary 1 of [2] demonstrates that the time to empty starting from x is stochastically dominated by the sum of $|x|$ independent random variables of finite mean. The time to couple to stationarity starting empty is a random variable of finite mean. The claim now follows easily.

To complete the proof of the theorem, we bound the remaining three terms. For all $1 \leqq i \leqq J$, we have

$$\sum_x (2\pi_x q^2_{xx^i} |\alpha_{x^ix}|)^{1/2} \leqq \gamma r_{0i} \sum_x (2\pi_x(C_1(|x|+1)+C_2))^{1/2}$$
$$< \infty.$$

Similarly, for all $1 \leqq i,\ j \leqq J$ we have

$$\sum_x (\pi_{x^i} q^2_{x^ix^j} |\alpha_{x^jx^i}|)^{1/2} \leqq \rho_i^{1/2} \mu_i r_{ij} \sum_x (2\pi_x(C_1(|x|+1)+C_2))^{1/2}$$
$$< \infty$$

and for all $1 \leqq i \leqq J$

$$\sum_x (\pi_{x^i} q^2_{x^ix} |\alpha_{xx^i}|)^{1/2} \leqq \rho_i^{1/2} \mu_i r_{i0} \sum_x (2\pi_x(C_1 |x| + C_2))^{1/2}$$
$$< \infty.$$

This completes the proof of Theorem 10.

6.1. *Computation of the covariance matrix of the traffic processes.* We now consider the problem of explicitly computing the covariance structure of all traffic processes of a Jackson network. Recalling that $A_{xy}(t)$ denotes the number of jumps from state x to state y up to time t, we let

$$A^{ij}(t) = \sum_x A_{x^ix^j}(t) \tag{47}$$

be the number of customers jumping from queue i into queue j between time 0 and time t, $1 \leqq i,\ j \leqq J$. Equation (47) also defines $A^{0j}(t)$ to be the arrival process into queue j and $A^{i0}(t)$ to be the external departure process from queue i, $1 \leqq i, j \leqq J$.

To formulate the result, define β_{ij} to be the expected amount of time that a customer will spend in node j before leaving the network, given that it entered the network at node i, $1 \leqq i,\ j \leqq J$. This quantity is computable from first-step equations for the substochastic routing matrix R. Let

$$\beta_{i0} = \beta_{0j} = \beta_{00} = 0, \qquad 1 \leqq i,\ j \leqq J.$$

Observe that $\beta_{ij}\mu_j$ is the total number of visits to node j by a customer starting at i. So, from first-step equations, we have

$$\beta_{ij}\mu_j = \delta_{ij} + \sum_{k=1}^{J} r_{ik}\beta_{kj}\mu_j.$$

So, in principle, the β_{ij} can be computed explicitly.

Recall that λ_i, $1 \leqq i \leqq J$ is defined by the flow balance equations (46). Let $\lambda_0 = \mu_0 = \gamma$ by convention. Recall that $x^i = x + e^i$, $1 \leqq i \leqq J$, and $x^0 = x$.

Theorem 11 (asymptotic covariances). *The asymptotic covariance between the counting processes A^{ij} and $A^{k\ell}$ defined by* (47), *is given by*

$$C^{ij,k\ell} = r_{ij}r_{k\ell}(\lambda_i\beta_{jk}\mu_k + \lambda_k\beta_{\ell i}\mu_i) + \lambda_i r_{ij}\delta_{ij,kl} \tag{48}$$

where $\delta_{ij,k\ell} = 1$ if $(i,j) = (k,\ell)$ and 0 *otherwise.*

Proof. From Theorem 10, i.e. existence of the weak limit in $D_{\ell^1}[0,\infty)$, and formulas (47) we have

$$\begin{aligned} C^{ij,k\ell} &= \sum_x \sum_y C_{x^ix^j,y^ky^\ell} \\ &= \sum_{x \neq y}\sum C_{x^ix^j,y^ky^\ell} + \sum_x C_{x^ix^j,y^ky^\ell} \qquad (49) \\ &= C_1^{ij,k\ell} + C_2^{ij,k\ell}. \end{aligned}$$

The quantities in the middle line are as in Equation (3) of Theorem 1. When $(i,j) = (k,\ell)$ the second term of (49) equals

$$C_2^{ij,ij} = \sum_x \pi_{x^i}q_{x^ix^j} = \sum_x \frac{\lambda_i}{\mu_i}\pi_x\mu_i r_{ij} = \lambda_i r_{ij}. \tag{50}$$

When $(i,j) \neq (k,\ell)$ this term is zero. For the first term, we have

$$\begin{aligned} C_1^{ij,k\ell} &= \sum_x \sum_y q_{x^ix^j}q_{y^ky^\ell}(\pi_{x^i}\alpha_{x^jy^k} + \pi_{y^k}\alpha_{y^\ell x^i}) \\ &= \mu_i r_{ij}\mu_k r_{k\ell}\sum_x\sum_y \Big\{\pi_{x^i}\lim_{t\to\infty} E_{x^j}[\tau_{y^k}(t) - \pi_{y^k}t] \qquad (51) \\ &\qquad + \pi_{y^k}\lim_{t\to\infty} E_{y^\ell}[\tau_{x^i}(t) - \pi_{x^i}t]\Big\}. \qquad (52) \end{aligned}$$

The limit and the double summation can be interchanged, see Lemma 5 below. Let

$\tau^k(t)$ be the busy time of node k up to time t. The first term in the summation above can be written as follows:

$$\begin{aligned}\sum_x \sum_y \pi_{x^i} \boldsymbol{E}_{x^j}[\tau_{y^k}(t) - \pi_{y^k} t] &= \sum_x \frac{\lambda_i}{\mu_i} \pi_x \boldsymbol{E}_{x^j}\Big(\tau^k(t) - \frac{\lambda_k}{\mu_k} t\Big) \\ &= \frac{\lambda_i}{\mu_i} \sum_x \pi_x [\boldsymbol{E}_{x^j} \tau^k(t) - \boldsymbol{E}_x \tau^k(t)].\end{aligned} \tag{53}$$

Similarly, for the second term:

$$\sum_x \sum_y \pi_{y^k} \boldsymbol{E}_{y^\ell}[\tau_{x^i}(t) - \pi_{x^i} t] = \frac{\lambda_k}{\mu_k} \sum_y \pi_y [\boldsymbol{E}_{y^\ell} \tau^i(t) - \boldsymbol{E}_y \tau^i(t)]. \tag{54}$$

Inserting (53) and (54) into (52) and interchanging the limit and summation once more, see Lemma 6 below, we get:

$$\begin{aligned} C_1^{ij,k\ell} &= \mu_i r_{ij} \mu_k r_{k\ell} \lim_{t\to\infty} \Big\{ \frac{\lambda_i}{\mu_i} \sum_x \pi_x [\boldsymbol{E}_{x^j} \tau^k(t) - \boldsymbol{E}_x \tau^k(t)] + \frac{\lambda_k}{\mu_k} \sum_y \pi_y [\boldsymbol{E}_{y^\ell} \tau^i(t) - \boldsymbol{E}_y \tau^i(t)] \Big\} \\ &= r_{ij} r_{k\ell} \Big\{ \lambda_i \mu_k \sum_x \pi_x \lim_{t\to\infty} [\boldsymbol{E}_{x^j} \tau^k(t) - \boldsymbol{E}_x \tau^k(t)] + \lambda_k \mu_i \sum_y \pi_y \lim_{t\to\infty} [\boldsymbol{E}_{y^\ell} \tau^i(t) - \boldsymbol{E}_y \tau^i(t)] \Big\} \\ &= r_{jj} r_{k\ell} (\lambda_i \beta_{jk} \mu_k + \lambda_k \beta_{\ell i} \mu_i). \end{aligned} \tag{55}$$

The last step is justified in Lemma 7 below. Putting (50) and (55) together we obtain the required formula (48).

We now proceed with the justification of the three steps in the proof of the theorem. First, the interchange of limit and double summation in (52) is a consequence of the dominated convergence theorem.

Lemma 5. *Let $f_{x,y}(t) = E_{x^j}(\tau_{y^k}(t) - \pi_{y^k} t)$. Then there exists $\hat{f}_{x,y}$ such that $|f_{x,y}(t)| \leqq \hat{f}_{x,y}$ and $\sum_x \sum_y \pi_x \hat{f}_{x,y} < \infty$.*

Proof of Lemma 5. Let $\{X^{x^j}(t)\}$, $\{X^\pi(t)\}$ be two independent copies of the Jackson network state process starting from state x^j and steady state, respectively. Let also M^{x^j} be their first meeting time. We can couple the processes after time M^{x^j} without changing the marginals. We have

$$\begin{aligned} f_{x,y}(t) &= \boldsymbol{E}\Big\{ \int_0^t 1\{X^{x^j}(s) = y^k\}\, ds - \int_0^t 1\{X^\pi(s) = y^k\}\, ds \Big\} \\ &= \boldsymbol{E} \int_0^{\min(t, M^{x^j})} [1\{X^{x^j}(s) = y^k\} - 1\{X^\pi(s) = y^k\}]\, ds. \end{aligned}$$

Therefore

$$|f_{x,y}(t)| \leqq \boldsymbol{E} \int_0^{M^{x^j}} [1\{X^{x^j}(s) = y^k\} + 1\{X^\pi(s) = y^k\}]\, ds = \hat{f}_{x,y}.$$

Furthermore,

$$\sum_x \pi_x \sum_y \hat{f}_{x,y} \leqq \sum_x \pi_x \boldsymbol{E}(2M^{x^j}).$$

The latter is equal to twice the expected meeting time of two independent stationary copies of the process, which is finite.

Next, the interchange of limit and summation in (55) is again a consequence of the dominated convergence theorem. The proof is similar to that of Lemma 5.

Lemma 6 *Let* $f_x(t) = \boldsymbol{E}_{x^j}\tau^k(t) - \boldsymbol{E}_x\tau^k(t)$. *Then there exists* $\hat{f}_x$, *such that* $|f_x(t)| \leqq \hat{f}_x$ *and* $\sum_x \pi_x \hat{f}_x < \infty$.

Finally, let us verify the last line of (55).

Lemma 7 *Let* β_{ij} *be the expected amount of time that a customer will spend in node* j *before leaving the network, given that it entered the network at node* i. *Then*

$$\lim_{t\to\infty} [\boldsymbol{E}_{x^i}\tau^j(t) - \boldsymbol{E}_x\tau^j(t)] = \beta_{ij}, \tag{56}$$

for all states x.

Proof of Lemma 7. The claim is obviously true if either i is zero or j is zero. So assume that they are both non-zero. Couple the initial states x and x^i by having an additional colored customer starting in node i. Give this customer *low priority* in all queues of its path in the network. Pathwise the left-hand-side time of (56), inside expectation, increases, while the right-hand-side time does not *only* when the colored customer is in node j and it is the only customer there. But the colored customer's path through the network, followed at the times when service is offered to it, is that of a single customer starting in node i in an isolated network. Also, service is only offered to it when it is in node j if there are no uncolored customers in node j. The claim follows.

6.2. *Some examples.* It is interesting to see what formulas (48) result in, in some specific cases.

1. *Departure process from a queue.* Consider the overall departure process from a specific queue, say queue 1. This is $\sum_{j=0}^J A^{1j}$. Its asymptotic autocovariance is thus given by the sum (see formula (48))

$$\sum_j \sum_\ell C^{1j,1\ell} = \sum_j \sum_\ell r_{1j} r_{1\ell}(\lambda_1\beta_{j1}\mu_1 + \lambda_1\beta_{\ell 1}\mu_1) + \sum_j \lambda_1 r_{1j} = 2\lambda_1\mu_1 \sum_j r_{1j}\beta_{j1} + \lambda_1. \tag{57}$$

2. *Arrival process into a queue.* Consider the overall arrival process into a specific queue, say queue 1. This is $\sum_{i=0}^J A^{i0}$. Its asymptotic autocovariance is thus given by the sum (see formula (48))

$$\sum_i \sum_k r_{i1} r_{k1}(\lambda_i\beta_{1k}\mu_k + \lambda_k\beta_{1i}\mu_i) + \sum_i \lambda_i r_{i1} = 2\lambda_1 \sum_k \beta_{1k}\mu_k r_{k1} + \lambda_1. \tag{58}$$

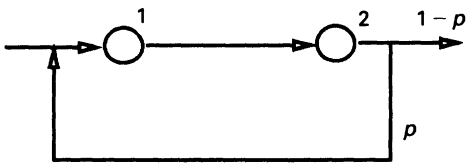

Figure 3. A simple Jackson network

Let us observe that the autocovariance of the arrival process in queue 1 should be the same as the autocovariance of the departure process from queue 1 *in reverse time.* Letting tilde over a quantity denote the quantity evaluated on the reverse time process, the autocovariance of the departure process from queue 1 *in reverse time* is, from (57),

$$2\lambda_1 \sum_k \tilde{r}_{1k} \tilde{\beta}_{k1} \mu_1 + \lambda_1. \tag{59}$$

It is easy to see that $\lambda_1 \tilde{r}_{1k} = \lambda_k r_{k1}$ and

$$\lambda_k \beta_{k1} \mu_1 = \lambda_1 \beta_{1k} \mu_k. \tag{60}$$

Substituting (60) into (59) we obtain (58), as we should.

6.3. *A simple Jackson network with two queues.* Consider the network of Figure 3. The external arrival rate is γ and p is the probability of feedback. The service rates are μ_1, μ_2. We first compute the β_{ij}:

$$\beta_{11}\mu_1 = \frac{1}{1-p} = \beta_{12}\mu_2 = \beta_{22}\mu_2,$$

$$\beta_{21}\mu_1 = \frac{p}{1-p}.$$

The average flows are $\lambda_1 = \lambda_2 = \gamma/(1-p)$. So the asymptotic covariance between the process that counts customers jumping from queue 1 into queue 2 and the feedback process is given by

$$C^{12,21} = r_{12} r_{21} (\lambda_1 \beta_{22} \mu_2 + \lambda_2 \beta_{11} \mu_1) = \frac{2\gamma p}{(1-p)^2}.$$

7. Concluding remarks

We have proved joint functional central limit theorems for the family of all the counting processes associated to the jumps of a uniform countable state Markov process. For finite state space the technique for proving such theorems is rather standard, but we have given a complete proof in order to facilitate the discussion of the subsequent results. A novelty in the results for infinite state space is our analysis of the function space topologies that one needs to use to derive such theorems; which topologies work depends on the underlying Markov process. For both finite and infinite state space processes, it is interesting and useful that we have simple and explicit formulas for the covariance matrix of the limiting Gaussian process. Further, the null space of this covariance matrix has been related in a simple way to the topology of the underlying rate diagram.

Our motivation in addressing such issues comes from traffic theory, where traffic processes in Markovian queueing networks are expressed as the counting processes associated to certain sums of jumps of an underlying Markov process. For Jackson networks, we have demonstrated the value of our results by deriving simple explicit formulas for the limiting covariance structure of the fluctuations of the various internode traffic processes.

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