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Tracking and Disturbance Rejection of MIMO Nonlinear Systems with a PI or PS Controller

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ABSTRACT

We study tracking and disturbance rejection for a class of MIMO nonlinear systems, with a *linear* proportional plus integral (PI) compensator, in the continuous time case, and a *linear* proportional plus sum (PS) compensator in the discrete time case. We show that if the nonlinear plant is *exponentially stable* and has a strictly increasing dc steady state I/O map then a simple PI or PS compensator can be used to yield a stable unity feedback closed loop system which asymptotically tracks reference inputs that tend to constant vectors and asymptotically rejects disturbances that tend to constant vectors. This extends earlier work of Desoer and Lin.

I. INTRODUCTION

I.1. Informal discussion of results

In this paper we study tracking and disturbance rejection for a class of nonlinear MIMO unity feedback systems, namely the system ${}^1S(\mathbf{N}, \frac{\varepsilon}{s}I+K)$ consisting of the given nonlinear plant $\mathbf{N}$ and the *linear* proportional plus (PI) compensator $\frac{\varepsilon}{s}I + K$ (see Fig. 1) and, in the discrete time case, the system ${}^1S(\mathbf{N}_D, \frac{\varepsilon z}{z-1}I+K)$ with the compensator of proportional plus sum type. The main results are Theorem 1 in continuous time and Theorem 3 in discrete time, which show, roughly speaking that if the nonlinear plant $\mathbf{N}$ is exp stable and has a strictly increasing d.c. steady-state I/O map, then for sufficient small $\varepsilon > 0$ and K appropriately chosen we achieve a stable unity feedback closed loop system which asymptotically tracks reference inputs which tend to constant vectors and asymptotically rejects disturbances which tend to constant vectors.

The key ideas behind Theorems 1 are contained in the estimates used to prove Theorem 2. A similar special case underlies the discrete time Theorem 3.

The assumptions we have made on $\mathbf{N}$ and $\mathbf{N}_D$ in the following are satisfied in practice by many plants. The results are of particular interest for process control where PI or PS control is used in practice to ensure disturbance rejection and asymptotic tracking.

II. THE CONTINUOUS TIME CASE

II.1. Assumptions in the nonlinear plant N

We assume that the nonlinear time-invariant MIMO plant $\mathbf{N}$ (see Fig. 1) can be described by the following equations:

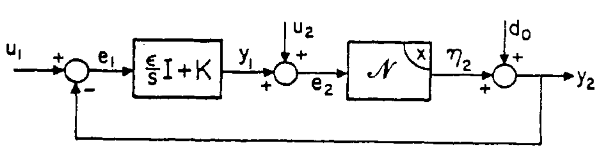

Fig. 1.

$$\dot{x}(t) = f(x(t), e_2(t)) \ ; \ \eta_2(t) = h(x(t)) \tag{2.1}$$

where $t \geq 0$, $e_2(t) \in \mathbb{R}^m$, $\eta_2(t) \in \mathbb{R}^m$, and $x(t) \in \mathbb{R}^n$.

Further $\mathbf{N}$ is assumed to satisfy the following conditions:

(N.1) $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ and $h : \mathbb{R}^n \to \mathbb{R}^m$ are C^1 functions, and $f(\vartheta_n, \vartheta_m) = \vartheta_n$, $h(\vartheta_n) = \vartheta_m$. (Together with N.4 below this ensures that for every piecewise continuous input $e_2(\cdot)$ and for every initial condition (s_o, t_o). Equation (2.1a) has a unique solution.

$$t \to x(t, t_o, t_o, e_2(\cdot)) \qquad \text{defined on } [t_o\ \infty).)$$

(N.2) There is a C^1-function $g : \mathbb{R}^m \to \mathbb{R}^n$ such that $\forall\, v \in \mathbb{R}^m$

$$f(\xi, v) = \vartheta_n \ \text{ iff } \ g(v) = \xi \ . \tag{2.2}$$

(N.3) The map $h \circ g : v \to h(g(v))$ is a bijection of $\mathbb{R}^m$ onto $\mathbb{R}^m$

(N.4) There exists $M > 0$ s.t. $\forall\, v \in \mathbb{R}^m$, $\forall\, \xi \in \mathbb{R}^n$.

$$|D_1 f(\xi, v)| < M \ ; \ |D_2 f(\xi, v)| < M \ . \tag{2.3}$$

(N.5) There is a constant A_h such that $\forall\, x_1, x_2 \in \mathbb{R}^n$.

$$|h(x_1) - h(x_2)| \leq A_h |x_1 - x_2| \ . \tag{2.4}$$

(N.6) There is a constant A_g such that $\forall\, \xi_1, \xi_2 \in \mathbb{R}^m$

$$|g(\xi_1) - g(\xi_2)| \leq A_g |\xi_1 - \xi_2| \ . \tag{2.5}$$

(N.7) There is a constant $m > 0$ such that $\forall\, \xi_1, \xi_2 \in \mathbb{R}^m$

$$\langle h \circ g(\xi_1) - h \circ g(\xi_2), \xi_1 - \xi_2 \rangle \geq m |\xi_1 - \xi_2|^2 \tag{2.6}$$

(N.8) There is a C^1 Lyapunov function $V : \mathbb{R}^n \to \mathbb{R}$ and constants a_v and M_v such that

$$A_v |x|^2 \leq V(x) \ ; \ |DV(x)| \leq M_v |x| \tag{2.7}$$

Further there is $\alpha > 0$ such that $\forall\, \xi \in \mathbb{R}^m$ we have, with $x(t)$ a state trajectory for the equation $\dot{x}(t) = f(x(t), \xi)$ (ξ constant)

$$\frac{d}{dt} V(x(t) - g(\xi)) \Big|_{t = t_o} \leq -\alpha v(x(t_0 - g(\xi)) \tag{2.8}$$

II.2. Allowable inputs in the continuous time case

We assume that for ${}^1S(\mathbf{N}, \frac{\varepsilon}{s}I+K)$ that the reference input u_1 and disturbances u_2, d_o satisfy the assumption:

$u_1(\cdot)$, $u_2(\cdot)\, d_o(\cdot) \in C^1$, and $\exists\ \bar{u}_1, \bar{u}_2, \bar{d}_o \in \mathbb{R}^m$ such that, as $t \to \infty$,

$$\begin{cases} u_1(t) \to \bar{u}_1 \\ u_2(t) \to \bar{u}_2 \\ d_o(t) \to \bar{d}_o \end{cases}, \text{ and } \begin{cases} \dot{u}_1(t) \to \vartheta_m \\ \dot{u}_2(t) \to \vartheta_m \\ \dot{d}_o(t) \to \vartheta_m \end{cases} \tag{2.9}$$

we also assume that for all $\varepsilon > 0$ and $K \in \mathbb{R}^{m \times m}$, the system ${}^1S(\mathbf{N}, \frac{\varepsilon}{s}I+K)$ is *reachable*, namely that, for all states (x_o, η_o), $(x_1, \eta_1) \in \mathbb{R}^n \times \mathbb{R}^m$, there exists inputs u_1, $u_2 \in C^1$, with compact support, say $[0 \ T]$, which take $(x(0), \eta(0)) = (x_o, \eta_o)$ to $(x(T), \eta(T)) = (x_1, \eta_1)$.

Theorem 1: Consider the system ${}^1S(\mathbf{N}, \frac{\varepsilon}{s}I+K)$ where $\mathbf{N}$ satisfies (N.1)-(N.8). U.t.c. if

(i) $K \in \mathbb{R}^{m \times m}$ is positive semidefinite

(ii) $|K|$ is small enough,

then there exists $\bar{\varepsilon} > 0$ such that, for all $\varepsilon \in (0, \bar{\varepsilon}]$, *for all initial conditions* $(x(0), \eta(0)) \in \mathbb{R}^n \times \mathbb{R}^m$ and *for all* $u_1(\cdot)$ $u_2(\cdot)$ and $d_o(\cdot)$ satisfying (2.9), the corresponding $e_1(\cdot)$, $e_2(\cdot)$, $x(\cdot)$ and $y_2(\cdot)$ are *bounded* and $e_1(t) \to \vartheta_m$ as $t \to \infty$.

The basic ideas required to prove Theorem 1 are already apparent in the proof of Theorem 2 below.

Theorem 2: Given that the nonlinear plant $\mathbf{N}$ satisfies (N.1)-(N.8). Consider the system ${}^1S(\mathbf{N}, \frac{\varepsilon}{s}I)$ shown in Fig. 1, with $K = \vartheta_{m \times m}$. U.t.c. $\exists$ $\varepsilon^* > 0$ such that, for all $\varepsilon \in (0, \varepsilon^*]$, for all $(x(0), e_2(0)) \in \mathbb{R}^n \times \mathbb{R}^m$ and for all $\bar{u}_1 \in \mathbb{R}^m$ the system ${}^1S(\mathbf{N}, \frac{\varepsilon}{s}I)$ has the state

$$(\bar{x}_1, \bar{e}_2) = (g((h \circ g)^{-1}(\bar{u}_1)), (h \circ g)^{-1}(\bar{u}_1))$$

as a globally exponentially stable equilibrium point.

Remarks: A key feature of our proof of Theorem 2 is that we rely on direct estimates based on the assumptions. In particular we avoid the use of singular perturbation techniques used by Desoer and Lin in the proof of a parallel result.

III. THE DISCRETE TIME CASE

III.1. Assumptions on the nonlinear plant N_D

Consider the system ${}^1S(\mathbf{N}_D, \frac{\varepsilon z}{z-1}I+K)$ of Fig. 2. We assume that the nonlinear plant $\mathbf{N}_D$ can be described by the following equations

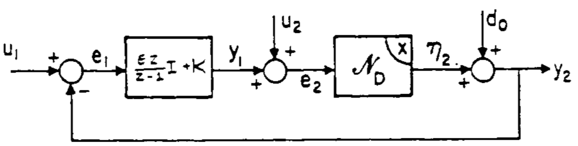

Fig. 2.

$$x(k+1) = f(x(k), e_2(k))\,;\ \eta_2(k) = h(x(k)) \tag{3.1}$$

where $k \geq 0$, $e_2(k) \in \mathbb{R}^m$, $\eta_2(k) \in \mathbb{R}^m$, and $x(k) \in \mathbb{R}^n$.

$\mathbf{N}_D$ is assumed to satisfy conditions $(\mathbf{N}_D \cdot 1)-(\mathbf{N}_D \cdot 4)$ directly parallel to $(\mathbf{N}.1)-(\mathbf{N}.4)$ and also the conditions.

($\mathbf{N}_D$.5) There is a constant A_h such that $\forall$ x_1, $x_2 \in \mathbb{R}^n$

$$|h(x_1)-h(x_2)|^2 \leq A_h |x_1-x_2|^2 \ . \tag{3.1}$$

($\mathbf{N}_D$.6) There is a constant A_g such that $\forall$ ξ_1, $\xi_2 \in \mathbb{R}^m$

$$|g(\xi_1)-g(\xi_2)|^2 \leq A_g |\xi_1-\xi_2|^2 \tag{3.2}$$

($\mathbf{N}_D$.7) There is a constant $m > 0$ such that $\forall$ ξ_1, $\xi_2 \in \mathbb{R}^m$

$$\langle h \circ g(\xi_1) - h \circ g(\xi_2)\,,\ \xi_1-\xi_2 \rangle \geq m\,|\xi_1-\xi_2|^2 \tag{3.3}$$

($\mathbf{N}_D$.8) There is a C^1 Lyapunov function $V: \mathbb{R}^n \to \mathbb{R}$ and constants a_v, A_v, and M_v such that.

$$a_v |x|^2 \leq V(x) \leq A_v |x|^2 \tag{3.4a}$$

$$|DV(x)|^2 \leq M_V |x|^2 \tag{3.4b}$$

Further, there is a constant $0 < \mu < 1$ such that $\forall$ $x \in \mathbb{R}^n$ and $\forall$ $\xi \in \mathbb{R}^m$ we have

$$V(f(x, \xi) - g(\xi)) \leq \mu V(x - g(\xi)) \tag{3.5}$$

III.2. Statement of Results

If we assume that there exist $\bar{u}_1$, $\bar{u}_2$, $\bar{d}_o \in \mathbb{R}^m$ such that as $k \to \infty$

$$u_1(k) \to \bar{u}_1 : u_2(k) \to \bar{u}_2\,;\ d_o(k) \to \bar{d}_o \tag{3.6}$$

and we assume that ${}^1S(\mathbf{N}_D, \frac{\varepsilon z}{z-1}I+K)$ is reachable, then we have

Theorem 3: If N_D satisfies the assumptions ($\mathbf{N}_D$.1)-($\mathbf{N}_D$.8), the inputs satisfy (3.6) and we have

(i) $K \in \mathbb{R}^{m \times m}$ is positive semidefinite

(ii) $|K|$ is small enough,

then there is $\bar{\varepsilon} > 0$, such that, for all $\varepsilon \in (0, \bar{\varepsilon}]$, *for all initial conditions* $(x(0), \eta(0)) \in \mathbb{R}^n \times \mathbb{R}^m$, the corresponding $e_1(\cdot)$, $e_2(\cdot)$, $x(0)$, and $y_2(\cdot)$ are bounded, and $e_1(k) \to \vartheta_m$ as $k \to \infty$.

Remarks. An analog to Theorem 2 underlies the proof of the above theorem. However, the estimates required are somewhat different. The results have not appeared before in the literature.

ACKNOWLEDGEMENT

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