

Functional Minimization for Fair Surface Design

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Abstract

This paper presents a simple-to-use mechanism for the creation of complex smoothly shaped surfaces of any genus or topology. The work described here is the result of research into the fairness of curves and surfaces specified through geometric interpolatory constraints. Constraints consist of positions and, optionally, surface normals and surface curvatures. The outcome of our investigation is a recommendation for the use of nonlinear optimization techniques that minimize a fairness functional based on the variation of curvature. The approach produces very high quality surfaces with predictable, intuitive behavior, while generating, where possible, simple shapes, such as cylinders, spheres, or tori which are commonly used in geometric modeling.

From a designer's point of view, this approach allows the specification of a desired surface in the most natural way. Though computationally intense, the techniques described have now become practical because of the wide availability of very fast work stations. As the processing power available on each desk-top further increases, the techniques described here will become real-time and interactive.

1.0 Introduction

In this paper we present a simple-to-use mechanism for the creation of complex, smoothly shaped models of any genus or topology. The shapes are compactly specified using geometric interpolatory constraints. The resulting models accurately reflect these specifications and are free of unwanted wrinkles, bulges, and ripples. When the given constraints indicate and/or permit, the resulting surfaces take on the desirable shapes of spheres, cylinders, cones, and tori. Specification of a desired shape is straight forward, allowing simple or complex shapes to be described easily and compactly. For example, a “suitcase corner”, the blend of three quarter cylinders of differing radii, is formed by specifying just six constraints (Fig. 1).

A Klein bottle is specified with equal ease; only twelve point constraints are used to model the surface shown in Figure 2.

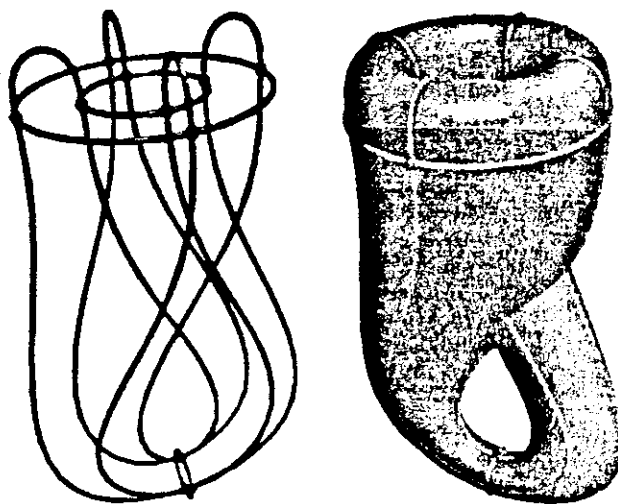


Figure 2. A Klein bottle. This model is defined by twelve constraint sets.

The work described here is the result of research into the fairness of curves and surfaces specified through geometric interpolatory constraints. These geometric constraints consist of points, surface normals, and surface curvatures. We use nonlinear optimization techniques to minimize a fairness functional subject to the given geometric constraints.

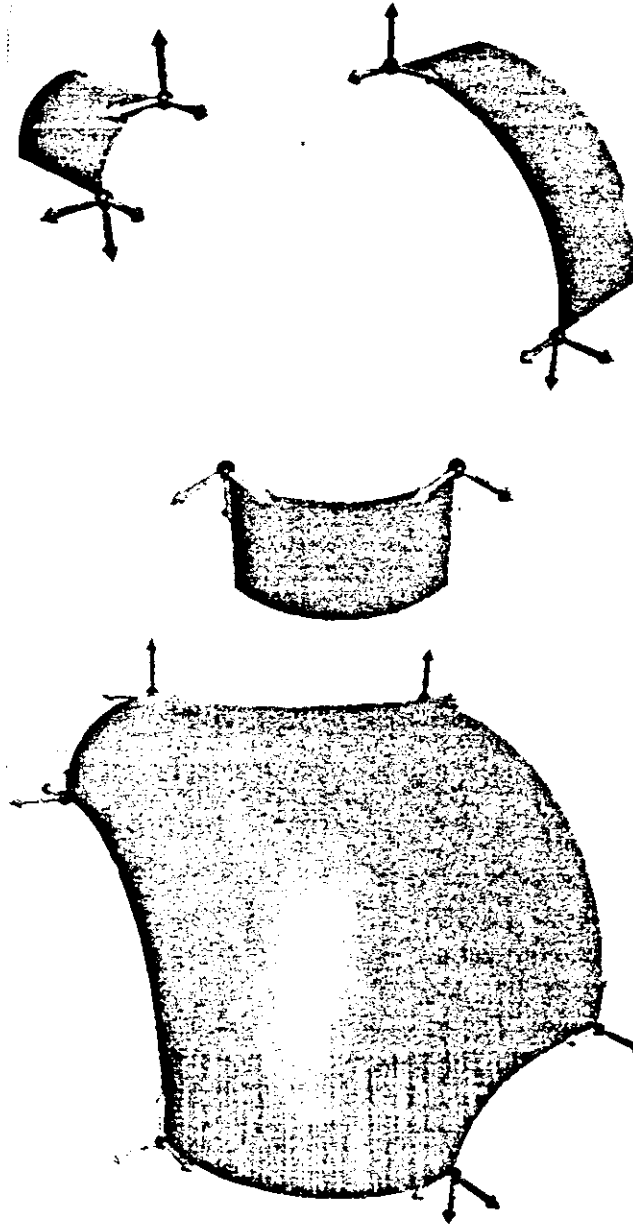


Figure 1. A suitcase corner. Fig. 1a illustrates the specification with normal and curvature constraints. Fig. 1b illustrates the resulting blend.

Once the geometric constraints are satisfied by construction, the techniques described here set the remaining surface parameters (degrees of freedom) to minimize our fairness functional while maintaining G^1 continuity using a penalty function. The problem of creating surfaces

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with G^1 continuity is very difficult to solve satisfactorily. Most techniques use heuristics to set extra degrees of freedom and relatively complex constructions to guarantee G^1 continuity; however they typically produce unnecessary and undesirable “wrinkles”. An extensive survey by members of the Graphics Group at the University of Washington [14] demonstrates these widely prevailing flaws.

The curve and surface functionals that we have derived minimize the *variation* of curvature; thus we refer to the curves as *minimal variation curves* (MVC¹) [16] and to the surfaces as *minimal variation surfaces* (MVS). In the case of curves, the integral of the squared magnitude of the derivative of curvature is minimized

$$\int \frac{d\kappa^2}{ds} ds. \quad (1)$$

Note that this integral evaluates to zero for circular arcs. For surfaces, the functional is the integral of the squared magnitude of the derivatives of normal curvature taken in the principle directions²

$$\int \frac{d\kappa_n^2}{d\hat{e}_1} + \frac{d\kappa_n^2}{d\hat{e}_2} dA.$$

Note that analogous to the MVC, the MVS functional evaluates to zero for spheres, cones, cylinders, and tori.

Section 2.0 reviews previous related work, discussing approaches, advantages, and shortcomings. Section 3.0 presents an overview of our approach outlining the steps taken to produce a surface model from specified constraints. Section 4.0 provides details of the representation of the surfaces, and a description of the optimization techniques used to compute them. Section 5.0 details the computation of the minimum energy networks used in our algorithm. Section

1. Emery Jou coined this name/acronym.
2. It is our convention that a “hat”, e.g. \hat{e} , indicates a unit vector.

6.0 presents a comparison of our approach with other methods; it exhibits examples of complex surfaces created from simple, compact specifications.

2.0 Previous Work

The work described in this paper touches on several problems and thus several areas of study. First, we discuss work on creating a G^1 surface out of a collection of non-degenerate polynomial patches. Second, we reference work on functional minimization, constrained optimization, and finite element analysis, all applied to surface design. The last portion of this section reviews minimum energy networks.

2.1 G^1 Continuity

Peters [20] provides a good classification and review of G^1 interpolation techniques. All of the methods discussed are constructive, using heuristics to set those degrees of freedom which are not fixed by continuity constraints or set as a side effect of the construction method. These methods rely on the computation of a network of curves that interpolate the data, subject to varying continuity and connectivity constraints. Peters has done a great deal of work on the construction of geometrically continuous surfaces. His most recent work outlines a method for creating " C^k " surfaces. Relevant to this discussion, G^1 surfaces [21] require quintic patches and a curve network maintaining G^2 continuity without any restrictions on the order of (i.e., the number of edges joining) the network nodes. Our work combines the construction of a G^1 continuous surface with the setting of the unconstrained degrees of freedom to form a fair G^1 continuous surface.

In related work, used in our algorithms, DeRose [8] presents the necessary and sufficient conditions for G^1 continuity between adjacent triangular and quadrilateral Bézier patches of equal degree.

2.2 Optimization, Minimization, and Finite Element Analysis

In [23] Pramila describes techniques for ship hull design that employ finite element analysis to minimize a quadratic functional approximating strain energy. Celniker [4] presents a free

form design system that uses finite element analysis to simulate physical models. Interactive deformation is carried out by simulating forces applied to the subject model. Surfaces are represented by triangular patches meeting with G^1 continuity. Approximations are used to model deformations. As a result, surfaces converge on their theoretical shape after multiple elements are inserted between constraints. Rando and Roulier [24] propose several specialized geometrically based fairness functionals. These functionals are referred to as “flattening”, “rounding”, and “rolling”. They apply these functionals to Bézier patches with some control points fixed and other free to vary, minimizing these functionals. Hagen and Schulze [10] use the calculus of variations to fit generalized Coons patches to three-dimensional data. The resulting patches minimize a strain energy fairness criterion. The analysis uses simplifying approximations to limit the complexity of calculations.

Our work uses higher order patches and the full nonlinear expression for the functional to achieve the highest possible surface quality from the fewest underlying patches.

2.3 Minimum Energy Networks

Nielson [18] introduced the minimum norm network (MNN) using linear energy terms to produce a G^1 network and a resulting G^1 surface. Pottmann [22] presents a generalization of MNN to produce a G^2 surface. Most recently we [17] describe an algorithm for the computation of a G^2 minimum energy network composed of MVC, curves minimizing (1) along the edges of the network. These MVC networks are of higher fairness and normally closer to the corresponding minimal variation surface.

3.0 Our Algorithm

We treat the problem of creating a surface interpolating a collection of geometric constraints as one of scattered data interpolation. The interpolation problem is broken into three steps (Fig. 3); 1) connectivity definition, 2) curve network computation, 3) patch blending. In accordance with the topology of the desired surface, the geometric constraints are first connected into a network of straight edges. A curve is then placed at each edge of the network, and an op-

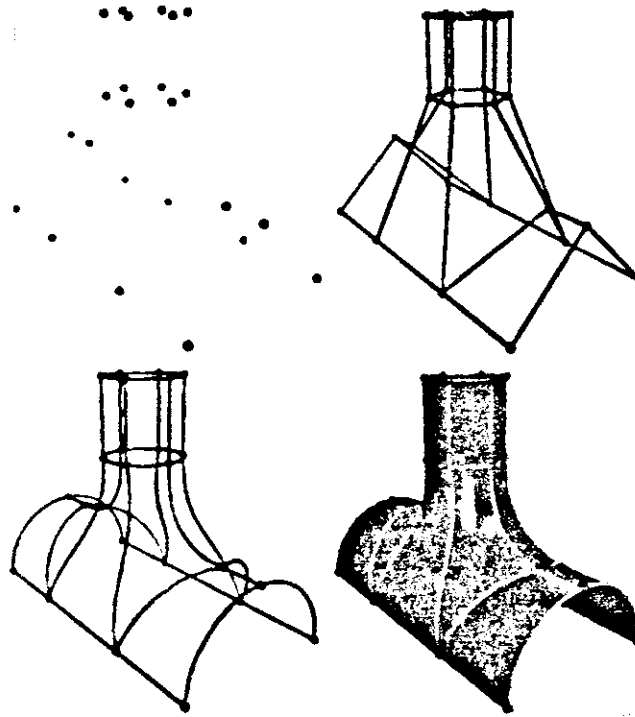


Figure 3. The blend of two pipes.

timized network is computed composed of minimal variation curves (MVC) subject to the specified geometric constraints and the additional constraint that the curve segments meet with second order geometric continuity, G^2 at the vertices. Finally, an interpolatory minimal variation surface (MVS) is computed, interpolating the MVC network with tangent continuity. In a first approach, the boundaries of the MVS patches are fixed, interpolating the previously constructed curve network. Alternatively, the surface calculation may use the MVC network as a starting point and modify its geometry during surface calculation. The latter approach yields even smoother surfaces at a substantially higher computational expense. The higher quality surface results because the curves of an MVC network resulting from a given constraint set do not always lie in the MVS resulting from the same set of constraints.

During the modeling process, the connectivity of the geometrical constraints is established as a natural outgrowth of the design process. The techniques described here are also amenable

to true scattered data interpolation, in which case connectivity must first be derived with some other method, possibly based on some minimal triangulation on the data points.

Our system is based on triangular and quadrilateral patches. All constraints are located at corners of these patches. Additional points and edges may be added to the network to create three- and four-sided openings only. These additional points do not represent constraints and are appropriately positioned by the curve network computation and patch blending phases of the construction.

The computation of a curvature continuous (G^2) MVC network is cast as a nonlinear optimization or finite element problem. First an initial shape of the curve network is computed using heuristics based on the geometry of the network. The optimization then proceeds from this starting point using standard gradient descent techniques. The geometric constraints and the second order continuity of the curve network are maintained by construction. During each iteration, the algorithm defines a direction normal to the surface, principle directions, and principle curvatures at each vertex of the network, and all the curve segments are constrained to remain consistent with them. Once the geometric and continuity constraints are satisfied, the gradient of the functional with respect to the remaining degrees of freedom is calculated, and those free parameters are adjusted iteratively to minimize the overall curvature variation functional.

Based on this MVC network, the computation of the MVS interpolatory surface is accomplished using constrained optimization. The geometric constraints are again imposed by construction similar to the MVC network. Patch-to-patch tangent continuity is imposed by means of a penalty function that is equal to zero when the patches composing the MV-surface meet with tangent continuity and proportionally greater than zero for any G^1 discontinuity. The use of penalty functions alone does not guarantee perfect tangent continuity. Exact tangent continuity may be achieved in a subsequent phase of optimization using LaGrange multipliers [6] or a continuous reduction to zero of the weight of the curvature variation term in the functional. In practice, it is rarely necessary to resort to this second phase because the surfaces resulting

from the first phase are of high quality and sufficiently close to tangent continuous. Mann and DeRose have shown *approximate* tangent continuity to be sufficient and, in fact, desirable in some applications [7].

4.0 Representation and Computation

As described in section 3.0 the computation of an MVS satisfying a set of given constraints is broken into several steps. In the remainder of this paper we will focus on the last phase of the algorithm where surface patches are fit to a G^2 MVC network. The curves may remain fixed or they may be used simply as a starting point for optimization. The choice between fixed and variable curves is up to the designer and does not affect the algorithms described here. For completeness, section 5.0 provides details of MVC network calculation.

The MVS is approximated by a quilt of parametric polynomial patches which interpolate the curve network, satisfy the geometric constraints, and meet with approximate tangent continuity. The surface functional is then minimized by varying the surface parameters that are not fixed by geometric constraints.

4.1 Bézier Patches

The curves of the network are represented by quintic Hermite polynomial segments; one segment replaces each edge of the network of constraints. Consequently, the patches making up the interpolatory surface are [bi-]quintic patches. Peters [21] has demonstrated that quintics are sufficient to achieve tangent continuity for all triangular-quadrilateral patch-patch combinations. One patch is used for each opening in the network. Though we have found single patches to have sufficient descriptive power, it is simple to subdivide network patches creating multiple patches per opening. The use of multiple patches improves the approximation of the theoretical MVS surface which in general has no closed form representation. Note that while Peters' construction requires that the curve network being interpolated have G^2 continuity, the interpolatory surface resulting from his construction is only G^1 across boundaries and at the

vertices of the network. In contrast, our surfaces are constrained to meet with G^2 continuity at the vertices of the network (see section 4.8).

Even though the boundary curves are in the Hermite form, we have chosen to use Bézier patches because of their superior numerical characteristics and because the tangent continuity conditions we use are particularly concise when formulated in terms of Bézier coefficients. Also Bézier patches are more amenable to rendering, and may be rendered directly by subroutines found in the graphics library of workstations such as the Silicon Graphics IRIS[®].

4.2 Fairness Functionals

Our choice of functional for minimization was prompted by the need for very high quality surfaces with predictable, intuitive behavior, and the desire to capture shapes commonly used in geometric modeling. The fairness of curves and surfaces has been studied extensively and has been shown to be closely related to how little and how smoothly a curve or surface bends. For an early and interesting reference see [2].

Work on the *fairness of curves* has traditionally focused on the minimization of *strain energy* or the arc length integral of the squared magnitude of curvature [15]

$$\int \kappa^2 ds.$$

We use an alternative fairness metric based on the minimization of the arc length integral of the squared magnitude of the *derivative* of curvature

$$\int \left(\frac{d\kappa}{ds}\right)^2 ds.$$

This new functional results in curves with noticeably smoother curvature plots [16], and it has the added benefit that circular arcs are formed when constraints permit, since according to this new functional they are optimally curved.

Traditional work on the *fairness of surfaces* also focuses on strain energy, minimizing the area integral of the sum of the principle curvatures squared [19, 13, 10]

$$\int \kappa_1^2 + \kappa_2^2 dA.$$

Our own approach again minimizes the *variation* of curvature, rather than its magnitude. For surfaces, we minimize the area integral of the sum of the squared magnitudes of the derivatives of the normal curvatures taken in the principle directions:

$$\int \frac{d\kappa_n^2}{d\hat{e}_1} + \frac{d\kappa_n^2}{d\hat{e}_2} dA. \quad (2)$$

The *normal curvature* at a point on a surface in a direction specified by a surface tangent vector is determined from the intersection curve of the surface with the plane spanned by the surface normal and the given tangent vector. The principle directions, \hat{e}_1 and \hat{e}_2 , and the principle curvatures, κ_1 and κ_2 , at a point on a surface are the directions and magnitudes of the minimum and maximum of all possible normal curvatures at that point [9] (Fig. 4).

Like the MVC functional, the MVS functional has associated shapes that are optimal in the sense that the functional evaluates to zero. In the case of the MVS functional, those shapes are spherical, cylindrical, conical, and toroidal, all of which have lines of principal curvature where the associated normal curvature remains constant. Lines of principal curvature follow the paths of minimum and maximum normal curvature across a surface.

4.3 Parametric Functionals

The fairness functional for surfaces (2) is defined in terms of an area integral. To evaluate the functional and its gradient in the context of the parametric polynomial surfaces patches described in section 4.1, the functional must be converted to a compatible form. Here we outline the calculations necessary to evaluate the functional. The fairness functional is computed for

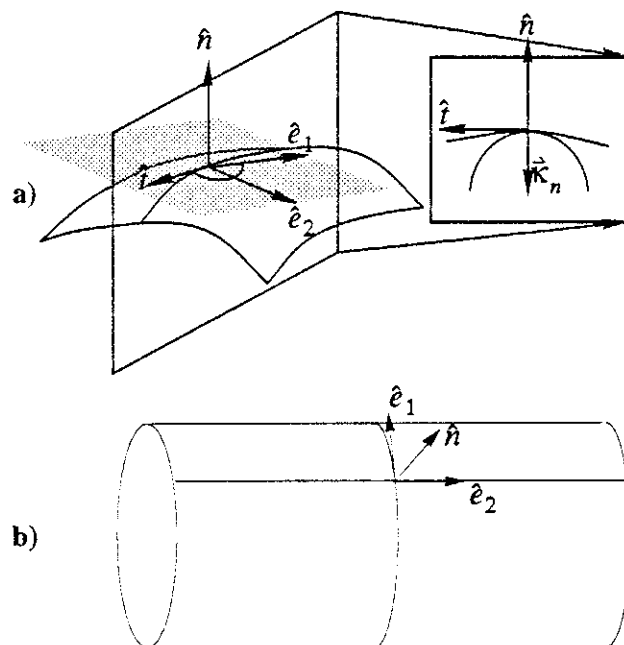


Figure 4. a. Normal curvature, the curvature of the curve formed by the intersection of the surface and a plane containing the normal and tangent. Figure 4b Principle directions and principle curvatures, the directions and magnitudes of the maximum and minimum normal curvature.

each patch, and the value of the functional for the whole surface is the sum of the values for each patch. The area based definition

$$\int \left(\frac{d\kappa_n^2}{d\hat{e}_1} + \frac{d\kappa_n^2}{d\hat{e}_2} \right) dA$$

is converted to integrals of functions of the independent parameters u and v in $\hat{S}(u, v)$. For quadrilateral patches, the bounds of the integrals are set to vary over the unit square, and the differential with respect to area is converted to differentials in u and v

$$\int_0^1 \int_0^1 \left(\frac{d\kappa_n^2}{d\hat{e}_1} + \frac{d\kappa_n^2}{d\hat{e}_2} \right) \|S_u \times S_v\| dudv$$

where

$$\|S_u \times S_v\| = \sqrt{EG - F^2},$$

and

$$\begin{aligned} E &= \hat{S}_u \cdot \hat{S}_u \\ F &= \hat{S}_u \cdot \hat{S}_v \\ G &= \hat{S}_v \cdot \hat{S}_v \end{aligned} \quad (3)$$

The variables E , F , and G , are from the first fundamental form from differential geometry [9].

The principal curvatures κ_1 and κ_2 are the normal curvatures in the principle directions. Thus the problem of computing $d\kappa_n/d\hat{e}_1$ and $d\kappa_n/d\hat{e}_2$ becomes one of computing $d\kappa_1/d\hat{e}_1$ and $d\kappa_2/d\hat{e}_2$. First we find expressions for these in terms of derivatives taken in the parametric directions

$$\begin{aligned} \frac{d\kappa_1}{d\hat{e}_1} &= \frac{d\kappa_1}{d\hat{u}} (\hat{e}_1 \cdot \hat{S}_u) + \frac{d\kappa_1}{d\hat{v}} (\hat{e}_1 \cdot \hat{S}_v) \\ \frac{d\kappa_2}{d\hat{e}_2} &= \frac{d\kappa_2}{d\hat{u}} (\hat{e}_2 \cdot \hat{S}_u) + \frac{d\kappa_2}{d\hat{v}} (\hat{e}_2 \cdot \hat{S}_v). \end{aligned}$$

where

$$\hat{S}_u = S_u / \|S_u\| \quad \hat{S}_v = S_v / \|S_v\|$$

Next we define the derivatives of κ_1 , κ_2 taken in the parametric *directions* using simple parametric derivatives:

$$\frac{d\kappa_i}{d\hat{u}} = \frac{d\kappa_i}{du} \frac{1}{\|\hat{S}_u\|} \quad \frac{d\kappa_i}{d\hat{v}} = \frac{d\kappa_i}{dv} \frac{1}{\|\hat{S}_v\|}.$$

Finally, the parametric derivatives of κ_1 and κ_2 are computed from an expression derived from the fact that the principle directions are the eigenvalues of the curvature tensor. The expression for the curvature tensor is

$$\begin{bmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{bmatrix},$$

where

$$\begin{aligned}
 a_{11} &= \frac{fF - eG}{EG - F^2} & a_{21} &= \frac{eF - fE}{EG - F^2} \\
 a_{12} &= \frac{gF - fG}{EG - F^2} & a_{22} &= \frac{fF - gE}{EG - F^2} \\
 e &= \hat{n} \cdot \vec{S}_{uu} \\
 f &= \hat{n} \cdot \vec{S}_{uv} \\
 g &= \hat{n} \cdot \vec{S}_{vv}
 \end{aligned}$$

$E, F,$ and G are defined as in equation (3), $e, f,$ and g are the terms of the second fundamental form from differential geometry [9].

Since κ_1, κ_2 and \hat{e}_1, \hat{e}_2 are the eigenvalues and eigenvectors of the curvature tensor, we get the following expressions:

$$\begin{aligned}
 \kappa_i &= \frac{a_{11} + a_{22} \pm \sqrt{a_{11}^2 + 4a_{12}a_{21} - 2a_{11}a_{22} + a_{22}^2}}{2} \\
 \hat{e}_i &= \{a_{11} - a_{22} \pm \sqrt{a_{11}^2 + 4a_{12}a_{21} - 2a_{11}a_{22} + a_{22}^2}, 2a_{12}\} \\
 \hat{e}_i &= \hat{e}_i / \|\hat{e}_i\|
 \end{aligned}$$

These expressions are in terms of the surface parameters u and v . Using the chain rule, it is simple to compute the required parametric derivatives. Note that in computing the parametric derivatives of $e, f,$ and g , it is helpful to have a simple way of computing \hat{n}_u and \hat{n}_v :

$$\begin{aligned}
 \hat{n}_u &= \kappa_1 (\vec{S}_u \cdot \hat{e}_1) \hat{e}_1 + \kappa_2 (\vec{S}_u \cdot \hat{e}_2) \hat{e}_2 \\
 \hat{n}_v &= \kappa_1 (\vec{S}_v \cdot \hat{e}_1) \hat{e}_1 + \kappa_2 (\vec{S}_v \cdot \hat{e}_2) \hat{e}_2
 \end{aligned}$$

4.4 Numerical Integration

In Section 4.3 we discussed a method for evaluating the quantity on the inside of the fairness integral (2). Because it is impractical to compute the integral analytically, we use numerical integration to evaluate the integral. Instead of using standard Gauss-Legendre quadrature, we use Lobatto quadrature [1]. Lobatto quadrature has approximately the same convergence and, un-

like Gauss-Legendre quadrature, samples the *perimeter* of the integration domain.

$$\int_0^1 f(x) dx \approx w_1 f(0.0) + \sum_{i=2}^{n-1} w_i f(x_i) + w_n f(1.0)$$

The first ten sets of abscissas and weight factors for Lobatto's integration formula are tabulated in [1]. The computation of other sets of weights and abscissas requires finding the roots of the first derivative of a Legendre polynomial. Mathematica [27] may be used to generate larger tables. Because finding the roots of high order polynomials is more difficult and prone to errors, the results calculated for a new table should be checked for accuracy. An alternative to computing higher order sets of weights and abscissas is to subdivide the domain and integrate over the subdomains.

We have found Lobatto's integration formula to be more effective than Gauss-Legendre quadrature for our application. As a default, we use ten integration points in each parametric direction a satisfactory number for the modeling problems we have encountered so far. If the number of sample points is reduced, the surface might form a cusp or crease between sample points where the integrator will not "see" it.

4.5 Differentiation

During the optimization process, it is necessary to compute the gradient of the functional with respect to all the available degrees of freedom. When computing the curve network, analytical partial derivatives were used in conjunction with numerical integration to compute the gradient. In the case of surfaces, the functional is of such complexity that it is impractical to compute the gradient in this fashion. Instead we use central differences [5] to compute the partial derivatives. The standard formula for computing the derivative of $f(a)$ with respect to a follows:

$$f'(a) = \frac{f(a+h) - f(a-h)}{2h}. \quad (4)$$

In order to get accurate derivative estimates, it is necessary to choose the difference value h carefully. An optimum value of h balances the trade-off between the discretization error resulting

from a large h and an increasing relative roundoff error resulting from too small a value for h . The analysis used to compute h is taken from [5]. First we find the approximate roundoff error in computing our functional. By computing the fairness functional, FF , in both single and double precision, we find the number of significant digits in the single precision calculation:

$$s_{\text{single}} = -\log_{10} \left| \frac{FF_{\text{single}} - FF_{\text{double}}}{FF_{\text{double}}} \right|$$

The roundoff error R in (4) is approximately

$$R = \pm \frac{2F \times 10^{-s}}{2h}.$$

The discretization error T is approximately

$$T = -\frac{1}{6}h^2.$$

To find the optimum h we must minimize

$$\frac{FF \times 10^{-s}}{h} + \frac{1}{6}h^2. \tag{5}$$

To find the value of h for which (5) is a minimum, we differentiate with respect to h and find the positive root.

$$-\frac{FF \times 10^{-s}}{h^2} + \frac{h}{3} = 0$$

$$h^3 = 3FF \times 10^{-s} \quad h = \sqrt[3]{3FF \times 10^{-s}}$$

Currently our calculations are carried out in double precision. Because we can only directly compute s_{single} , we presume a value of $s_{\text{double}} = 9$ by extrapolation. We recalculate h for each new value of the functional. For example, $FF = 10.0$ yields $h = 0.006$. Note that because the functional and its derivatives are computed on a patch by patch basis, the value of FF , and thus h , is set on a patch by patch basis, too.

4.6 Tangent Continuity

We have described how our functional is evaluated. To complete the objective function that is being minimized, we add a penalty for lack of G^1 continuity. In [8] DeRose sets forth the

necessary and sufficient conditions for G^1 continuity. The G^1 conditions take the form of a series of formulas, eq_m , all of which must be zero for G^1 continuity to exist. Using the notation of DeRose $N_{F'}$, $N_{G'}$, $N_{H'}$ refer to the degree of the cross-boundary tangent functions (F' , G') and the degree of the tangent function (H') along the boundary (Fig. 5). For example, a pair of abutting bi-quintic patches have $N_{F'} = N_{G'} = 5$, $N_{H'} = 4$ and formulas

$$eq_m = \sum_{j+k+l=m} |F_j^*, G_k^*, H_l^*| = 0$$

$$m = 0 \dots D \quad D = N_{F'} + N_{G'} + N_{H'}$$

where

$$F_j^* = \binom{N_{F'}}{j} F'_j$$

$$G_k^* = \binom{N_{G'}}{k} G'_k \quad H_l^* = \binom{N_{H'}}{l} H'_l$$

where the F'_j , G'_k , H'_l are difference vectors as shown in Figure 5. The result per shared

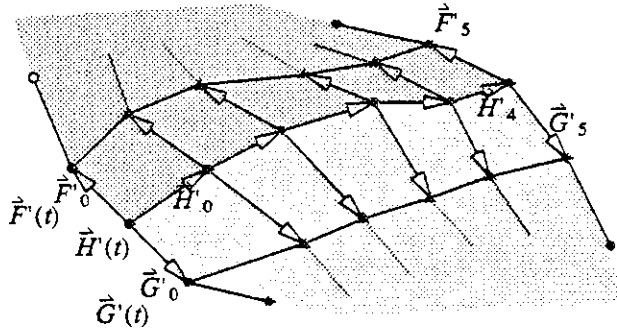


Figure 5. Difference vectors for a pair of bi-quintic patches

boundary, for our example of bi-quintic patches, is a set of fifteen equations, made up of one hundred eighty 3×3 determinants. In computing the penalty function, we square these values yielding

$$\sum_{m=0}^D \left(\sum_{j+k+l=m} |F_j^*, G_k^*, H_l^*| \right)^2. \quad (6)$$

This penalty function is computed for every G^1 patch-patch boundary and added to the fairness

functional forming the objective function that is minimized.

The complexity of solving this system of equations has been outlined by a number of authors [11, 21, 25, 26]. The complexity arises because the corner interior control points (\blacktriangle Fig. 6) appear in the cross-boundary equations for multiple sides. This multiple appearance couples different patch-patch continuity equations and thereby creates a global system of equations with a very large number of variables. In the context of the optimization described here, it is impractical and unproductive to solve this explicitly.

4.7 Initialization

The gradient descent scheme described in section 4.0 starts with an initial surface and iteratively refines that surface until the optimal surface is reached. In this section we discuss ways of finding a suitable initial surface. In terms of the desired optimization, the goal is to find an initial point in the proper “valley” of the solution space where the desired surface is found as the minimal point. The optimization requires that initial values be provided for any parameters not explicitly set. The use of an optimized curve network initializes the control points on the perimeter of each patch. The interior control points are set so as to: 1) achieve approximate G^1 continuity and 2) zero the high order derivatives at the patch corners. The first step initializes the twelve control points adjacent to the perimeter (\blacktriangle), and the second step initializes the four points in the center of the patch (\blacksquare) (Fig. 6). The heuristic used to position

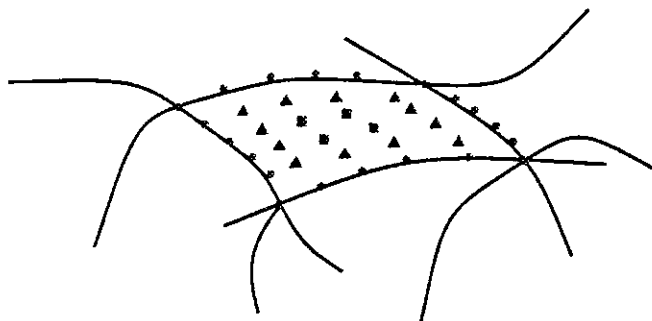


Figure 6. The 20 perimeter control points (\bullet), the 12 adjacent control points (\blacktriangle), and the 4 central control points (\blacksquare).

the control points adjacent to the perimeter uses linear interpolation of the normal vectors and magnitudes of difference vectors. Figure 7 and equation (7) demonstrate the approach, with

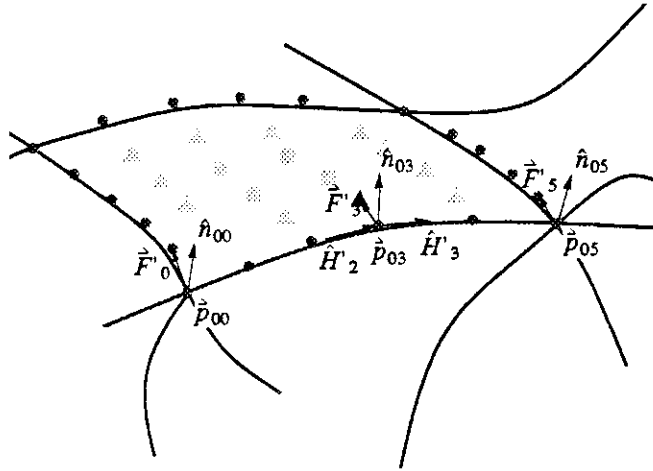


Figure 7. The initialization of Bézier control points adjacent to the perimeter of a patch. Here \vec{p}_{13} is initialized by linearly interpolating the surface normal vectors at the corners and the magnitudes of the corresponding difference

the calculation of \vec{p}_{13} , (n.b. $\|\hat{H}'_i\| = 1$):

$$\begin{aligned}
 \vec{p}_{13} &= \vec{p}_{03} + \vec{F}'_3 \\
 \vec{F}'_3 &= \|\vec{F}'_3\| \hat{F}'_3 \\
 \|\vec{F}'_3\| &= \frac{2}{5} \|\vec{F}'_0\| + \frac{3}{5} \|\vec{F}'_5\| \\
 \hat{F}'_3 &= \frac{\hat{H}'_2 \times \hat{n}_{03} + \hat{H}'_3 \times \hat{n}_{03}}{\|\hat{H}'_2 \times \hat{n}_{03} + \hat{H}'_3 \times \hat{n}_{03}\|} \quad (7) \\
 \hat{n}_{03} &= \frac{\frac{2}{5} \hat{n}_{00} + \frac{3}{5} \hat{n}_{05}}{\|\frac{2}{5} \hat{n}_{00} + \frac{3}{5} \hat{n}_{05}\|}
 \end{aligned}$$

Alternatively, one could also use the construction due to Jörg Peters [21].

4.8 G^2 Vertices

Rather than impose G^2 cross boundary continuity at the cost of many more equations to evaluate, we construct the network of patches to meet with G^2 continuity at the vertices, and we maintain this continuity by construction during the minimization process. Because of the nature of the fairness functional, the G^2 continuity tends to “propagate” along the patch-patch boundaries. A comparison of surfaces with and without G^2 vertex continuity shows those with G^2 continuity have superior overall curvature distribution.

The construction used to maintain G^2 vertex continuity of the surface is a simple extension of the construction used to maintain G^2 compatibility of the MVC network (see section 5.0). An additional step is carried out after the principle curvatures at the vertices of the network have been established. This extra step of the construction requires that the twist vectors at the incident patch corners be compatible with the established curvatures, and thus $f = \hat{n} \cdot \vec{S}_{uv}$. The value of f is computed from the formulas for mean and Gaussian curvature:

$$\text{Gaussian} = \kappa_1 \kappa_2 = \frac{LN - M^2}{EG - F^2}$$

and

$$\text{mean} = \kappa_1 + \kappa_2 = \frac{NE - 2MF + LG}{EG - F^2},$$

where

$$\begin{aligned} E &= \vec{S}_u^2 & e &= \hat{n} \cdot \vec{S}_{uu} \\ F &= \vec{S}_u \cdot \vec{S}_v & f &= \hat{n} \cdot \vec{S}_{uv} \\ G &= \vec{S}_v^2 & g &= \hat{n} \cdot \vec{S}_{vv} \end{aligned}$$

The twist vector is adjusted to satisfy $f = \hat{n} \cdot \vec{S}_{uv}$. This is accomplished by forcing the tip of \vec{S}_{uv} to lie in the plane perpendicular to \hat{n} offset by distance f (Fig. 8).

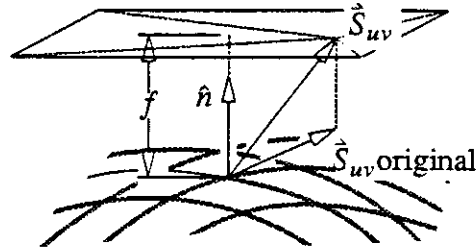


Figure 8. Calculation of a compatible twist. The tip of the twist vector is projected onto the plane perpendicular to the normal offset from the vertex by a distance f .

5.0 MVC Network Computation

MVC networks are used to initialize the boundaries of the patches making up the surface network. Many of the techniques used in computing the network are used in the computation of an MVS where the initial shape of the curve network is allowed to change. In this section we outline the methods used to compute the G^2 curve network.

5.1 Network Representation and Continuity

The network of curves is defined via the parameters of a second order “fitting” surface description at each vertex of the network and a description of how each curve segment emerges from within the fitting surfaces specified at its endpoints. Each fitting surface is defined by the vertex position \vec{p} , the conjugate directions \hat{w}_1, \hat{w}_2 , and the normal curvatures in these directions $\kappa_{w_1}, \kappa_{w_2}$. Conjugate directions are equivalent to principle directions in that, coupled with the associated curvatures, they fully characterize the curvature of a surface at a point. Conjugate directions are more amenable to optimization because they do not have to be constrained to mutual orthogonality. The network is represented by quintic Hermite curves. These Hermite curves are defined by the positions and first two parametric derivatives at their endpoints. Each curve in the network is defined by the position \vec{p} , tangent direction \hat{t} , and three scalar parameters, m, α, c , at each endpoint. The mapping from these values to the parameters defining the

corresponding Hermite curve is

$$\begin{aligned}
 \vec{P} &= \vec{p} & \vec{P}' &= m^2 \hat{t} & \vec{P}'' &= m^4 \hat{\kappa} + \alpha m^2 \hat{t} \\
 \hat{\kappa} &= \kappa_n \hat{n} + c \hat{b} & \hat{b} &= \hat{n} \times \hat{t} \\
 \kappa_n &= \hat{t} \cdot [\mathbf{K}] \cdot \hat{t}
 \end{aligned} \tag{8}$$

$$[\mathbf{K}] = \begin{bmatrix} \hat{w}_1 \\ \hat{w}_2 \\ \hat{n} \end{bmatrix}^{-1} \cdot \begin{bmatrix} \kappa_{w_1} & 0 & 0 \\ 0 & \kappa_{w_2} & 0 \\ 0 & 0 & 0 \end{bmatrix} \cdot \left(\begin{bmatrix} \hat{w}_1 \\ \hat{w}_2 \\ \hat{n} \end{bmatrix}^{-1} \right)^T$$

During the optimization process, those variables not fixed by constraints are iteratively adjusted to minimize the MVC functional. At each iteration step, \hat{w}_1, \hat{w}_2 are renormalized, and \hat{t} is projected onto the plane spanned by \hat{w}_1, \hat{w}_2 and also renormalized. It is this normalization step in combination with the construction outlined in equation (8) that guarantees G^2 continuity is maintained.

5.2 Network Initialization

The curve network must be initialized to some reasonable values before optimization may proceed. First a vertex normal vector is initialized, then the tangent vectors of the incident curves are computed, next the principle directions and curvatures are defined, and finally each curves' scalar coefficients are initialized.

The vertex normal is initialized as a weighted average of the incident face normals (weighted inversely proportional to the area of the incident face, i.e. the smaller the face the greater the weight [3]) (Fig. 9). The tangent vectors of curves incident to a vertex are set to the direction of the incident chords projected onto the plane defined by the vertex position \vec{p}_i and normal \hat{n} (Fig. 10).

Once vertex normal vectors and incident tangent directions have been computed, the principle curvatures and principle directions at a vertex are calculated. Our approach uses a least squares fit of sample tangent directions and normal curvatures to compute the principle directions and curvatures. The initialization of these values is very important to the speed of convergence. First consider the situation shown in Figure 10. A junction is shown with a number of

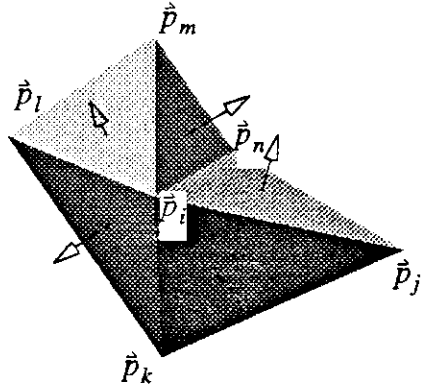


Figure 9. Vertex normal vector calculation. The normal is initialized to the average of the incident face normals inversely weighted by area.

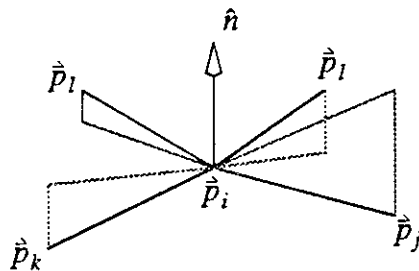


Figure 10. Tangent initialization. The projection of incident chords onto the plane defined by the normal

incident edges. For each edge we calculate the curvature implied by that edge emanating from the vertex. Starting with edge \vec{p}_l we reflect the associated tangent \hat{i}_l through the normal and fit a circle through its end point, its reflection's end point, and the base of the normal. The radius of the resulting circle is the radius of curvature (Fig. 11). Repeating this procedure for each of



Figure 11. Calculating an approximate radius of curvature in the direction of \hat{i}_l .

the incident edges, provides a set of sample tangent directions and normal curvatures. This set is used to compute a least squares fit for the principle directions and principle curvatures of the fitting surface at the vertex.

We start with the expression for normal curvature expressed with respect to any convenient basis in the plane defined by the normal,

$$\kappa_n = \hat{t} \cdot [K] \cdot \hat{t}$$

$$[K] = \begin{bmatrix} \hat{e}_{1,x} & \hat{e}_{1,y} \\ -\hat{e}_{1,y} & \hat{e}_{1,x} \end{bmatrix}^{-1} \cdot \begin{bmatrix} \kappa_1 & 0 \\ 0 & \kappa_2 \end{bmatrix} \cdot \left(\begin{bmatrix} \hat{e}_{1,x} & \hat{e}_{1,y} \\ -\hat{e}_{1,y} & \hat{e}_{1,x} \end{bmatrix}^{-1} \right)^T$$

and extract the tangent components, to produce an over determined set of linear equations:

$$\begin{bmatrix} \hat{i}_{0,x}^2 & \hat{i}_{0,x}\hat{i}_{0,y} & \hat{i}_{0,y}^2 \\ \hat{i}_{1,x}^2 & \hat{i}_{1,x}\hat{i}_{1,y} & \hat{i}_{1,y}^2 \\ \vdots & \vdots & \vdots \\ \hat{i}_{m,x}^2 & \hat{i}_{m,x}\hat{i}_{m,y} & \hat{i}_{m,y}^2 \end{bmatrix} \cdot \begin{bmatrix} \hat{e}_{1,x}^2 \kappa_{\hat{e}_1} + \hat{e}_{1,y}^2 \kappa_{\hat{e}_2} \\ 2\hat{e}_{1,x}\hat{e}_{1,y}(\kappa_{\hat{e}_1} - \kappa_{\hat{e}_2}) \\ \hat{e}_{1,x}^2 \kappa_{\hat{e}_2} + \hat{e}_{1,y}^2 \kappa_{\hat{e}_1} \end{bmatrix} = \begin{bmatrix} \kappa_{n,0} \\ \kappa_{n,1} \\ \vdots \\ \kappa_{n,m} \end{bmatrix} \quad (9)$$

$$Ax = b$$

The general formula for computing the least squares solution to this type of system is

$A^T A \bar{x} = A^T b$ where \bar{x} is the least squares solution for "x" in equation (9). Having solved for \bar{x} we have three equations and four unknowns

$$\begin{bmatrix} \hat{e}_{1,x}^2 \kappa_1 + \hat{e}_{1,y}^2 \kappa_2 \\ 2\hat{e}_{1,x}\hat{e}_{1,y}(\kappa_1 - \kappa_2) \\ \hat{e}_{1,x}^2 \kappa_2 + \hat{e}_{1,y}^2 \kappa_1 \end{bmatrix} = \begin{bmatrix} \bar{x}_0 \\ \bar{x}_1 \\ \bar{x}_2 \end{bmatrix}$$

Adding the fact that $\hat{e}_{1,x}^2 + \hat{e}_{1,y}^2 = 1$, allows us to solve for the principle directions and principle curvatures.

Finally the scalars associated with each curve are set as follows: m is set to the chord length, and α, c are set to zero.

5.3 Optional Network Constraints

Since the quality of the network directly impacts the quality of the resulting surface, we present an optional heuristic constraint. A very successful method for improving the quality of a curve network is to force curve segments into G^2 continuity through the vertices of the network. Curves are made G^1 continuous by having them share tangent vectors. G^2 continuity is imposed by forcing the two curves making up the joint to also share the bi-normal component, c from (8).

During initialization, shared tangents are set to the average of the individual tangents computed by chord projection. Figure 12 illustrates curve continuity applied to points on the surface

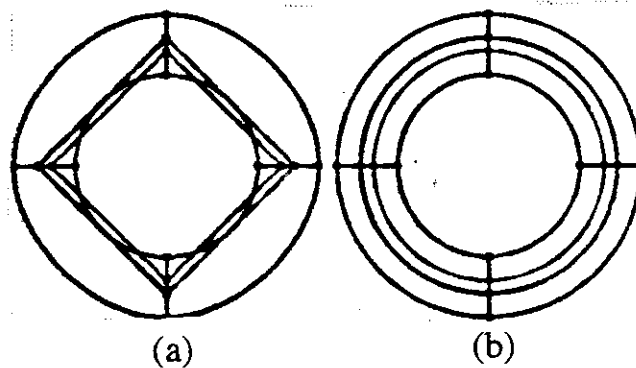


Figure 12. A network through points on a torus without (a) and with (b) continuous curves through vertices.

of a torus. The figure shows the two resulting networks after initialization. Figure 12b illustrates the improvement to the network when G^2 continuity is imposed.

6.0 Examples: A Comparison of Functionals

6.1 Spheres

In Figure 13 we compare the MVS functional with three other functionals. The surfaces are pseudo-colored using mean and Gaussian curvature to index a color map. In the upper left of the figure, only the G^1 penalty function was minimized when fitting a surface to the points of a

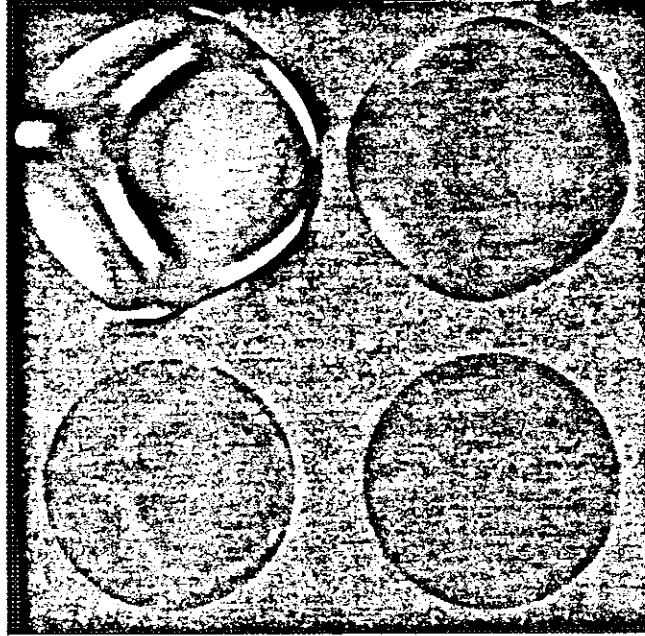


Figure 13. Cube Surfaces. Surfaces interpolating points on a cube illustrate the differences among different methods. Pseudo-coloring according to mean and Gaussian curvature exposes the differences between objective functions. Upper left - G^1 penalty, upper right - linearized strain energy, lower left - strain energy, lower right - MVS.

cube. The upper right illustrates the result of using a linearized approximation to strain energy; curvature distribution is improved. Next at the lower left, true strain energy is minimized producing a surface with fairly uniform curvature. Finally, in the lower right of the figure, an MVS surface fitted to the corners of a cube produces a very close approximation to a sphere.

6.2 Three Handles

Figure 14 illustrates the application of MVC to a more complicated example. The frame shown in the upper left is the MVC network interpolated to created the G^1 MVS. The tangent continuity is demonstrated by the surface rendered with lines of reflection, upper right. In the lower half of Figure 14, strain energy and the MVS functional are compared. The differences

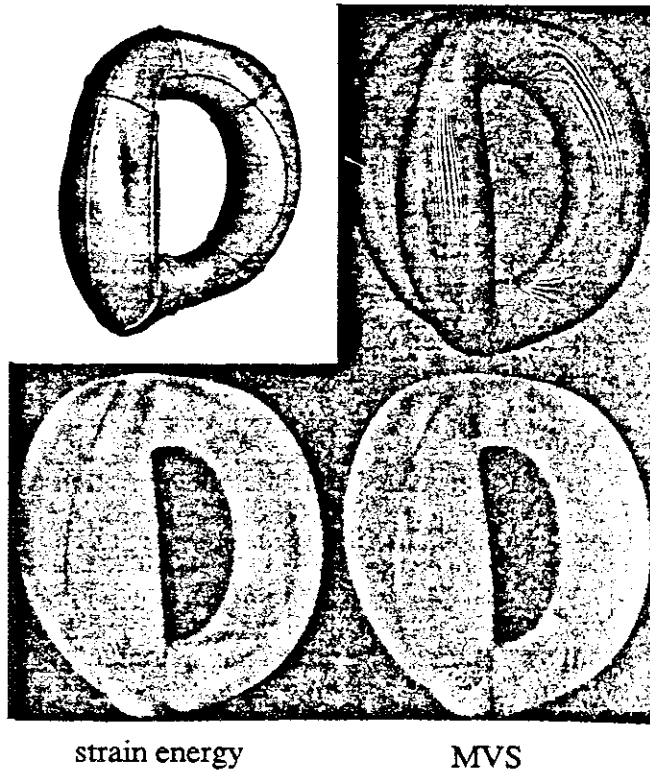


Figure 14. Three Handles. Minimization of strain energy is compared with the MVS surface starting with the same MVC network.

are subtle, curvature varies more smoothly and is distributed more evenly over the MVS surface.

6.3 Tetrahedral Frame

As our last example, an MVS surface fit to a tetrahedral frame, (Fig. 15). Proceeding clockwise from the upper left: the fitted surface with MVC network, individual patches demarcated by black borders, simple lighted rendering, lines of reflection. This last example demonstrates the versatility and power of MVS to solve a very difficult interpolation problem.

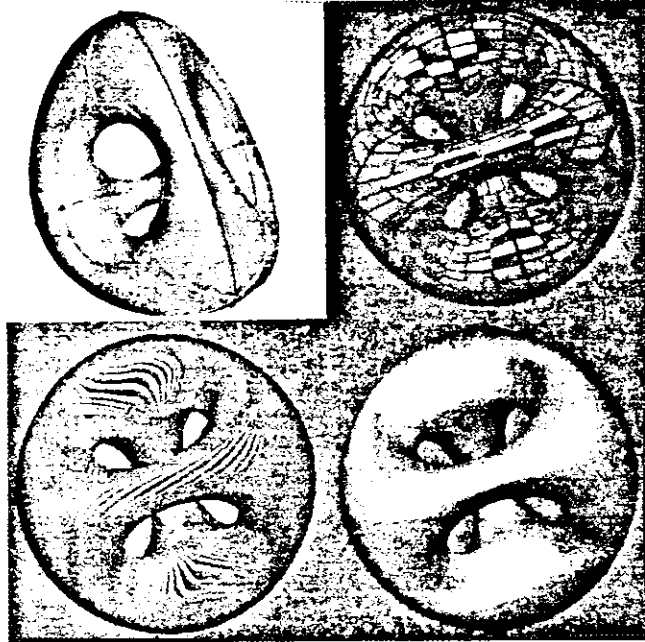


Figure 15. TetraThing. Upper left - the MVC network. Upper right - individual patches demarcated by black boundaries. Lower left - lines of reflection indicating G^1 continuity. Lower right - a simple shaded rendering.

7.0 Conclusions

Constructing a network of G^1 continuous surface patches is known to be a difficult task, and it is even harder to shape such a network into a satisfactory surface. The use of a general optimization procedure with suitable penalty functions greatly simplifies both tasks.

In this paper we have described a conceptually simple yet powerful technique for surface modeling. Nonlinear optimization is used to minimize a fairness functional while maintaining geometric and continuity constraints. The functional of choice is the variation of curvature. This choice has the advantage that it leads to regular shapes commonly used in geometric modeling; spheres, cylinders, and tori are formed in response to a compatible set of constraints. The minimization of our fairness functional also produces very fair free-form surfaces. This allows the designer to specify technical and artistic shapes in the most natural way for a given design problem.

Surfaces are represented by a patch work of quintic polynomial Bézier patches. The use of quintic patches makes the satisfaction of geometric constraints simple, direct, and exact. The use of a quintic patch also tends to minimize the *number* of patches needed to solve a given problem. The use of Bézier patches eases numerical problems and simplifies communication with other modeling systems.

The techniques described here are computationally very expensive. They have only become practical since the wide availability of very fast workstations. As computers increase in speed, these techniques will become even more attractive.

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