Keplerian Orbits

Isaac Newton derived his now classical differential equation $d^2\mathbf{z}/d\tau^2 = -\mathbf{z}/\|\mathbf{z}\|^3$ from his laws of gravity and motion to explain Johannes Kepler's laws of planetary motion in elliptical orbits. Nowadays the differential equation serves to test numerical methods for trajectory calculations. The function $\mathbf{z}(\tau)$ can be a real vector in 3-space or 2-space, or a complex number; time τ is real. The Euclidean length $\|\mathbf{z}\| := \sqrt{\mathbf{z}^{`}\cdot\mathbf{z}}$ if $\mathbf{z}$ is a vector; $\|\mathbf{z}\| := |\mathbf{z}|$ if $\mathbf{z}$ is a complex number. The path traced by $\mathbf{z}(\tau)$ is a lighter particle's orbit around a heavier attracting body and lies in a plane that includes their center of mass, which was put at the origin $\mathbf{o}$ of the plane's coordinate system. To further simplify the differential equation, the masses and the gravitational constant have been absorbed into $\mathbf{z}$ and τ to render them dimensionless.

The differential equation keeps two things called "integrals" constant. To explain them we write

$$d\mathbf{z}/d\tau = \mathbf{w}\ ; \quad d\mathbf{w}/d\tau = -\mathbf{z}/\|\mathbf{z}\|^3\ ;$$

this converts the second-order differential equation into a pair of first order equations. $\mathbf{w}$ is the particle's velocity and also its linear momentum. Its *Angular Momentum* is $\Gamma := \mathbf{z}\times\mathbf{w}$ (their cross-product), a 3-vector for 3-space, a scalar for 2-space and for the complex plane, where $\Gamma := Im(\bar{\mathbf{z}}\cdot\mathbf{w})$; here $\bar{\mathbf{z}}$ is the complex conjugate of $\mathbf{z}$. The particle's *Kinetic Energy* is $\|\mathbf{w}\|^2/2$; its gravitational *Potential Energy* is $-1/\|\mathbf{z}\|$; its *Total Energy* is $E := \|\mathbf{w}\|^2/2 - 1/\|\mathbf{z}\|$.

The differential equation keeps its "integrals" E and Γ constant.

These constants are determined by substituting for $\mathbf{z}$ and $\mathbf{w}$ their initial values, say $\mathbf{z}^\circ$ and $\mathbf{w}^\circ$ at time $\tau = 0$, and in turn they determine the shape of the orbit, a conic section of eccentricity $\varepsilon := \sqrt{(1 + 2E\cdot\|\Gamma\|^2)}$ with focus at $\mathbf{o}$. This ε must be real because, when $E < 0$ and so $\|\mathbf{z}\|\cdot\|\mathbf{w}\|^2 < 2$, then $1 \ge \varepsilon^2 = 1 + 2E\cdot\|\Gamma\|^2 \ge (\|\mathbf{z}\|\cdot\|\mathbf{w}\|^2 - 1)^2 \ge 0$ since $\|\Gamma\|^2 \le (\|\mathbf{z}\|\cdot\|\mathbf{w}\|)^2$. Consequently the orbit's shape depends solely upon Γ and ε as determined from $\mathbf{z}^\circ$ and $\mathbf{w}^\circ$:

- If $-\infty < -1/(2\|\Gamma\|^2) \le E < 0$ then $0 \le \varepsilon < 1$ and the orbit is circular or elliptical with period $T := 2\pi/(-2E)^{3/2}$, determined from the formula for τ below.
- If $E = 0$ the orbit is cometary, an unbounded parabola with $\varepsilon = 1$, unless $\|\Gamma\| = 0$.
- If $E > 0$ the orbit is a branch of an hyperbola, and $\varepsilon > 1$.

Assume $\|\Gamma\| \ne 0$ lest the orbit degenerate to a line segment, finite if $E < 0$ or $\mathbf{z}^{\circ`}\cdot\mathbf{w}^\circ \le 0$.

Each of a differential equation's integrals reduces its order, dimension or degrees of freedom. Consequently the orbit can be expressed implicitly in polar coordinates (r, θ) in the orbit's plane, which is perpendicular to Γ in 3-space. θ is the angle subtended at $\mathbf{o}$ by the orbit's arc joining $\mathbf{z}$ to the orbit's *pericentre*, the orbit's point nearest $\mathbf{o}$. Complex $\mathbf{z}(\tau) = r(\tau)\cdot\exp(\imath\cdot\theta(\tau))$.

$$\|\mathbf{z}(\tau)\| := r(\tau) := \|\Gamma\|^2/\big(1 + \varepsilon\cdot\cos\theta(\tau)\big) \quad \text{wherein } \theta = \theta(\tau) \text{ satisfies } \dots$$

$$d\theta/d\tau = (1 + \varepsilon\cdot\cos\theta)^2/\|\Gamma\|^3 \text{ for counter-clockwise rotation; and if } E < 0 \text{ then } \dots$$

$$(-2E)^{3/2}\cdot\tau = \theta - 2\cdot\arctan\left(\frac{\varepsilon\cdot\sin\theta}{\sqrt{(1-\varepsilon^2)} + 1 + \varepsilon\cdot\cos\theta}\right) - \frac{\sqrt{(1-\varepsilon^2)}\cdot\varepsilon\cdot\sin\theta}{1 + \varepsilon\cdot\cos\theta}\ ,$$

assuming that initially $\theta = 0$ and $\mathbf{z}$ is at the pericentre when $\tau = 0$. Otherwise, if the orbit is launched elsewhere, an origin different from $\tau = 0$ must be determined from measurements of $\mathbf{z}^{\circ`}\cdot\mathbf{w}^\circ$ (or $Re(\bar{\mathbf{z}}^\circ\cdot\mathbf{w}^\circ)$), whose sign distinguishes outward bound (+) from inward bound (–), and if 0 then $\|\Gamma\|^2 - \|\mathbf{z}^\circ\|$ whose sign distinguishes pericentre (+) from apocentre (–).

Exercises

Use representations of position **z** and velocity **w** as complex variables to answer the questions.

1. The foregoing formulas were developed for a counter-clockwise orbit. How do the initial values **z**° and **w**° determine whether the orbit is clockwise or counter-clockwise, and how must the formulas be changed if the orbit is clockwise?

2. Provide formulas that offset the origin for the time variable τ to accommodate arbitrary starting values **z**° and **w**° when the eccentricity $\varepsilon < 1$.

3. Devise and test a MATLAB program to compute **z**(τ) given **z**°, **w**° and τ when $\varepsilon < 1$. You may use MATLAB's fzero function. How do you test a program when you have no other way to compute the desired result? How accurate do you think your program is, and why?

4. How does the explicit formula relating τ to θ change for parabolic and hyperbolic orbits (when $\varepsilon \ge 1$) ? You may consult a table of integrals or an automated algebra system.

5. Derive the equation $\Gamma^2 = |\mathbf{z}| + \varepsilon \cdot Re(\mathbf{z})$ of the orbit, and translate it into Cartesian (x, y) coordinates so that, given a numerical value for either x or y, you can compute the other's two values. Then derive a formula $\mathbf{w} = \imath \cdot (\varepsilon + \mathbf{z}/|\mathbf{z}|)/\Gamma$ to compute it numerically from Γ, ε and **z** .

6. When $\varepsilon < 1$, the trigonometric formula above computes τ numerically from E, ε and θ. How should time τ be computed more directly from E, Γ, ε and position **z** instead? At most one trigonometric function need be invoked if the following formulas are used:

$$\text{Let} \quad \Psi := \frac{y\cdot|\mathbf{z}|\cdot(\sqrt{1-\varepsilon^2} + 1 - \varepsilon)}{(|\mathbf{z}| + x)\cdot(|\mathbf{z}|\cdot\sqrt{1-\varepsilon^2} + \Gamma^2)} = \frac{(|\mathbf{z}| - x)\cdot|\mathbf{z}|\cdot(\sqrt{1-\varepsilon^2} + 1 - \varepsilon)}{y\cdot(|\mathbf{z}|\cdot\sqrt{1-\varepsilon^2} + \Gamma^2)} \;; \quad \text{then}$$

$$(-2E)^{3/2}\cdot\tau = 2\cdot\arctan(\Psi) - y\cdot\varepsilon\cdot\sqrt{(1-\varepsilon^2)}/\Gamma^2 \;\text{ determines }\; \tau \pm (\text{integer})\cdot T\,.$$

Can you confirm these formulas?

7. How sensitive is an elliptical orbit to perturbations in its initial conditions? To explore this question, imagine two orbits initiated one at **z**° and **w**° and the other at **z**° + δ**z**° and **w**° + δ**w**° with infinitesimal perturbations δ**z**° and δ**w**° so tiny that we can ignore their squares and higher powers. Denote one orbit by **z**(τ) and **w**(τ) , and the other by **z**(τ) + δ**z**(τ) and **w**(τ) + δ**w**(τ) . As τ increases towards $+\infty$, how fast can $|\delta\mathbf{z}(\tau)|$ and $|\delta\mathbf{w}(\tau)|$ grow? Can they grow exponentially? Or can they grow at most like some power of τ, and if so, which? These questions deserve only qualitative answers since small non-infinitesimal perturbations cannot grow beyond the diameters of the ellipses.