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NOTE

FAST UNIFORM GENERATION OF REGULAR GRAPHS

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Abstract. An algorithm is presented which randomly selects a labelled graph with specified vertex degrees from a distribution which is arbitrarily close to uniform. The algorithm is based on simulation of a rapidly convergent stochastic process, and runs in polynomial time for a wide class of degree sequences, including all regular sequences and all n-vertex sequences with no degree exceeding $\sqrt{n/2}$. The algorithm can be extended to cover the selection of a graph with given degree sequence which avoids a specified set of edges. One consequence of this extension is the existence of a polynomial-time algorithm for selecting an f-factor in a sufficiently dense graph. A companion algorithm for *counting* degree-constrained graphs is also presented; this algorithm has exactly the same range of validity as the one for selection.

1. Synopsis

This paper addresses the following problem: given a sequence $\boldsymbol{d} = (d_1, \ldots, d_n)$ of non-negative integers, select, uniformly at random, a labelled undirected graph on n vertices whose vertex degrees are precisely $d_1, \ldots, d_n$. We shall exhibit an algorithm which, for a wide class of degree sequences, efficiently selects such a graph from a distribution which is arbitrarily close to uniform. The class includes all regular degree sequences, and all n-vertex sequences with maximum degree $\sqrt{n/2}$. Here, as elsewhere in the paper, the word "efficient" is used to describe algorithms whose execution time is bounded by a polynomial in n.

This problem has recently received a considerable amount of attention, motivated in part by the increasingly important role played by the random regular graph model in probabilistic graph theory [4]. The earliest technique for uniform generation of graphs with given degree sequence is implicit in the work of Bender and Canfield [2] and Bollobás [3], and is made explicit in an algorithm presented by Wormald [15]; this algorithm is efficient only for degrees up to $O((\log n)^{1/2})$ in the regular case (and in practice only for very small degrees). Wormald [15] also presents specialised exact algorithms for cubic and 4-regular graphs, but these do not seem to generalise to higher degrees. By an indirect method based on approximate counting, the present authors [14] give an almost uniform generation algorithm which runs in polynomial time for regular graphs up to degree $O(n^{1/3})$. Finally, two very recent papers [6, 12] handle degrees up to $o(n^{1/5})$ and $o(n^{1/3})$, respectively, using more direct methods. (In the latter case, the output distribution of graphs is *exactly* uniform.) We should also mention that generators for labelled graphs with given degrees can be used in the uniform generation of *unlabelled* graphs [16].

The approach we describe here not only handles *all* regular degree sequences for the first time, but also has the merit of being extremely simple to describe. It is based on a Markov chain simulation technique which has recently proved to be a powerful tool for the random generation of various combinatorial structures [5, 7, 14]. The idea is as follows: for a degree sequence $\boldsymbol{d}$, let $\mathscr{G}(\boldsymbol{d})$ denote the set of labelled undirected graphs with vertex set $\{1, \ldots, n\}$ in which the degree of vertex i is d_i. We set up a Markov chain $\mathscr{M}\mathscr{C}(\boldsymbol{d})$ whose states include the elements of $\mathscr{G}(\boldsymbol{d})$, together with some auxiliary structures, and whose transitions correspond to simple random perturbations (edge additions and deletions). This process will turn out to converge asymptotically to a stationary distribution which is uniform over the states. Moreover, and crucially, for many values of $\boldsymbol{d}$ the convergence will turn out to be *fast* in the sense that the distribution gets very close to uniform after only polynomially many steps. This strong property is known as *rapid mixing* [1, 13]. Thus we can generate elements of $\mathscr{G}(\boldsymbol{d})$ almost uniformly by simulating the evolution of $\mathscr{M}\mathscr{C}(\boldsymbol{d})$ for some small number of steps and outputting the final state.

The Markov chain we use is one which we have already studied extensively in a different context in [7], and has as its states the perfect and "near-perfect" matchings of a given graph. This chain is relevant here because we may identify elements of $\mathscr{G}(\boldsymbol{d})$ with perfect matchings in a suitably constructed graph. We may then appeal to the analysis of [7] to establish the rapid mixing property, under a certain condition on the degree sequence $\boldsymbol{d}$. The methods of [7] also yield an efficient algorithm for testing this condition for an *arbitrary* degree sequence. Moreover, we can show that the condition is certainly satisfied by all k-regular sequences for $k \leq n/2$, and all n-vertex sequences with maximum degree $\sqrt{n/2}$. By complementation, we can therefore handle *all* regular degree sequences. Our technique can be extended to handle the uniform generation of graphs with given degree sequence which avoid a specified set of edges (a so-called *excluded graph*); for this the only additional condition we require is that the maximum vertex degree of the excluded graph should not be too large.

We also observe that the generation procedures lead immediately to efficient algorithms for estimating the number of graphs on a given degree sequence $\boldsymbol{d}$, under the same conditions on $\boldsymbol{d}$. This problem has also been the subject of much recent research: asymptotic estimates have been obtained for a class of sequences with maximum degree $o(n^{1/2})$ [10]. (Graphs of high degree are studied in [11].) The range of validity of our algorithmic method is therefore much wider.

2. Degree sequences and matchings

A *generation problem* is defined by a function, $\mathscr{S}$, which maps a set of *instances* to a set of possible *solutions*. For example, the function $\mathscr{S}$ may have as its domain the set of all undirected graphs, and may associate with each graph Γ the set of all perfect matchings in Γ. We shall assume that each problem instance x has a well-defined *size*. An *almost uniform generator* for $\mathscr{S}$ is a probabilistic algorithm which, given an instance x and a positive real *bias* ε, outputs an element of $\mathscr{S}(x)$ such that the probability of each element appearing approximates $|\mathscr{S}(x)|^{-1}$ within ratio $1+\varepsilon$.[1] The generator is *fully polynomial* if its execution time is bounded by a polynomial in $\lg \varepsilon^{-1}$ and the size of x. (For a fuller treatment of almost uniform generation see [8, 14].)

Recall that $\mathscr{G}(\boldsymbol{d})$ denotes the set of all labelled graphs with degree sequence $\boldsymbol{d}=(d_1,\ldots,d_n)$. We assume in the sequel that the sequence $\boldsymbol{d}$ is *graphical*, that is to say $|\mathscr{G}(\boldsymbol{d})|>0$; note that there are simple polynomial-time algorithms for testing this condition (see, for example, exercise 7.50 of [9]). Define $\mathscr{G}'(\boldsymbol{d})$ to be $\bigcup_{\boldsymbol{d}'} \mathscr{G}(\boldsymbol{d}')$, where the union ranges over vectors $\boldsymbol{d}'\in\mathbb{N}^n$ which satisfy $\boldsymbol{d}'\leqslant \boldsymbol{d}$ and $\sum_{i=1}^n |d_i-d_i'|\leqslant 2$. Call a class of degree sequences *P-stable* if there exists a polynomial p such that $|\mathscr{G}'(\boldsymbol{d})|/|\mathscr{G}(\boldsymbol{d})|\leqslant p(n)$ for every sequence $\boldsymbol{d}=(d_1,\ldots,d_n)$ in the class. Informally, a degree sequence $\boldsymbol{d}$ is P-stable if $|\mathscr{G}(\boldsymbol{d})|$ does not change radically when $\boldsymbol{d}$ is slightly perturbed. Although the class of *all* graphical degree sequences is not P-stable, there are natural subclasses which are. We shall return to this issue in the next section.

Our aim is to construct a fully polynomial almost uniform generator for $\mathscr{G}(\boldsymbol{d})$, which is valid for all sequences $\boldsymbol{d}$ within a specified P-stable class. Our approach is to reduce this generation problem to the previously studied problem of generating perfect matchings in an undirected graph. For any undirected graph Γ, let $\mathscr{M}(\Gamma)$ denote the set of perfect matchings in Γ, and let $\mathscr{M}^-(\Gamma)$ denote the set of *near-perfect matchings* in Γ, i.e., those matchings which leave unmatched precisely two vertices of Γ. From Theorem 3.6 of [7], and the comment immediately preceding that theorem, we know the following.

Theorem 2.1. *Let $q(\cdot)$ be any polynomial. There is a fully polynomial almost uniform generator for $\mathscr{M}(\Gamma)$, which is valid for all m-vertex graphs Γ which satisfy $|\mathscr{M}^-(\Gamma)|/|\mathscr{M}(\Gamma)|\leqslant q(m)$.*

As indicated earlier, the generator is obtained by simulating a rapidly mixing Markov chain with state space $\mathscr{M}(\Gamma)\cup\mathscr{M}^-(\Gamma)$ and uniform stationary distribution. The bound on the ratio $|\mathscr{M}^-(\Gamma)|/|\mathscr{M}(\Gamma)|$ is necessary both to establish the rapid mixing property and to ensure that the chain visits perfect matchings reasonably often. A full discussion can be found in [7].

We are now in a position to state and prove the main result of this section.

[1] For non-negative real numbers a, $\tilde{a}$, ε, we say that $\tilde{a}$ *approximates* a *within ratio* $1+\varepsilon$ if $a(1+\varepsilon)^{-1}\leqslant \tilde{a}\leqslant a(1+\varepsilon)$.

Theorem 2.2. *There is a fully polynomial almost uniform generator for $\mathcal{G}(\boldsymbol{d})$ provided the degree sequence $\boldsymbol{d}$ is drawn from some* P-*stable class.*

Proof. For given degree sequence $\boldsymbol{d}=(d_1,\ldots,d_n)$, let $\Gamma=\Gamma(\boldsymbol{d})$ be the undirected graph with vertex set

$$V(\Gamma)=\{v_{ik}: 1\leq i\leq n \text{ and } 1\leq k\leq d_i\}\cup\{u_{ij}: 1\leq i,j\leq n \text{ and } i\neq j\}$$

and edge set

$$E(\Gamma)=\{(v_{ik},u_{ij}): 1\leq i,j\leq n, 1\leq k\leq d_i, \text{ and } i\neq j\}$$
$$\cup\{(u_{ij},u_{ji}): 1\leq i,j\leq n \text{ and } i\neq j\}.$$

The intention is to set up a correspondence between perfect matchings M in Γ and elements of $\mathcal{G}(\boldsymbol{d})$. Informally, Γ contains an edge (u_{ij},u_{ji}) corresponding to each potential edge (i,j) in a graph $G\in\mathcal{G}(\boldsymbol{d})$; the *presence* of the edge (u_{ij},u_{ji}) in M models the *absence* of the edge (i,j) in G. Additionally there are n *clusters* of vertices of the form $\{v_{ik}: 1\leq k\leq d_i\}$ which, together with their incident edges, enforce the degree constraints at each vertex i in G.

Let ϕ be the function, from matchings in Γ to (undirected) graphs on vertex set $\{1,\ldots,n\}$, which maps the matching $M\subseteq E(\Gamma)$ to the graph with edge set

$$\{(i,j): i\neq j \text{ and } (u_{ij},u_{ji})\notin M\}.$$

Denote the set of all perfect matchings in Γ by $\mathcal{M}=\mathcal{M}(\Gamma)$. It is a straightforward task to verify that $\phi(\mathcal{M})=\mathcal{G}(\boldsymbol{d})$ and, moreover, that each graph in $\mathcal{G}(\boldsymbol{d})$ is the image of precisely $\prod_{i=1}^{n} d_i!$ elements of $\mathcal{M}$. Thus, to generate elements of $\mathcal{G}(\boldsymbol{d})$ almost uniformly, it is enough to generate perfect matchings in $\Gamma(\boldsymbol{d})$ almost uniformly. By Theorem 2.1, this will be possible provided $|\mathcal{M}^-(\Gamma(\boldsymbol{d}))|/|\mathcal{M}(\Gamma(\boldsymbol{d}))|\leq q(m)$, where $m=|V(\Gamma)|$ and q is some fixed polynomial. (The polynomial q will depend on the polynomial p in the definition of P-stable.)

Call a matching M of Γ *normalised* iff either (i) M is a perfect matching, or (ii) M is a near-perfect matching whose unmatched vertices are both cluster vertices. Denote the set of all normalised matchings of Γ by $\mathcal{N}(\Gamma)$. Let $M\in\mathcal{M}^-(\Gamma)$ be a matching in which the vertex u_{ij} is unmatched. By adding the edge (u_{ij},u_{ji}) to M, and removing the edge from M which was previously incident at u_{ji}, we succeed in moving one unmatched vertex into the set of cluster vertices. Two such operations are sufficient to normalise any near-perfect matching. (If vertices u_{ij} and u_{ji} are both unmatched, then M can be normalised by adding the single edge (u_{ij},u_{ji}).)

The normalising operation maps at most n^2 distinct matchings onto a single normalised matching; hence $|\mathcal{M}^-(\Gamma)|\leq n^2|\mathcal{N}(\Gamma)|$. It is straightforward to check that $\phi(\mathcal{N}(\Gamma))=\mathcal{G}'(\boldsymbol{d})$ and that each element of $\mathcal{G}'(\boldsymbol{d})$ is the image of at most $\prod_{i=1}^{n} d_i!$ elements of $\mathcal{N}(\Gamma)$. Putting these facts together we have

$$\frac{|\mathcal{M}^-(\Gamma)|}{|\mathcal{M}(\Gamma)|}\leq\frac{n^2|\mathcal{N}(\Gamma)|}{|\mathcal{M}(\Gamma)|}\leq\frac{n^2|\mathcal{G}'(\boldsymbol{d})|}{|\mathcal{G}(\boldsymbol{d})|}\leq n^2p(n).$$

The proof is completed by appealing to Theorem 2.1; the degree of the polynomial q in the statement of that theorem can be taken as $\lceil \frac{1}{2} \deg p \rceil + 1$. □

Whenever an efficient almost uniform generator exists for a class of combinatorial structures, we may reasonably expect that an efficient approximation algorithm will exist for counting those structures [8]. Let f be a function from problem instances to natural numbers. A ***randomised approximation scheme*** for f is a probabilistic algorithm which, when presented with an instance x and a real number $\varepsilon > 0$, outputs a number which, with probability at least $\frac{3}{4}$, approximates $f(x)$ within ratio $(1+\varepsilon)$. The approximation scheme is *fully polynomial* if it runs in time bounded by a polynomial in ε^{-1} and the size of x. From Corollary 3.7 of [7] we learn the following.

Theorem 2.3. *Let $q(\cdot)$ be any polynomial. There is a fully polynomial randomised approximation scheme for $|\mathcal{M}(\Gamma)|$, which is valid for all m-vertex graphs Γ which satisfy $|\mathcal{M}^{-}(\Gamma)|/|\mathcal{M}(\Gamma)| \leq q(m)$.*

We therefore have immediately, by the reduction of Theorem 2.2, the following theorem.

Theorem 2.4. *There is a fully polynomial randon ised approximation scheme for $|\mathcal{G}(\boldsymbol{d})|$ provided the degree sequence $\boldsymbol{d}$ is drawn from some* P*-stable class.*

The proof of Theorem 2.2 hinged on the fact that a polynomial bound on the ratio $|\mathcal{G}'(\boldsymbol{d})|/|\mathcal{G}(\boldsymbol{d})|$ implies a polynomial bound on the ratio $|\mathcal{M}^{-}(\Gamma(\boldsymbol{d}))|/|\mathcal{M}(\Gamma(\boldsymbol{d}))|$; in fact the reverse implication also holds. P-stability is therefore not merely a sufficient, but also a necessary condition for our reduction to be applicable.

It is interesting to observe that, in Theorems 2.2 and 2.4, it is not necessary to know *a priori* the specific polynomial p implicit in the requirement of P-stability. A polynomial time randomised algorithm presented in Section 5 of [7] may be used to estimate the ratio $|\mathcal{M}^{-}(\Gamma)|/|\mathcal{M}(\Gamma)|$ within a constant factor; the resulting estimate then determines how long the Markov chain simulation should run before the probability distribution over states is sufficiently uniform.

In the next section, we will develop a simple numerical condition on degree sequences which is sufficient to guarantee P-stability.

Remark. The algorithms we have described involve transforming a given degree sequence $\boldsymbol{d}$ into a graph $\Gamma(\boldsymbol{d})$, and simulating a Markov chain which has a natural interpretation in terms of matchings in $\Gamma(\boldsymbol{d})$. We might ask whether a more direct attack is possible. For a given degree sequence $\boldsymbol{d}$, consider the Markov chain with state space $\mathcal{G}'(\boldsymbol{d})$ and transitions as follows: in state $G \in \mathcal{G}'(\boldsymbol{d})$, select an ordered pair i, j of vertices uniformly at random and then

(i) if $G \in \mathcal{G}(\boldsymbol{d})$ and (i, j) is an edge of G, delete (i, j) from G,

(ii) if $G \notin \mathcal{G}(\boldsymbol{d})$, the degree of i in G is less than d_i, and (i, j) is not an edge of G, add (i, j) to G; if this causes the degree of j to exceed d_j, select an edge (j, k) uniformly at random and delete it.

It is easy to verify that this Markov chain converges to a uniform stationary distribution. (This follows because it is irreducible, aperiodic and symmetric.) Using techniques very similar to those developed in [7, 13], it can be shown that the chain is rapidly mixing under certain conditions on $\boldsymbol{d}$: in particular, the criterion given in Theorem 3.1 of the next section is certainly sufficient. This Markov chain therefore yields an alternative, and arguably more natural version of the generation algorithm described here. However, we prefer not to obscure our presentation with a detailed analysis of the modified chain.

3. A criterion for P-stability

For a graphical degree sequence $\boldsymbol{d}=(d_1,\dots,d_n)$, define $d_{\max}=\max_i d_i$ and $e(\boldsymbol{d})=\frac{1}{2}\sum_{i=1}^n d_i$. Note that $e(\boldsymbol{d})$ is integral. We now derive a very useful sufficient condition for $\boldsymbol{d}$ to be P-stable in terms of the quantities $d_{\max}$ and $e(\boldsymbol{d})$.

Theorem 3.1. *The class of all degree sequences $\boldsymbol{d}$ which satisfy $e(\boldsymbol{d})>d_{\max}^2-d_{\max}$ is P-stable.*

Proof. Let $\boldsymbol{d}$ be a degree sequence satisfying the above condition. We will show how to associate with each graph $G\in\mathscr{G}'(\boldsymbol{d})$ a graph $\bar{G}\in\mathscr{G}(\boldsymbol{d})$ which is "close to" G, in the sense that G can be transformed into $\bar{G}$ via a simple edge exchange operation. The result will then follow from the observation that no graph in $\mathscr{G}(\boldsymbol{d})$ can be close to too many graphs in $\mathscr{G}'(\boldsymbol{d})$.

If $G\in\mathscr{G}(\boldsymbol{d})$ we simply set $\bar{G}=G$, so assume that $G\in\mathscr{G}'(\boldsymbol{d})-\mathscr{G}(\boldsymbol{d})$. We describe the operation first in the case that G has degree sequence $\boldsymbol{d}'$ of the form

$$d_i'=\begin{cases} d_i-1 & \text{for } i\in\{k,l\},\\ d_i & \text{otherwise.}\end{cases}$$

If G does not contain the edge (k,l), we just add this edge to G to form $\bar{G}$. If on the other hand the edge (k,l) is already in G, we look for a pair x, y of vertices such that (x,y) is an edge of G and

(i) x, y, k, l are all distinct,

(ii) (x,k) and (y,l) are not edges of G.

Then the graph $\bar{G}$ is formed by adding to G the edges (x,k) and (y,l) and deleting the edge (x,y). We claim that such a pair x, y can always be found. To see this, note that there are $2e(\boldsymbol{d}')=2(e(\boldsymbol{d})-1)$ candidates for the ordered pair x, y among endpoints of edges of G, some of which are excluded by requirements (i) and (ii). Elementary counting reveals that the number of candidates excluded by (i) is at most $2(d_k'+d_l')-2\leqslant 2(2d_{\max}-3)$. Similarly, the number excluded by (ii) is at most $2(d_{\max}-2)(d_{\max}-1)$. The total number of excluded candidates therefore does not exceed $2(d_{\max}^2-d_{\max}-1)$. It follows that a suitable pair x, y can be found provided that

$$2(e(d)-1)>2(d_{\max}^2-d_{\max}-1),$$

which is equivalent to the condition on d stipulated in the theorem.

It remains to describe $\bar{G}$ when the degree sequence of G is

$$d_i' = \begin{cases} d_i - 2 & \text{for } i = k, \\ d_i & \text{otherwise.} \end{cases}$$

In this case, we seek an edge (x, y) of G for which (i) $x, y \neq k$, and (ii) (x, k), (y, k) are not edges of G. The graph $\bar{G}$ is then obtained from G by adding the edges (x, k) and (y, k) and deleting (x, y). Using similar reasoning to the above, the reader may easily verify that a suitable edge (x, y) always exists under the stated condition on d.

Now for any graph $H \in \mathcal{G}(d)$, define the set

$$\mathcal{K}(H) = \{G \in \mathcal{G}'(d) \colon \bar{G} = H\}.$$

Note that the sets $\mathcal{K}(H)$ partition $\mathcal{G}'(d)$. It is a straightforward task to verify that each element of $\mathcal{K}(H)$ can be coded by a unique tuple (x, y, k, l), and hence that $|\mathcal{K}(H)| \leq n^4$. We therefore conclude that $|\mathcal{G}'(d)|/|\mathcal{G}(d)| \leq n^4$, so this class of degree sequences is indeed P-stable. □

Informally, Theorem 3.1 says that a degree sequence belongs to a P-stable class provided its maximum and average degrees do not differ by too much. Let us mention two important types of degree sequence which satisfy this condition:

(i) Let $d = (k, k, \ldots, k)$, i.e., d is k-regular, with $k \leq n/2$. Then setting $e(d) = nk/2$ and $d_{\max} = k$, we see at once that the hypothesis of Theorem 3.1 is satisfied.

(ii) Suppose all degrees in d lie in the range $[1, \sqrt{n/2}]$. Then setting $e(d) \geq \frac{1}{2}(d_{\max} + n - 1)$ and $d_{\max} \leq \sqrt{n/2}$, the hypothesis of Theorem 3.1 is again seen to hold.

Note also that there is an obvious bijection between the sets $\mathcal{G}(d)$ and $\mathcal{G}(\bar{d})$, where $\bar{d}$ is the complement of d, i.e., $\bar{d}_i = n - 1 - d_i$. Hence for the purposes of generation and counting d and $\bar{d}$ are equivalent. We summarise these observations in the following corollary.

Corollary 3.2. *There exists a fully polynomial almost uniform generator for* $\mathcal{G}(d)$, *and a fully polynomial randomised approximation scheme for* $|\mathcal{G}(d)|$, *provided the degree sequence* d *satisfies* $e(d) > d_{\max}^2 - d_{\max}$. *In particular, such algorithms exist for*

(i) *all regular degree sequences,*

(ii) *all n-vertex sequences with maximum degree at most* $\sqrt{n/2}$ *and no isolated vertices (and hence, by complementation, all n-vertex sequences with minimum degree at least* $n - \sqrt{n/2} - 1$ *and no vertices of degree* $n-1$).

Remark. The class of *all* graphical degree sequences is not P-stable. To see this, consider the family of degree sequences on $2k$ vertices of the form

$$d^{(k)} = (1, 2, \ldots, k-1, k, k, k+1, \ldots, 2k-1)$$

for $k=3,4,\ldots$. Observe that the selection of a graph with any given degree sequence $\boldsymbol{d}=(d_1,\ldots,d_n)$ can be viewed recursively as follows:

(1) select a set of d_n neighbours for vertex n;

(2) recursively select a graph with degree sequence $\boldsymbol{d}'=(d'_1,\ldots,d'_{n-1})$, where $\boldsymbol{d}'$ is obtained from $\boldsymbol{d}$ by deleting the final component and decrementing by 1 the components which correspond to the d_n chosen neighbours for vertex n.

Adopting this view, it is clear that there is a unique graph with degree sequence $\boldsymbol{d}^{(k)}$.

Now modify $\boldsymbol{d}^{(k)}$ by decrementing by 1 the final two components. Note that graphs on the modified degree sequence are members of $\mathscr{G}'(\boldsymbol{d}^{(k)})$. Applying the recursive selection procedure we therefore have

$$\begin{aligned}
|\mathscr{G}'(\boldsymbol{d}^{(k)})| &\geqslant |\mathscr{G}(1,2,\ldots,k,k,k+1,\ldots,2k-3,2k-3,2k-2)| \\
&\geqslant |\mathscr{G}(1,1,2,\ldots,k-1,k-1,k,\ldots,2k-4,2k-4)| \\
&\geqslant |\mathscr{G}(1,1,1,2,\ldots,k-2,k-2,k-1,\ldots,2k-5)| \\
&\geqslant 3\times|\mathscr{G}(1,1,1,2,\ldots,k-3,k-3,k-2,\ldots,2k-7)| \\
&\geqslant 3^2\times|\mathscr{G}(1,1,1,2,\ldots,k-4,k-4,k-3,\ldots,2k-9)| \\
&\ \ \vdots \\
&\geqslant 3^{k-3}|\mathscr{G}(1,1,1,1)| \\
&= 3^{k-2}.
\end{aligned}$$

(The factor of 3 arises at each stage from the freedom to choose one of three degree-one vertices to be adjacent to the vertex of largest degree.) The ratio $|\mathscr{G}'(\boldsymbol{d}^{(k)})|/|\mathscr{G}(\boldsymbol{d}^{(k)})|$ is exponential in k, and hence in $n=2k$, the number of vertices.

By appropriately refining the above counterexample, it is possible to demonstrate that Theorem 3.1 is quite close to being a best possible characterisation of P-stability in terms of the quantities $e(\boldsymbol{d})$ and $d_{\max}$.

4. Excluded graphs

As remarked in the Synopsis, our technique can be extended to the generation of graphs with given degree sequence which avoid a specified set of edges. Suppose $\boldsymbol{d}=(d_1,\ldots,d_n)$ is a degree sequence and X is a graph with vertex set $\{1,\ldots,n\}$. Let $\mathscr{G}(\boldsymbol{d},X)$ denote the set of all graphs $G\in\mathscr{G}(\boldsymbol{d})$ for which the edge sets of G and X are disjoint. Our goal is to provide a fully polynomial almost uniform generator for $\mathscr{G}(\boldsymbol{d},X)$, subject to appropriate conditions on $\boldsymbol{d}$ and X. The motivation for considering this generalisation is twofold. First, it encompasses some problems of independent interest, such as generating f-factors in an undirected graph. Second, the generalisation extends the class of structures under consideration from one, namely $\mathscr{G}(\boldsymbol{d})$, which is apparently not self-reducible in the sense of Schnorr [8] to one, $\mathscr{G}(\boldsymbol{d},X)$, which definitely is.

The required modification to the reduction of Section 2 is straightforward: selected parts of the graph Γ are simply omitted in accordance with the specified excluded graph X. More precisely, the definition of the vertex set of $\Gamma=\Gamma(\boldsymbol{d}, X)$ is amended to read

$$V(\Gamma)=\{v_{ik}: 1 \leqslant i \leqslant n \text{ and } 1 \leqslant k \leqslant d_i\}$$
$$\cup \{u_{ij}: 1 \leqslant i, j \leqslant n,\ i \neq j \text{ and } (i, j) \notin E(X)\},$$

where $E(X)$ denotes the edge set of X. The edge set of Γ is also revised, so that it includes precisely those edges of the old edge set which have both endpoints in the new vertex set. As before, there is a correspondence between elements of $\mathscr{G}(\boldsymbol{d}, X)$ and perfect matchings in $\Gamma(\boldsymbol{d}, X)$.

Proceeding by analogy with the definition of P-stability in Section 2, define $\mathscr{G}'(\boldsymbol{d}, X)$ to be the set of all graphs G in $\mathscr{G}'(\boldsymbol{d})$ for which the edge sets of G and X are disjoint. Say that a class of degree sequence/excluded graph pairs $(\boldsymbol{d}, X)$ is *P-stable* if there is a polynomial p such that $|\mathscr{G}'(\boldsymbol{d}, X)|/|\mathscr{G}(\boldsymbol{d}, X)| \leqslant p(n)$ for every pair $(\boldsymbol{d}, X)$ in the class. (Here, n denotes the number of components in the degree sequence $\boldsymbol{d}$.) An analogue of Theorem 2.2 holds for the extended notion of P-stability, with the pair $(\boldsymbol{d}, X)$ taking the role of the singleton $\boldsymbol{d}$; indeed the proof of Theorem 2.2 can be carried across virtually unchanged. Now let $x_{\max}$ denote the maximum vertex degree of the excluded graph X. Working through the proof of Theorem 3.1, taking into account the excluded graph X, we obtain the following theorem.

Theorem 4.1. *The class of all degree sequence/excluded graph pairs* $(\boldsymbol{d}, X)$ *which satisfy* $e(\boldsymbol{d}) > d_{\max}(d_{\max} + x_{\max} - 1)$ *is* P-*stable.*

Notice that this is a proper extension of the previous result, since we can recover Theorem 3.1 by setting $X=\emptyset$ and $x_{\max}=0$. The analogue of Corollary 3.2 is the following.

Corollary 4.2. *There exists a fully polynomial almost uniform generator for* $\mathscr{G}(\boldsymbol{d}, X)$, *and a fully polynomial randomised approximation scheme for* $|\mathscr{G}(\boldsymbol{d}, X)|$, *provided the pair* $(\boldsymbol{d}, X)$ *satisfies* $e(\boldsymbol{d}) > d_{\max}(d_{\max} + x_{\max} - 1)$.

It is worth noting two special cases of Corollary 4.2. Setting $\boldsymbol{d}=(1, 1, \ldots, 1)$ confirms the existence of a fully polynomial almost uniform generator for the set of perfect matchings in a graph G, provided the *minimum* degree of G exceeds $\frac{1}{2}n-1$. This fact had been observed earlier [7]. However, going further and setting $\boldsymbol{d}=(f, f, \ldots, f)$ we obtain a new result: there exists a fully polynomial almost uniform generator for the set of f-factors in a graph G, provided the minimum degree of G exceeds $\frac{1}{2}n+f-2$.

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