Free-Form Deformation of Solid Geometric Models

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ABSTRACT

A technique is presented for deforming solid geometric models in a free-form manner. The technique can be used with any solid modeling system, such as CSG or B-rep. It can deform surface primitives of any type or degree: planes, quadrics, parametric surface patches, or implicitly defined surfaces, for example. The deformation can be applied either globally or locally. Local deformations can be imposed with any desired degree of derivative continuity. It is also possible to deform a solid model in such a way that its volume is preserved.

The scheme is based on trivariate Bernstein polynomials, and provides the designer with an intuitive appreciation for its effects.

CR Categories and Subject Descriptors: I.3.5 [**Computer Graphics**]: Computation Geometry and Object Modeling - Curve, surface, solid, and object representations; Hierarchy and geometric transformations.

KEYWORDS: Solid geometric modeling, free-form surfaces, deformations.

1. INTRODUCTION

The fields of solid modeling and surface modeling have been developing rather independently over the past fifteen years [Requicha '82], [Varady '84]. Surface modeling has dealt primarily with parametric surface patches. These patches are generally referred to as free-form surfaces, or sculptured surfaces, which suggest that they can be shaped with flexibility akin to clay in a sculptor's hands. For this reason, planes, quadrics and tori are generally not considered to be free-form. Most solid modeling systems use surfaces that are planar, quadric or toroidal. Recently, the capability of defining fillets and blended surfaces has also been introduced [Hoffmann '85], [Middleditch '85], [Rockwood '86].

The problem of defining a solid geometric model of an object bounded by free-form surfaces has long been identified as an important research problem. Most of the approaches to this problem can be classified into one of three categories:

1. Combining *existing* free-form surface and solid modeling techniques. This extends the surface domain of a solid modeling system to include free-form parametric surface patches. It is currently the most popular approach and some applications can be found in [Kalay '82], [Jared '84], [Chiyokura '83], [Varady '84], [Riesenfeld '83], [Sarraga '84], [Steinberg '84], [Thomas '84], and [Kimura '84]. This method must overcome several difficulties such as ensuring representational validity in using the free-form surfaces in a general manner. These problems are described in [Requicha '82].

2. Trivariate parametric *hyperpatch*. The hyperpatch is used as a solid modeling primitive. This method has been used for years by the analysis community and has many applications such as finite element mesh generation [Stanton '77], [Casale '85]. [Farouki '85] discusses adding a fourth parameter of time to create a time-space swath useful for motion definition.

3. Implicit surfaces. There has been limited investigation of modeling directly with volumes bounded by implicit or algebraic surfaces. Calculating curves of surface intersection and deciding whether a point lies inside a volume is easier with this definition, especially when the surfaces are of low degree. However, free-form shape definition lends itself more naturally to parametric equations

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than to implicit equations. Sabin was one of the early investigators of modeling with algebraic surfaces [Sabin '68]. [Ricci '73], [Barr '81], [Rockwood '86], [Owen '86], [Hoffmann '85] and [Blinn '82] explore modeling implicit surfaces other than quadrics. [Sederberg '85] discusses modeling with piecewise algebraic surface patches.

This paper presents an approach to free-form solid modeling which does not fall cleanly into any of the above three categories, although it developed most directly out of the ideas in [Sederberg '85]. This technique is referred to as free-form deformation or FFD, and can be thought of as a method for sculpturing solid models. It is shown to be of value both as a design method, and as a representation for free-form solids. Indeed, the sculpturing metaphor is stronger for solids than for surfaces because a lump of clay or a block of marble is a solid.

Several researchers have promoted this sculpturing metaphor for geometric modeling, noting that it is a natural and familiar mode of thought for a designer or stylist. For example, [Parent '77] discusses a "computer graphics sculptor's studio" for defining polygonal objects, and [Brewer '77] describes a planar shaping tool for manipulating sculptured surfaces. Other "lump-of-clay" modeling techniques are surveyed in [Cobb '84]. None of these sculpturing techniques are directly applicable to solid geometric modeling. Parent's paper deals with polygonal data, and Brewer's work deals with a class of parametric surface patches.

FFD involves a mapping from $\mathbf{R}^3$ to $\mathbf{R}^3$ through a trivariate tensor product Bernstein polynomial. An earlier use of $\mathbf{R}^3$ to $\mathbf{R}^3$ mapping is found in Barr's innovative paper on regular deformations of solids [Barr '84]. While not a free-form modeling technique, Barr's idea of twisting, bending and tapering of solid primitives is a powerful and elegant design tool. Brief mention of deformation is also made in [Sabin '70] and in [Bézier '74]. Trivariate hyperpatches also are an $\mathbf{R}^3$-$\mathbf{R}^3$ map, but the result is a distorted cube with six four sided faces.

FFD is a remarkably versatile tool. It can be applied to CSG based solid models as well as those using Euler operators. It can sculpt solids bounded by *any* analytic surface: planes, quadrics, parametric surfaces patches, or implicit surfaces. Furthermore, its application is not restricted to solid models, but it can also sculpt surfaces or polygonal data.

FFD can be applied locally while maintaining derivative continuity with adjacent, undeformed regions of the model. It can also be applied hierarchically, with each application being analogous to a sweep of the sculptor's hands. Constraints can be placed on the FFD to control the degree to which the volume of the solid changes, and in fact, there exist free-form deformations which are perfectly volume preserving.

[Veenman '82] suggests that the free-form surfaces used in practical engineering design fall into four categories: Aesthetic surfaces (the main design requirement is visual appearance); fairings or duct surfaces (a surface transition between two other surfaces of different cross-section); blends and fillets (smooth the intersection of two other surfaces); and functional or fitted surfaces (high geometric constraint imposed to satisfy some functional requirement, such as a turbine blade). FFDs can create aesthetic surfaces and fairings. It is also possible to synthesize fillets in certain situations, but a general fillet and blending capability is not claimed. However, FFD can be used in conjunction with any fillet and blend formulation, such as those discussed in [Hoffmann '85], [Middleditch '85] and [Rockwood '86]. Functional surfaces are not discussed, although [Sabin '70] reports that a type of small displacement FFD is useful in the design of airplane wings.

This paper assumes that the reader is familiar with Bezier curves and surface patches. Necessary background can be found in [Boehm '84]. Basic understanding of solid modeling is also presumed, such as discussed in [Requicha '82].

Section 2 presents the mathematical formulation of FFD. Section 3 discusses local deformations and continuity control. Section 4 looks at volume change, and Section 5 presents several examples which illustrate the flexibility of FFD. Section 6 summarizes.

2. FORMULATING FREE-FORM DEFORMATIONS

A good physical analogy for FFD is to consider a parallelpiped of clear, flexible plastic in which is embedded an object, or several objects, which we wish to deform. The object is imagined to also be flexible, so that it deforms along with the plastic that surrounds it.

Figure 1 illustrates this analogy using several objects embedded in clear, flexible plastic. In Figure 2, the plastic has been deformed and the embedded spheres and cubes are deformed in a manner that is intuitively consistent with the motion of the plastic.

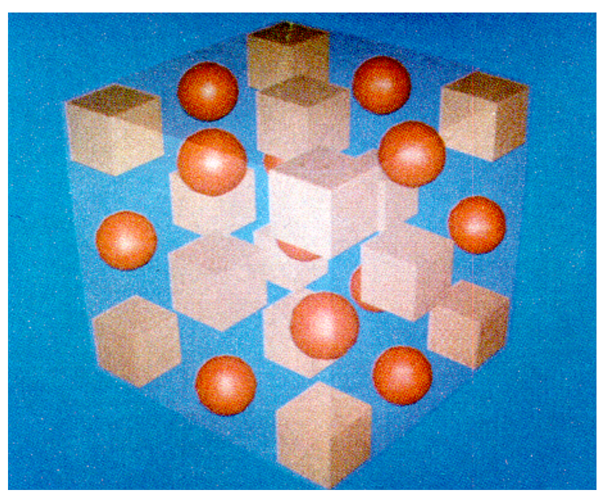

Fig 1. Undeformed Plastic

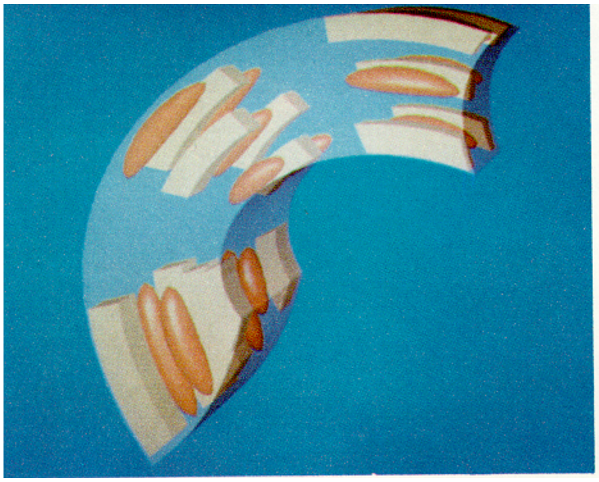

Fig 2. Deformed Plastic

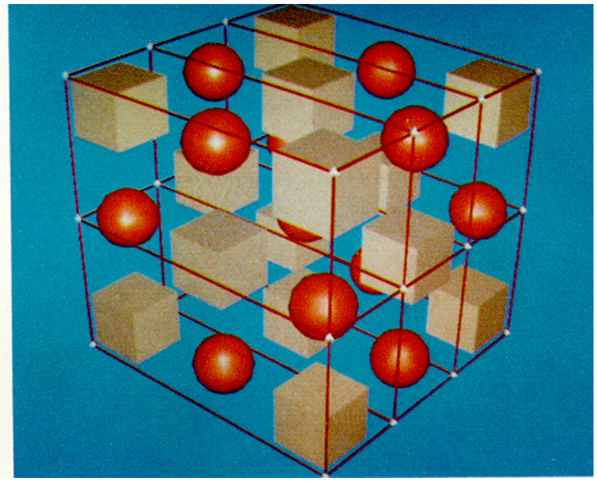

Fig. 4 Undisplaced Control Points

Mathematically, the FFD is defined in terms of a tensor product trivariate Bernstein polynomial. We begin by imposing a local coordinate system on a parallelpiped region, as shown in Figure 3. Any point $\mathbf{X}$ has (s,t,u) coordinates in this system such that

$$\mathbf{X} = \mathbf{X}_0 + s\mathbf{S} + t\mathbf{T} + u\mathbf{U}.$$

The (s,t,u) coordinates of $\mathbf{X}$ can easily be found using linear algebra. A vector solution is

$$s=\frac{\mathbf{T}\times\mathbf{U}\,(\mathbf{X}-\mathbf{X}_0)}{\mathbf{T}\times\mathbf{U}\cdot\mathbf{S}},\; t=\frac{\mathbf{S}\times\mathbf{U}\cdot(\mathbf{X}-\mathbf{X}_0)}{\mathbf{S}\times\mathbf{U}\cdot\mathbf{T}},\; u=\frac{\mathbf{S}\times\mathbf{T}\cdot(\mathbf{X}-\mathbf{X}_0)}{\mathbf{S}\times\mathbf{T}\cdot\mathbf{U}} \quad (1)$$

Note that for any point interior to the parallelpiped that $0<s<1$, $0<t<1$ and $0<u<1$.

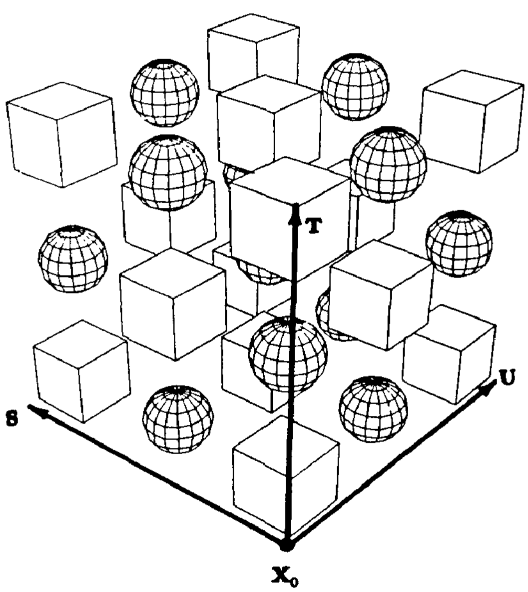

Fig. 3 s,t,u Coordinate System

We next impose a grid of control points $\mathbf{P}_{ijk}$ on the parallelpiped. These form $l+1$ planes in the $\mathbf{S}$ direction, $m+1$ planes in the $\mathbf{T}$ direction, and $n+1$ planes in the $\mathbf{U}$ direction. In Figure 4, l=1, m=2, and n=3. The control points are indicated by small white diamonds, and the red bars indicate the neighboring control points. These points lie on a lattice, and their locations are defined

$$\mathbf{P}_{ijk} = \mathbf{X}_0 + \frac{i}{l}\mathbf{S} + \frac{j}{m}\mathbf{T} + \frac{k}{m}\mathbf{U}.$$

The deformation is specified by moving the $\mathbf{P}_{ijk}$ from their undisplaced, latticial positions. The deformation function is defined by a trivariate tensor product Bernstein polynomial. The deformed position $\mathbf{X}_{ffd}$ of an arbitrary point $\mathbf{X}$ is found by first computing its (s,t,u) coordinates from equation (1), and then evaluating the vector valued trivariate Bernstein polynomial:

$$\mathbf{X}_{ffd}=\sum_{i=0}^{l}\binom{l}{i}(1-s)^{l-i}s^i\left[\sum_{j=0}^{m}\binom{m}{j}(1-t)^{m-j}t^j\left[\sum_{k=0}^{n}\binom{n}{k}(1-u)^{n-k}u^k\mathbf{P}_{ijk}\right]\right] \quad (2)$$

where $\mathbf{X}_{ffd}$ is a vector containing the Cartesian coordinates of the displaced point, and where $\mathbf{P}_{ijk}$ is a vector containing the Cartesian coordinates of the control point.

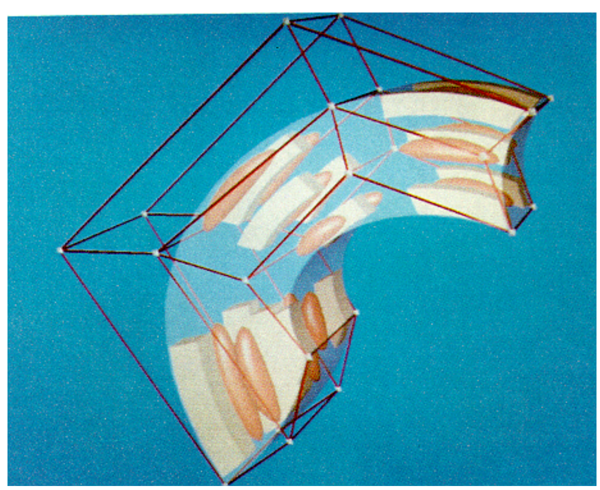

Fig. 5 Control Points in Deformed Position

The control points $\mathbf{P}_{ijk}$ are actually the coefficients of the Bernstein polynomial. As in the case of Bezier curves and surface patches, there are meaningful relationships between the deformation and the control point placement. Note from Fig. 5 that the 12 edges of the parallelpiped are actually mapped into Bezier curves, defined by the control points which initially lie on the respective edges. Also, the six planar faces map into tensor product Bezier surface patches, defined by the control points which initially lie on the respective faces.

This deformation could be formulated in terms of other polynomial bases, such as tensor product B-splines or non-tensor product Bernstein polynomials. We feel that the basis we have chosen provides the clearest discussion.

2.1. Deformation Domain

Consider the versatility of this FFD. Although the purpose of this paper is to establish FFD as a viable tool for solid modeling, we note that it can be applied to virtually any geometric model. Figures 6 and 7 show deformed polygonal data. Only the polygon vertices are transformed by the FFD, while maintaining the polygon connectivity. Deformation of polygonal data is discussed more thoroughly in [Sederberg '86].

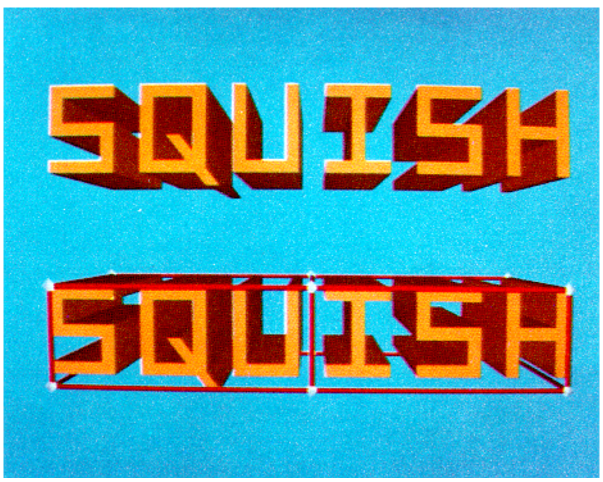

Fig. 6 Undeformed Polygons

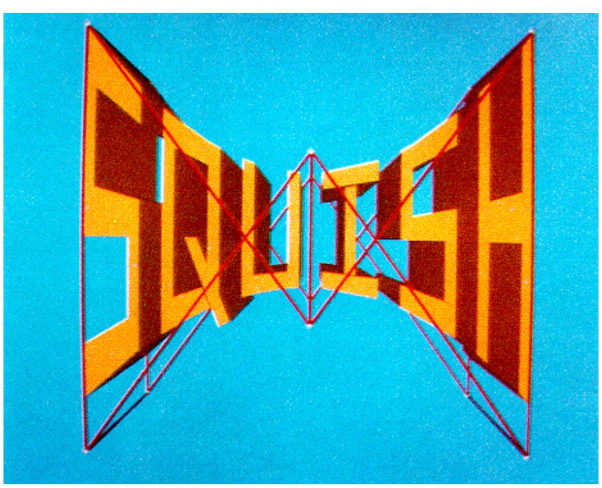

Fig. 7 Deformed Polygons

Figures 8 and 9 illustrate a sphere intersected by a plane, both deformed simultaneously by the same FFD. The sphere and the plane could each be expressed in parametric equations, or in implicit equations. The FFD can be applied with equal validity to either representation. A very important characteristic of FFD is that a deformed parametric surface remains a parametric surface. This is easy to see. If the parametric surface is given by $x = f(\alpha,\beta)$, $y = g(\alpha,\beta)$ and $z = h(\alpha,\beta)$ and the FFD is given by $\mathbf{X}_{ffd} = \mathbf{X}(x,y,z)$, then the deformed parametric surface patch is given by $\mathbf{X}_{ffd}(\alpha,\beta) = \mathbf{X}(f(\alpha,\beta),g(\alpha,\beta),h(\alpha,\beta))$. This is a simple composition.

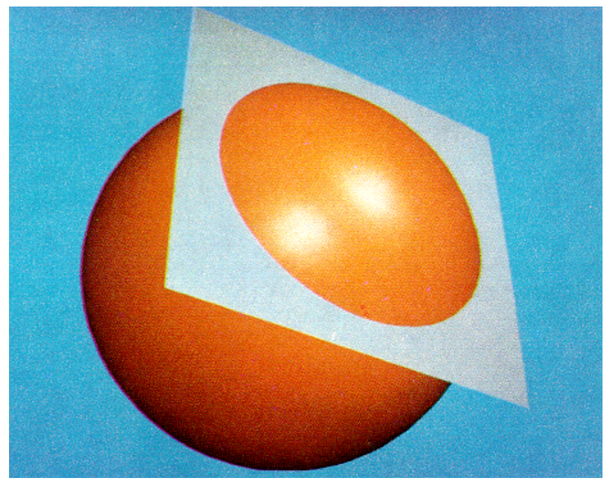

Fig. 8 Sphere and Plane

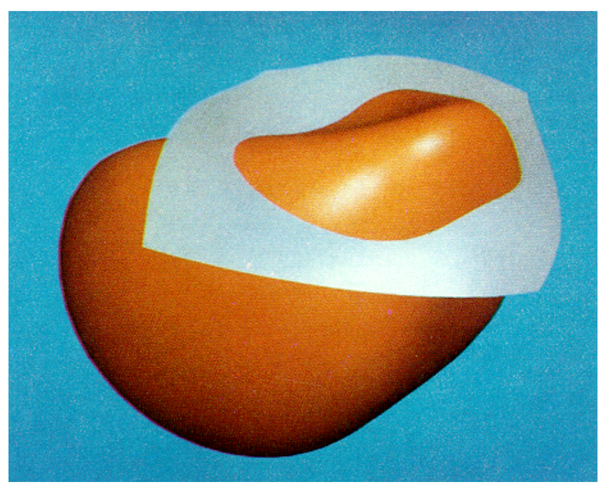

Fig. 9 Deformed Sphere and Plane

An important corollary to this is that parametric curves remain parametric under FFD. In Figure 8, the curve of intersection between the sphere and plane is a circle, which can be expressed parametrically in terms of quadratic rational polynomials. In Figure 9, that deformed circle is still a parametric curve. This fact suggests important possibilities for solid modeling. For example, if one performs FFD in a CSG modeling environment only after all boolean operations are performed, and the primitive surfaces are planes or quadrics, then all intersection curves would be parametric, involving rational polynomials and possibly square roots.

Quadrics and planes make excellent primitives because they possess both implicit and parametric equations. The parametric equation enables rapid computation of points on the surface, and the implicit equation provides a simple point classification test - is a point inside, outside, or on the surface. To classify a point on a deformed quadric, one must first compute its s,t,u coordinates and substitute them into the implicit equation. The s,t,u coordinates can be found by subdividing the control point lattice, or by trivariate Newton iteration (see [Parry '86]). This inverse mapping requires significant computation, and can be a source of robustness problems, especially if the Jacobian of the FFD changes sign (see section 4).

3. CONTINUITY CONTROL

It is possible to apply two or more FFDs in a piecewise manner so as to maintain cross-boundary derivative continuity. We will discuss this continuity in terms of a local surface parametrization. This covers the general case, since all surfaces possess a local parametrization. Denote the local parameters by v,w and the surface by $(s,t,u) = (s(v,w),t(v,w),u(v,w))$. Imagine two adjacent FFDs $\mathbf{X}_1(s_1,t_1,u_1)$ and $\mathbf{X}_2(s_2,t_2,u_2)$ which share a common boundary $s_1 = s_2 = 0$. The first derivatives of the deformed surface can be found using the chain rule:

$$\frac{\partial \mathbf{X}_i(v,w)}{\partial v} = \frac{\partial \mathbf{X}_i}{\partial s}\cdot\frac{\partial s}{\partial v} + \frac{\partial \mathbf{X}_i}{\partial t}\cdot\frac{\partial t}{\partial v} + \frac{\partial \mathbf{X}_i}{\partial u}\cdot\frac{\partial u}{\partial v}$$

$$\frac{\partial \mathbf{X}_i(v,w)}{\partial w} = \frac{\partial \mathbf{X}_i}{\partial s}\cdot\frac{\partial s}{\partial w} + \frac{\partial \mathbf{X}_i}{\partial t}\cdot\frac{\partial t}{\partial w} + \frac{\partial \mathbf{X}_i}{\partial u}\cdot\frac{\partial u}{\partial w}.$$

Note that $\frac{\partial s}{\partial v}$, $\frac{\partial t}{\partial v}$, $\frac{\partial u}{\partial v}$, $\frac{\partial s}{\partial w}$, $\frac{\partial t}{\partial w}$ and $\frac{\partial u}{\partial w}$ are all independent of the deformation. Thus, sufficient conditions for first derivative continuity are that

$$\frac{\partial \mathbf{X}_1(0,t,u)}{\partial s} = \frac{\partial \mathbf{X}_2(0,t,u)}{\partial s},$$

$$\frac{\partial \mathbf{X}_1(0,t,u)}{\partial t} = \frac{\partial \mathbf{X}_2(0,t,u)}{\partial t}, \qquad (3)$$

$$\frac{\partial \mathbf{X}_1(0,t,u)}{\partial u} = \frac{\partial \mathbf{X}_2(0,t,u)}{\partial u}.$$

These conditions (and those for higher derivative continuity) can be shown to be straightforward extensions of the continuity conditions for Bezier curves and tensor product Bezier surfaces, which are explained in [Boehm '84]. We will denote continuity by C^k, which means that two adjacent FFDs are geometrically continuous to the k^{th} derivative.

Consider the two adjacent undeformed FFD lattices in the upper right of Figure 10 which have a plane of control points in common. These two FFDs are C^0 if the common control points remain coincident, as in the upper left of Figure 10a. Sufficient conditions for C^1 are illustrated in the bottom of Figure 10a.

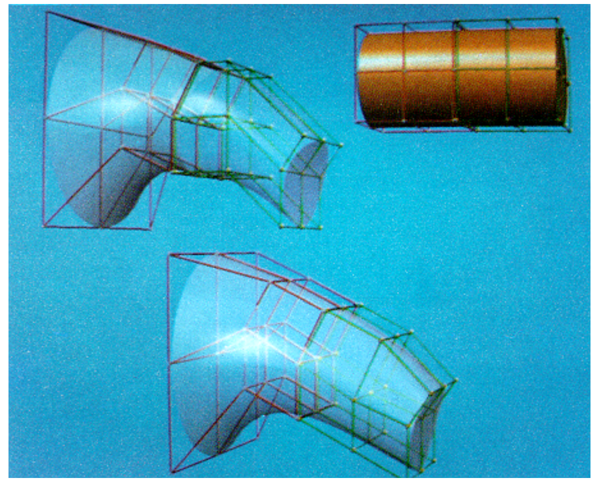

Fig. 10a Control Points for C^0 and C^1 FFDs

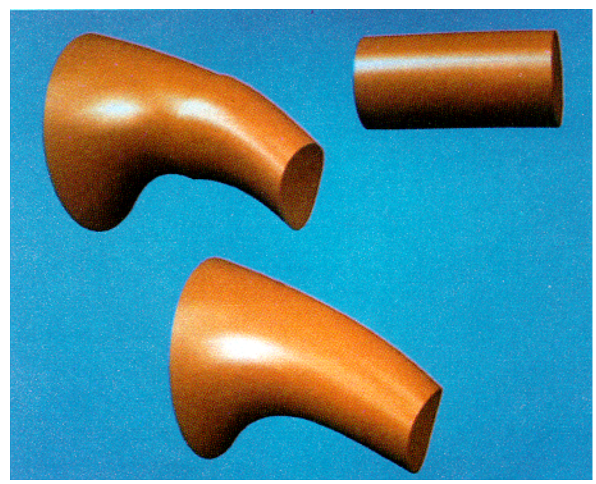

Fig. 10b C^0 and C^1 FFDs

3.1. Local Deformations

A special case of continuity conditions enables us to perform a local, isolated deformation. In this case, we might imagine that the neighboring FFD with which we wish to maintain C^k is simply an undeformed lattice. We consider the problem of maintaining C^k along the plane where one face of the FFD intersects the geometric model. It is easy to show that sufficient conditions for a C^k local deformation are simply that the control points on the k planes adjacent to the interface plane are not moved. This is illustrated in Figures 11 and 12. Of course, C^k can be maintained across more than one face by imposing these conditions for each face that the surface intersects.

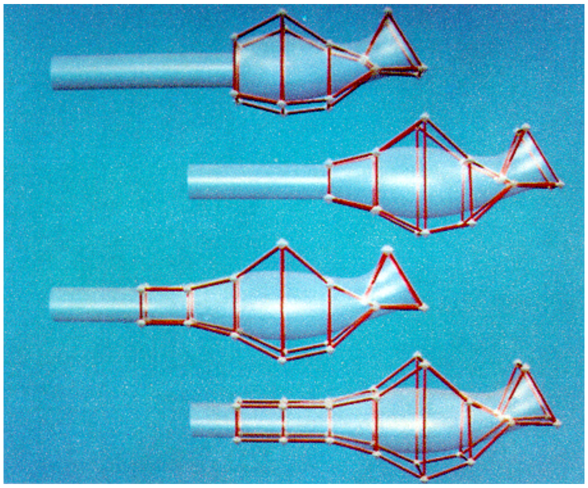

Fig. 11 Local C^k Control Points

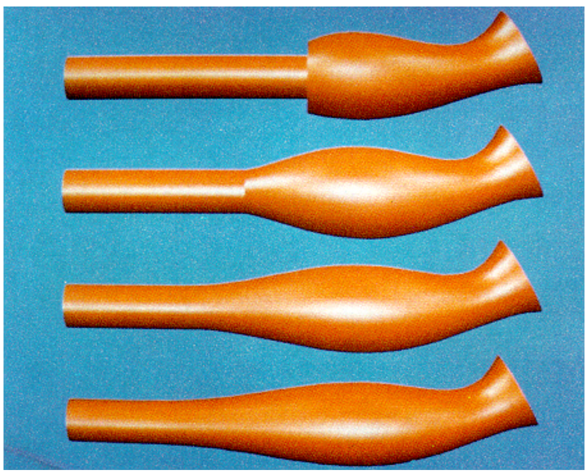

Fig. 12 C^{-1}, C^0, C^1 and C^2 Local Deformations

This local application lends to the FFD a capability which makes the technique strongly analogous to sculpting with clay. These local deformations can be applied hierarchically, which imparts exceptional flexibility and ease of use to the technique.

4. VOLUME CHANGE

Another reason that FFD is so nicely applicable to solid modeling is that it provides us with control over the volume change that a solid experiences under FFD. The volume change that a FFD imposes on each differential element is given by the Jacobian of the FFD. If the FFD is given by

$$\mathbf{F}(x,y,z) = (F(x,y,z), G(x,y,z), H(x,y,z))$$

then the Jacobian is the determinant

$$Jac(\mathbf{F}) = \begin{vmatrix} \frac{\partial F}{\partial x} & \frac{\partial F}{\partial y} & \frac{\partial F}{\partial z} \\ \frac{\partial G}{\partial x} & \frac{\partial G}{\partial y} & \frac{\partial G}{\partial z} \\ \frac{\partial H}{\partial x} & \frac{\partial H}{\partial y} & \frac{\partial H}{\partial z} \end{vmatrix}$$

If the volume of any differential element before deformation is $dx \cdot dy \cdot dz$, then after deformation, its volume is $Jac(\mathbf{F}(x,y,z)) \cdot dx \cdot dy \cdot dz$. The volume of the entire deformed solid is simply the triple integral of this differential volume over the volume enclosed by the undeformed surface. Thus, if we can obtain a bound on $Jac(\mathbf{F})$ over the region of deformation, we will have a bound on the volume change. Such a bound is conveniently provided if $Jac(\mathbf{F})$ is expressed as a trivariate Bernstein polynomial. Then, the largest and smallest polynomial coefficients provide upper and lower bounds on the volume change.

A noteworthy result is that there exists a family of *volume preserving* FFDs, which means $Jac(\mathbf{F}) \equiv 1$. Any solid model will retain its original volume under such a transformation. Figures 13a and 13b illustrate a 12 oz. Coke can before and after a volume preserving FFD. The deformed can still holds exactly 12 oz.! Space limitations prevent a more complete discussion here of volume preservation, but the details can be found in [Sederberg '86b]. The usefulness of the set of volume preserving FFDs is yet to be determined.

Fig. 13a 12 oz. Coke Can

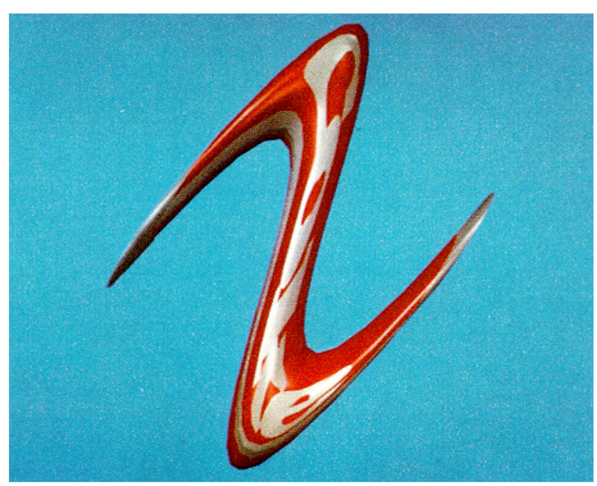

Fig. 13b Still 12 oz.

5. APPLICATIONS

We conclude by demonstrating some of the flexibility of FFD. Figures 14-16 demonstrate how the technique can be applied hierarchically to mold a rounded bar into a telephone handset. Figure 14 shows a local C^1 FFD which draws a mouthpiece from the undeformed bar. The earphone is formed in like manner, and Figure 15 shows a global FFD to impart a slight curvature to the handset. Figure 16 shows the final result. Note that the final telephone is a *free-form solid model*. The original bar in Figure 14 is modeled as a solid using an implicit equation, and each FFD merely modifies the geometry, without altering the integrity of the solid model. Thus, the hierarchical FFD formulation fully enables the computation of mass properties and point classification. We are also impressed by the ease with which the phone was designed. With only a few hours of experience under our belts, the phone was produced in a single design iteration!

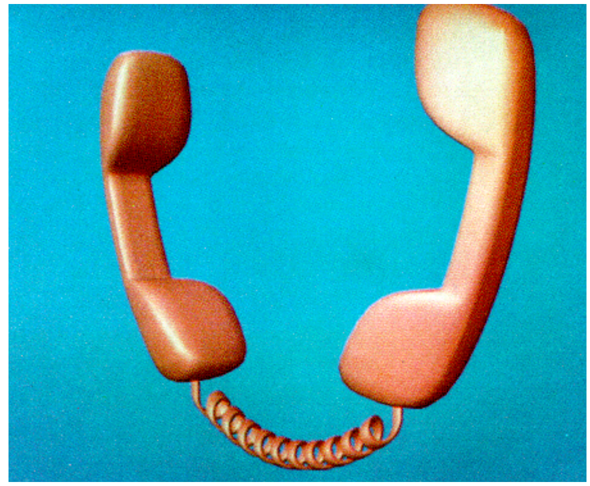

Fig. 16 Final Product

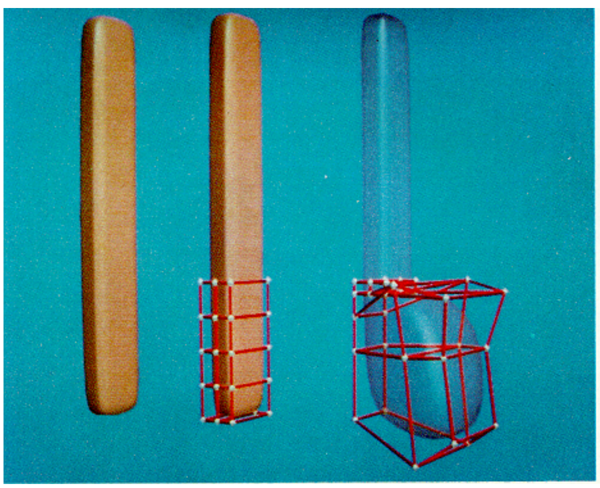

Fig. 14 Local FFD

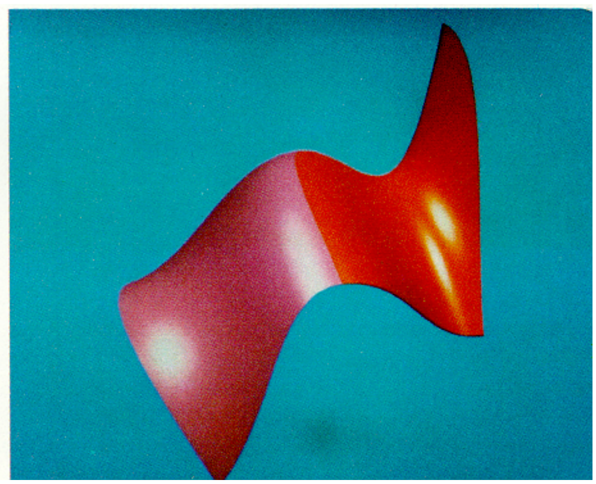

Fig. 17 Two C^1 Bicubic Patches

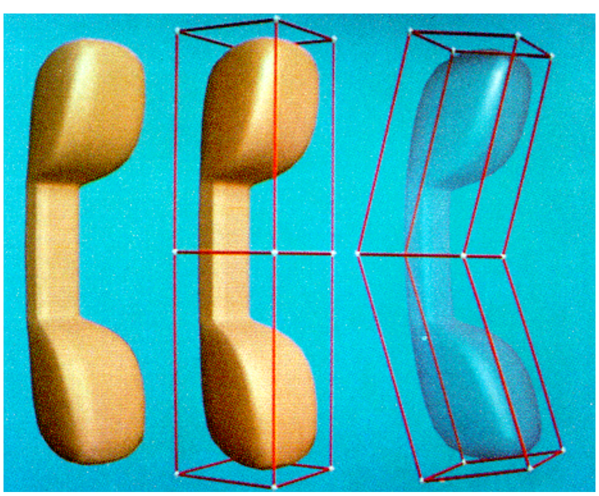

Fig. 15 Global FFD

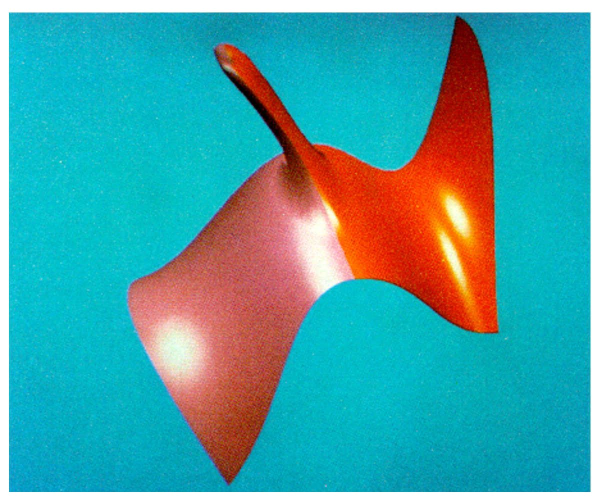

Fig. 18 C^1 FFD

Figures 17 and 18 further illustrate the "lump of clay" metaphor. Two slope continuous bicubic patches have an FFD applied which straddles the common boundary of the two patches. The resulting "tongue" is slope continuous with both patches, and the seam along the tongue is also slope continuous. Each half of the tongue itself is also a parametric surface! This example illustrates another important characteristic of FFD: it cares little about the underlying surface patch topology. To understand the importance of this, consider how the tongue would be created using conventional surface patches. It would probably require 6-8 patches, some of which may have to be non-four sided.

Fig. 19 Trophy

Figure 19 shows a trophy whose handles were created by applying a single FFD to a cylinder. The handles were then joined to the surface of revolution using a boolean sum. Again, the handles are modeled as solids. Since the underlying cylinder primitive has both a parametric and an implicit formulation, the handle surface has a parametric expression as well.

Figures 20-22 show how FFD can be used as a fairing or duct surface. The two cylinders, one with an elliptical cross-section, and the other with a peanut shaped cross-section, were both formed using FFDs applied to circular cylinders. The transitional duct surface is also created by applying a FFD to a circular cylinder, so as to be C^1 with the two neighbors.

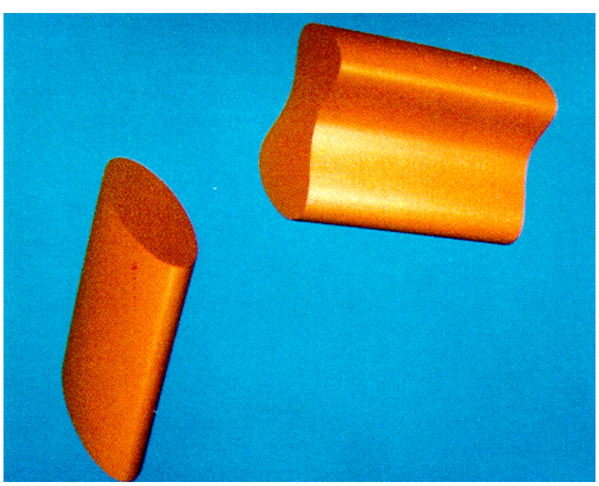

Fig. 20 Dissimilar Cylinders

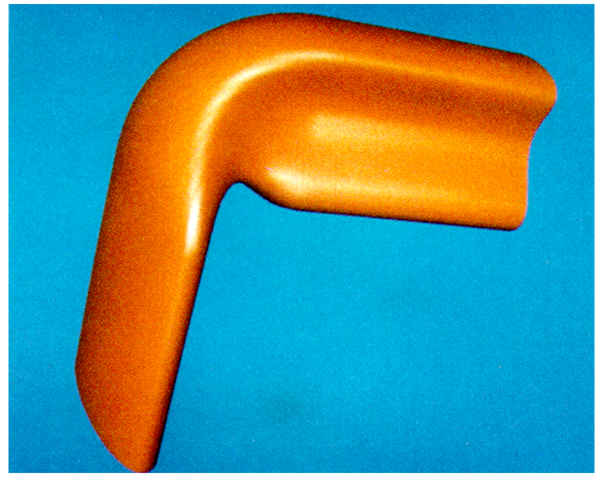

Fig. 21 C^1 Fairing

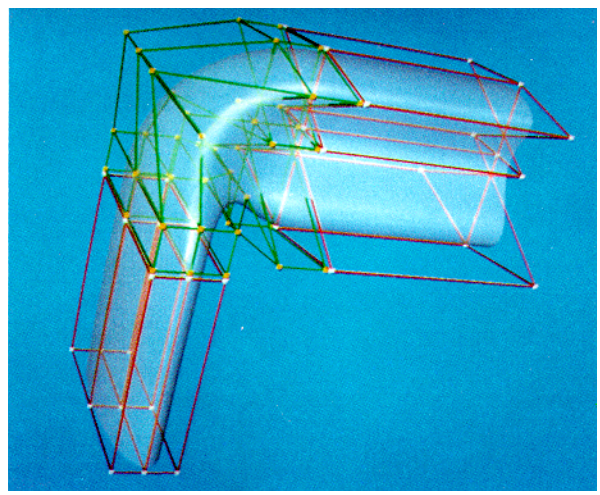

Fig. 22 Control Points

Fig. 23 It's the Real Thing

Finally, Figure 23 shows an artistic display of a deformed and undeformed Coke can and bottle.

6. CONCLUSIONS

Our brief experience with FFDs persuades us that they have great potential for defining free-form solid models. It's strength and versatility can be summarized as follows:

1. It can be used with any solid modeling scheme.
2. It works with surfaces of any formulation or degree.
3. It can be applied locally or globally, and with derivative continuity.
4. We have found it to be very easy to use. The informal response of some professional stylists is that the strong sculpturing metaphor seems natural and familiar to them.
5. In addition to solid modeling, it can be applied to surfaces or polygonal models.
6. It provides indication of the degree of volume change, and a class of FFDs are even volume preserving.
7. Parametric curves and surfaces remain parametric under FFD.
8. It can be used for aesthetic surfaces, many fairing surfaces, and probably many functional surfaces.

There are limitations to the technique. We can currently identify the following:

1. It cannot perform general filleting and blending.
2. Local FFD forms a planar boundary with the undeformed portion of the object. To create an arbitrary boundary curve, one would have to begin with a FFD which is already in a deformed orientation, and then deform it some more. This would be quite costly.
3. Operations on trivariate Bernstein polynomials, such as subdivision, are much more costly than operations on bivariates.

With more experience, the lists of strengths and weaknesses will both undoubtedly grow.

Due to page limits, several important topics have not been discussed, such as display techniques. This can be found in [Parry '86].

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