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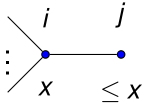

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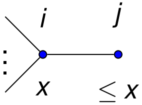

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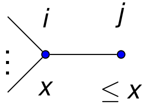

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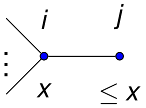

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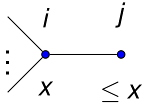

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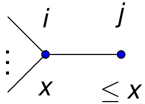

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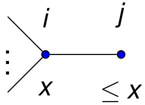

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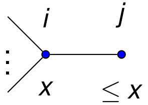

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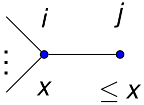

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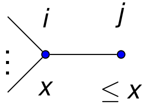

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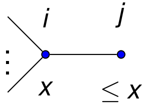

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Formally

Cheeger's Inequality

$$\frac{1 - \lambda_2}{2} \leq h(G) \leq \sqrt{2(1 - \lambda_2)}$$

Spectral Gap and Conductance.

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$$\phi(G) = min_{x \in \{0,1\}^V - \{\mathbf{0},\mathbf{1}\}} \frac{n \sum_{i,j} M_{ij}(x_i - x_j)^2}{\sum_{i,j}(x_i - x_j)^2}$$

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Proof:

$$\sum_{i,j} (x_i - x_j)^2 = \sum_{i,j} x_i^2 + x_j^2 - 2x_i x_j$$
$$= 2n \sum_i x_i^2 - 2(\sum_i x_i)^2 = 2n \sum_i x_i^2 = 2nx^T x$$

We used $x \perp \mathbf{1} \Rightarrow \sum_i x_i = 0$ $\square$

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$$\begin{aligned}
\sum_{i,j} M_{ij}(x_i - x_j)^2 &= \sum_{i,j} M_{ij}(x_i^2 + x_j^2) - 2\sum_{i,j} M_{ij} x_i x_j \\
&= \sum_i \sum_{j \sim i} \frac{1}{d}(x_i^2 + x_j^2) - 2x^T M x \\
&= 2 \sum_{(i,j) \in E} \frac{1}{d}(x_i^2 + x_j^2) - 2x^T M x \\
&= 2\sum_i x_i^2 - 2x^T M x = 2x^T x - 2x^T M x
\end{aligned}$$

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Hooray!! We get the easy part of Cheeger $\frac{1-\lambda_2}{2} \le h(G)$

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Now let's get to the hard part of Cheeger $h(G) \leq \sqrt{2(1-\lambda_2)}$.

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What will be a good t?

We don't know. Try all possible thresholds ($n-1$ possibilities), and hope there is a t leading to a good cut!

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Main Lemma: $G = (V, E)$, d-regular
$x \in \mathbb{R}^V, x \perp \mathbf{1}, \delta = \frac{\sum_{i,j} M_{ij}(x_i - x_j)^2}{\frac{1}{n}\sum_{i,j}(x_i - x_j)^2}$
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Note: Applying the Main Lemma with the 2^{nd} eigenvector v_2, we have $\delta = 1 - \lambda_2$, and $h(G) \leq h(S) \leq \sqrt{2(1-\lambda_2)}$. Done!

Proof of Main Lemma

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Probabilistic Argument: Construct a distribution D over $\{S_1, \ldots, S_{n-1}\}$ such that

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$$\exists S \qquad \tfrac{1}{d}|E(S, V-S)| - \sqrt{2\delta}\, min(|S|, |V-S|) \le 0$$

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Take D as the distribution over $S_1, \ldots, S_{n-1}$ resulted from the above procedure.

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Can check

$$\mathbb{E}_{S\sim D}[T_i] = Pr[T_i] = x_i^2$$

$$\begin{aligned}\mathbb{E}_{S\sim D}[min(|S|,|V-S|)] &= \mathbb{E}_{S\sim D}[\sum_i T_i] \\ &= \sum_i \mathbb{E}_{S\sim D}[T_i] \\ &= \sum_i x_i^2\end{aligned}$$

Goal: $\frac{\mathbb{E}_{S\sim D}[\frac{1}{d}|E(S,V-S)|]}{\mathbb{E}_{S\sim D}[min(|S|,|V-S|)]} \leq \sqrt{2\delta}$

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$$\begin{cases} x_i, x_j \text{ same sign:} & Pr[T_{i,j}] = |x_i^2 - x_j^2| \\ x_i, x_j \text{ different sign:} & Pr[T_{i,j}] = x_i^2 + x_j^2 \end{cases}$$

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$$\begin{aligned} \mathbb{E}_{S\sim D}[\frac{1}{d}|E(S,V-S)|] &= \frac{1}{2}\sum_{i,j} M_{ij}\mathbb{E}[T_{i,j}] \\ &\leq \frac{1}{2}\sum_{i,j} M_{ij}|x_i - x_j|(|x_i| + |x_j|) \end{aligned}$$

Cauchy-Schwarz Inequality

$|a \cdot b| \leq \|a\|\|b\|$, as $a \cdot b = \|a\|\|b\| cos(a, b)$

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$$\begin{aligned}
\mathbb{E}_{S \sim D}[\frac{1}{d}|E(S, V-S)|] &= \frac{1}{2}\sum_{i,j} M_{ij}\mathbb{E}[T_{i,j}] \\
&\leq \frac{1}{2}\sum_{i,j} M_{ij}|x_i - x_j|(|x_i| + |x_j|) \\
&= \frac{1}{2} a \cdot b \\
&\leq \frac{1}{2}\|a\|\|b\|
\end{aligned}$$

Recall $\delta = \frac{\sum_{i,j} M_{ij}(x_i - x_j)^2}{\frac{1}{n}\sum_{i,j}(x_i - x_j)^2}, a_{ij} = \sqrt{M_{ij}}|x_i - x_j|, b_{ij} = \sqrt{M_{ij}}|x_i| + |x_j|$

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$$
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\|a\|^2 &= \sum_{i,j} M_{ij}(x_i - x_j)^2 = \frac{\delta}{n}\sum_{i,j}(x_i - x_j)^2 \\
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$$\begin{aligned}
\|b\|^2 &= \sum_{i,j} M_{ij}(|x_i| + |x_j|)^2 \\
&\leq \sum_{i,j} M_{ij}(2x_i^2 + 2x_j^2) \\
&= 4\sum_i x_i^2
\end{aligned}$$

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Numerator:

$$\mathbb{E}_{S\sim D}[\frac{1}{d}|E(S,V-S)|] = \leq \frac{1}{2}\|a\|\|b\|$$
$$\leq \frac{1}{2}\sqrt{2\delta\sum_i x_i^2}\sqrt{4\sum_i x_i^2} \qquad = \sqrt{2\delta}\sum_i x_i^2$$

Recall **Denominator:**

$$\mathbb{E}_{S\sim D}[min(|S|,|V-S|)] = \sum_i x_i^2$$

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Thus $\exists S_i$ such that $h(S_i) \leq \sqrt{2\delta}$, which gives $h(G) \leq \sqrt{2(1-\lambda)}$ □

Cycle

Tight example for hard part of Cheeger?

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$\frac{\mu}{2}$

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$\frac{\mu}{2} = \frac{1-\lambda_2}{2}$

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$\frac{\mu}{2} = \frac{1-\lambda_2}{2} \leq h(G)$

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$$\frac{\mu}{2} = \frac{1-\lambda_2}{2} \leq h(G) \leq \sqrt{2(1-\lambda_2)} = \sqrt{2\mu}$$

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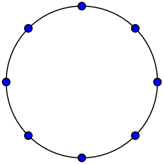
Will show other side of Cheeger is asymptotically tight.



Cycle on n nodes.

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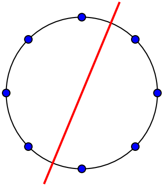

Cycle on n nodes.

Edge expansion:Cut in half.

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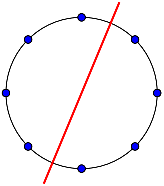

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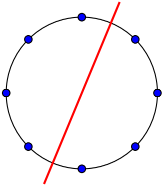

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$|S| = \frac{n}{2}$, $|E(S, \overline{S})| = 2$

$\rightarrow h(G) = \frac{4}{n}$.

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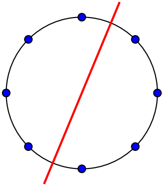

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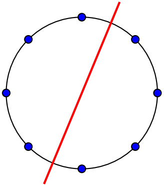

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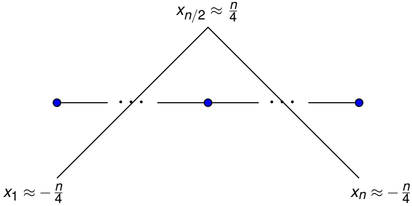

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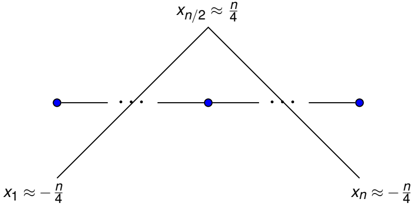

Hit with M.

$$(Mx)_i = \begin{cases} -n/4 + 1/2 & \text{if } i = 1, n \\ n/4 - 1 & \text{if } i = n/2 \\ x_i & \text{otherwise} \end{cases}$$

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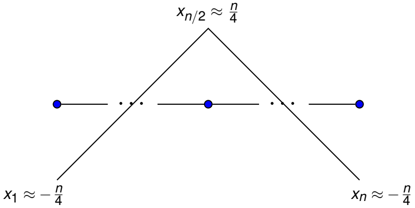

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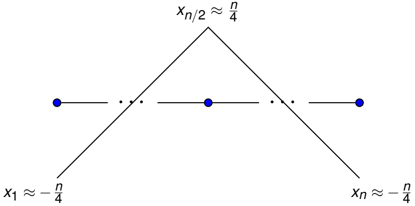

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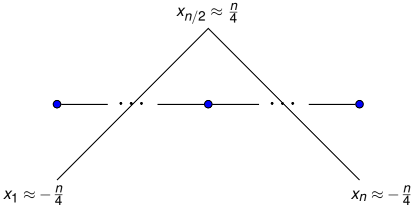

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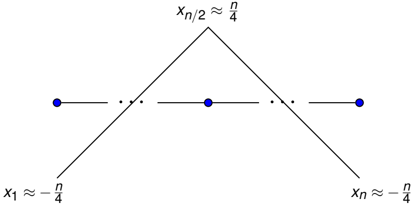

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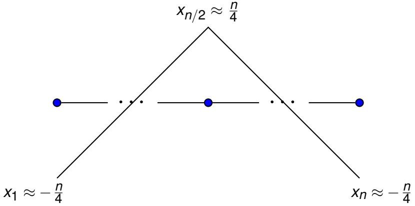

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Asymptotically tight example for upper bound for Cheeger
$h(G) \leq \sqrt{2(1 - \lambda_2)} = \sqrt{2\mu}$.

Sum up.

$1 - \lambda_2$ as a relaxation of $\phi(G)$.

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Example: Cycle, Cheeger hard part is asymptotic tight .

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