Welcome back.

Today.

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Sampling combinatorial structures.

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Random Walks.

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Random Walks.
Spectral Gap/Mixing Time.

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Expansion/Spectral Gap.

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Example: partial orders.

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Rayleigh quotient.

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$h(G)$ large

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$h(G)$ large $\rightarrow$ well connected

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Sampling: Random element of subset $S \subset \{0,1\}^n$ or $\{0,\ldots,k\}^n$.

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Random walk to do both for some interesting sets S.

Convex Bodies.

$S \subset [k]^n$ is grid points inside Convex Body.

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Ex: Numerically integrate convex function in d dimensions.

Compute $\sum_i v_i Vol(f(x) > v_i)$ where $v_i = i\delta$.

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Choose random point in $[k]^n$ and check if in P.
Works.

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But P could be exponentially small compared to $|[k]^n|$.

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Takes a long time.

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One neighbor in each direction for each dimension
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Start at a grid point, and take a (random) walk.

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How long does this take?

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How long does this take? More later.

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How long does this take? More later.
But remember power method...

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But remember power method...which finds first eigenvector.

Spanning Trees.

Problem: How many?

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Another Problem: find a random one.

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Algorithm:

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Algorithm:
Start with spanning tree.

Spanning Trees.

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Algorithm:
  Start with spanning tree.
  Repeat:

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Algorithm is random walk on BIG GRAPH (sample space graph.)

Spin systems.

Each element of S may have associated weight.

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Sample element proportional to weight.

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Example?

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Example? 2 or 3 dimensional grid of particles.

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Energy on local interactions: $E = \sum_{(i,j)} -\sigma_i \sigma_j$.

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Metropolis Algorithm:
At x, generate y with a single random flip.

Spin systems.

Each element of S may have associated weight.

Sample element proportional to weight.

Example? 2 or 3 dimensional grid of particles. Particle State ± 1.
System State $\{-1,+1\}^n$.

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"Ferromagnetic regime": same spin is good.

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Markov Chain on statespace of system.

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Fix-it-up chappie!

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Distance to uniform: $d_1(v_t, \pi) = \sum_i |(v_t)_i - \pi_i|$

"Rapidly mixing": $d_1(v_t, \pi) \leq \varepsilon$ in **poly**$(\log N, \frac{1}{\varepsilon})$ time.

When is chain rapidly mixing?

Another measure: $d_2(v_t, \pi) = \sum_i ((v_t)_i - \pi_i)^2$.

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"Lazy" random walk: With probability $1/2$ stay at current vertex.

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Rapidly mixing with big ($\geq \frac{1}{p(n)}$) spectral gap.

Rapid mixing, volume, and surface area..

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Grid graph on grid points inside convex body.

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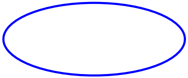

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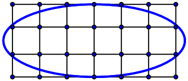

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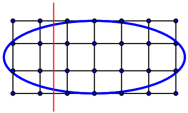

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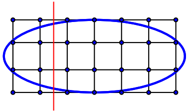

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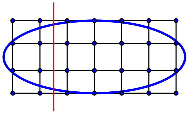

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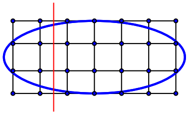

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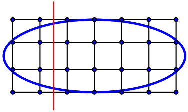

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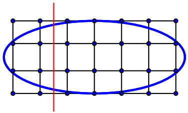

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$\rightarrow$ Rapidly mixing chain:

Sum up.

Sampling by random walks.

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Example: partial orders.

See you on Thursday.