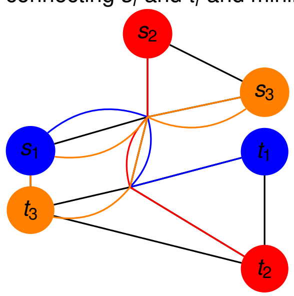
# CS270: Lecture 2.

Admin:
Check Piazza. There is a poll on bspace.
Today:

- Finish Path Routing.
- Games

# Path Routing.

Given $G=(V,E)$, $(s_1,t_1),\dots,(s_k,t_k)$, find a set of $k$ paths connecting $s_i$ and $t_i$ and minimize max load on any edge.

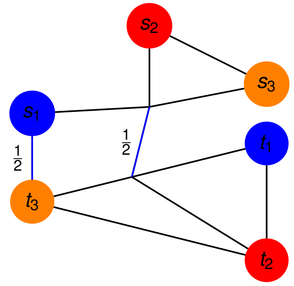



~~Value: 3~~

Value: 2

# Toll problem.

Given $G=(V,E)$, $(s_1,t_1),\dots,(s_k,t_k)$, find a set of $k$ paths assign one unit of "toll" to edges to maximize total toll for connecting pairs.




Assign $\frac{1}{11}$ on each of 11 edges.
Total toll: $\frac{3}{11}+\frac{3}{11}+\frac{3}{11}=\frac{9}{11}$
Can we do better?

Assign $1/2$ on these two edges.
Total toll: $\frac{1}{2}+\frac{1}{2}+\frac{1}{2}=\frac{3}{2}$

# Toll is lower bound on Path Routing.

From before:
Max bigger than minimum weighted average:
$\max_e c(e) \ge \sum_e c(e)d(e)$
Total length is total congestion: $\sum_e c(e)d(e) = \sum_i d(p_i)$
Each path, $p_i$, in routing has length $d(p_i) \ge d(s_i,t_i)$.

$$\max_e c(e) \ge \sum_e c(e)d(e) = \sum_i d(p_i) \ge \sum_i d(s_i,t_i).$$

A toll solution is lower bound on any routing solution.
Any routing solution is an upper bound on a toll solution.

# Algorithm.

Assign tolls.
How to route? Shortest paths!
Assign routing.
How to assign tolls? Higher tolls on congested edges.
Toll: $d(e) \propto 2^{c(e)}$.

Equilibrium:
The shortest path routing has has $d(e) \propto 2^{c(e)}$.

The routing does not change, the tolls do not change.

# How good is equilibrium?

Path is routed along shortest path and $d(e) \propto 2^{c(e)}$.
For $e$ with $c(e) \le c_{max} - 2\log m$; $2^{c(e)} \le 2^{c_{max}-2\log m} = \frac{2^{c_{max}}}{m^2}$.

$$
\begin{aligned}
c_{opt} &\ge \sum_i d(s_i,t_i) = \sum_e d(e)c(e) \\
&= \sum_e \frac{2^{c(e)}}{\sum_{e'} 2^{c(e')}} c(e) = \frac{\sum_e 2^{c(e)} c(e)}{\sum_e 2^{c(e)}} \quad \text{Let } c_t = c_{max} - 2\log m. \\
&\ge \frac{\sum_{e:c(e)>c_t} 2^{c(e)} c(e)}{\sum_{e:c(e)>c_t} 2^{c(e)} + \sum_{e:c(e)\le c_t} 2^{c(e)}} \\
&\ge \frac{(c_t)\sum_{e:c(e)>c_t} 2^{c(e)}}{(1+\frac{1}{m})\sum_{e:c(e)>c_t} 2^{c(e)}} \\
&\ge \frac{(c_t)}{1+\frac{1}{m}} = \frac{c_{max} - 2\log m}{(1+\frac{1}{m})}
\end{aligned}
$$

Or $c_{max} \le (1+\frac{1}{m})c_{opt} + 2\log m$.
(Almost) within $2\log m$ of optimal!

## The end: sort of.

Got to here in class. Feel free to continue reading.

## Getting to equilibrium.

Maybe no equilibrium!

Approximate equilibrium:

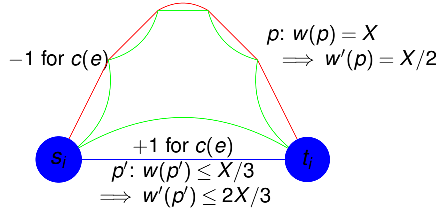
Each path is routed along a path with length within a factor of 3 of the shortest path and $d(e) \propto 2^{c(e)}$.

Lose a factor of three at the beginning.
$c_{opt} \geq \sum_i d(s_i,t_i) \geq \frac{1}{3}\sum_e d(p_i)$.
We obtain $c_{max} = 3(1+\frac{1}{m})c_{opt} + 2\log m$.

This is worse!
What do we gain?

## An algorithm!

Algorithm: reroute paths that are off by a factor of three.
(Note: $d(e)$ recomputed every rerouting.)




Potential function: $\sum_e w(e)$, $w(e) = 2^{c(e)}$
Moving path:
Divides $w(e)$ along long path (with $w(p)$ of $X$) by two.
Multiplies $w(e)$ along shorter ($w(p) \leq X/3$) path by two.

$-\frac{X}{2} + \frac{X}{3} = -\frac{X}{6}$.

Potential function decreases. $\implies$ termination and existence.

## Tuning...

Replace $d(e) = (1+\varepsilon)^{c(e)}$.

Replace factor of 3 by $(1+2\varepsilon)$

$c_{max} \leq (1+2\varepsilon)c_{opt} + 2\log m/\varepsilon$.. (Roughly)

Fractional paths?

## Wrap up.

Dueling players:
Toll player raises tolls on congested edges.
Congestion player avoids tolls.

Converges to near optimal solution!

A lower bound is "necessary" (natural),
and helpful (mysterious?)!

## Strategic Games.

$N$ players.

Each player has strategy set. $\{S_1,\ldots,S_N\}$.

Vector valued payoff function: $u(s_1,\ldots,s_n)$ (e.g., $\in \Re^N$).

Example:

2 players

Player 1: { **D**efect, **C**ooperate }.
Player 2: { **D**efect, **C**ooperate }.

Payoff:

| | **C** | **D** |
|---|---|---|
| **C** | (3,3) | (0,5) |
| **D** | (5,0) | (1,1) |

## Famous because?

| | C | D |
|---|---|---|
| C | (3,3) | (0,5) |
| D | (5,0) | (.1.1) |

What is the best thing for the players to do?

Both cooperate. Payoff $(3,3)$.

If player 1 wants to do better, what does he do?

Defects! Payoff $(5,0)$

What does player 2 do now?

Defects! Payoff $(.1,.1)$.

Stable now!

Nash Equilibrium: neither player has incentive to change strategy.

## Digression..

What situations?

Prisoner's dilemma:
Two prisoners separated by jailors and asked to betray partner.

Basis of the free market.
Companies compete, don't cooperate.
No Monopoly:
E.G., OPEC, Airlines, .
Should defect.
Why don't they?
Free market economics ...not so much?
More sophisticated models ,e.g, iterated dominance, coalitions, complexity..
Lots of interesting Game Theory!

This class(today): simpler version.

## Two Person Zero Sum Games

2 players.

Each player has strategy set:
$m$ strategies for player 1 $n$ strategies for player 2

Payoff function: $u(i,j) = (-a,a)$ (or just $a$).
"Player 1 pays $a$ to player 2."

Zero Sum: Payoff for any pair of strategies sums to 0.

Payoffs by $m$ by $n$ matrix: $A$.
Row player minimizes, column player maximizes.

Roshambo: rock,paper, scissors.

| | R | P | S |
|---|---|---|---|
| R | 0 | 1 | -1 |
| P | -1 | 0 | 1 |
| S | 1 | -1 | 0 |

Any Nash Equilibrium?

$(R,R)$? no. $(R,P)$? no. $(R,S)$? no.

## Mixed Strategies.

| | | R | P | S |
|---|---|---|---|---|
| | | $.3\overline{3}$ | $.3\overline{3}$ | $.3\overline{3}$ |
| R | $.3\overline{3}$ | 0 | 1 | -1 |
| P | $.3\overline{3}$ | -1 | 0 | 1 |
| S | $.3\overline{3}$ | 1 | -1 | 0 |

How do you play?

Player 1: play each strategy with equal probability.
Player 2: play each strategy with equal probability.

Definitions.

**Mixed strategies:** Each player plays distribution over strategies.

**Pure strategies:** Each player plays single strategy.

## Payoffs: Equilibrium.

| | | R | P | S |
|---|---|---|---|---|
| | | $.3\overline{3}$ | $.3\overline{3}$ | $.3\overline{3}$ |
| R | $.3\overline{3}$ | 0 | 1 | -1 |
| P | $.3\overline{3}$ | -1 | 0 | 1 |
| S | $.3\overline{3}$ | 1 | -1 | 0 |

Payoffs? Can't just look it up in matrix!.

Average Payoff. Expected Payoff.

Sample space: $\Omega = \{(i,j) : i,j \in [1,..,3]\}$
Random variable $X$ (payoff).

$$E[X] = \sum_{(i,j)} X(i,j) Pr[(i,j)].$$

Each player chooses independently:
$Pr[(i,j)] = \frac{1}{3} \times \frac{1}{3} = \frac{1}{9}$.

$$E[X] = 0.$$ [1]

[1] Remember zero sum games have one payoff.

## Equilibrium

| | | R | P | S |
|---|---|---|---|---|
| | | $.3\overline{3}$ | $.3\overline{3}$ | $.3\overline{3}$ |
| R | $.3\overline{3}$ | 0 | 1 | -1 |
| P | $.3\overline{3}$ | -1 | 0 | 1 |
| S | $.3\overline{3}$ | 1 | -1 | 0 |

Will Player 1 change strategy? Mixed strategies uncountable!

Expected payoffs for pure strategies for player 1.

Expected payoff of Rock? $\frac{1}{3} \times 0 + \frac{1}{3} \times 1 + \frac{1}{3} \times -1 = 0$.
Expected payoff of Paper? $\frac{1}{3} \times -1 + \frac{1}{3} \times 0 + \frac{1}{3} \times 1 = 0$.
Expected payoff of Scissors? $\frac{1}{3} \times 1 + \frac{1}{3} \times -1 + \frac{1}{3} \times 0 = 0$.

No better pure strategy. $\implies$ No better mixed strategy!

Mixed strat. payoff is weighted av. of payoffs of pure strats.

$E[X] = \sum_{(i,j)} (Pr[i] \times Pr[j]) X(i,j) = \sum_i Pr[i] (\sum_j Pr[j] \times X(i,j))$

Mixed strategy can't be better than the best pure strategy.

Player 1 has no incentive to change! Same for player 2.

Equilibrium!

## Another example plus notation.

Rock, Paper, Scissors, prEempt.
PreEmpt ties preEmpt, beats everything else.
Payoffs.

| | R | P | S | E |
|---|---|---|---|---|
| R | 0 | 1 | -1 | 1 |
| P | -1 | 0 | 1 | 1 |
| S | 1 | -1 | 0 | 1 |
| E | -1 | -1 | -1 | 0 |

Equilibrium? **(E,E)**. Pure strategy equilibrium.
Notation: Rock is 1, Paper is 2, Scissors is 3, prEmpt is 4.
Payoff Matrix.

$$A = \begin{bmatrix} 0 & 1 & -1 & 1 \\ -1 & 0 & 1 & 1 \\ 1 & -1 & 0 & 1 \\ -1 & -1 & -1 & 0 \end{bmatrix}$$

Lecture 2 ended here..and Lecture 3 reviewed a few of the previous slides and continued into lecture 3 notes.

## Playing the boss...

Row has extra strategy:Cheat.
Ties with rock and scissors, beats paper. (Scissors, or no rock!)
Payoff matrix:
Rock is strategy 1, Paper is 2, Scissors is 3, and Cheat is 4 (for row.)

$$A = \begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

Note: column knows row cheats.
Why play?
Row is column's advisor.
... boss.

## Two person zero sum games.

$m \times n$ payoff matrix $A$.

Row mixed strategy: $x = (x_1, \ldots, x_m)$.
Column mixed strategy: $y = (y_1, \ldots, y_n)$.

Payoff for strategy pair $(x, y)$:

$$p(x, y) = x^t A y$$

That is,

$$\sum_i x_i \left( \sum_j a_{i,j} y_j \right) = \sum_j \left( \sum_i x_i a_{i,j} \right) y_j.$$

Recall row minimizes, column maximizes.

Equilibrium pair: $(x^*, y^*)$?

$$(x^*)^t A y^* = \max_y (x^*)^t A y = \min_x x^t A y^*.$$

(No better column strategy, no better row strategy.)

## Equilibrium: play the boss...

$$A = \begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

Equilibrium:
Row: $(0, \frac{1}{3}, \frac{1}{6}, \frac{1}{2})$. Column: $(\frac{1}{3}, \frac{1}{2}, \frac{1}{6})$.

Payoff? Remember: weighted average of pure strategies.

Row Player.

Strategy 1: $\frac{1}{3} \times 0 + \frac{1}{2} \times 1 + \frac{1}{6} \times -1 = \frac{1}{3}$
Strategy 2: $\frac{1}{3} \times -1 + \frac{1}{2} \times 0 + \frac{1}{6} \times 1 = -\frac{1}{6}$
Strategy 3: $\frac{1}{3} \times 1 + \frac{1}{2} \times -1 + \frac{1}{6} \times 0 = -\frac{1}{6}$
Strategy 4: $\frac{1}{3} \times 0 + \frac{1}{2} \times 0 + \frac{1}{6} \times -1 = -\frac{1}{6}$

Payoff is $0 \times \frac{1}{3} + \frac{1}{3} \times (-\frac{1}{6}) + \frac{1}{6} \times (-\frac{1}{6}) + \frac{1}{2} \times (-\frac{1}{6}) = -\frac{1}{6}$

Column player: every column payoff is $-\frac{1}{6}$.

Both only play optimal strategies! Complementary slackness.

Why not play just one? Change payoff for other guy!

## Equilibrium.

Equilibrium pair: $(x^*, y^*)$?

$$p(x, y) = (x^*)^t A y^* = \max_y (x^*)^t A y = \min_x x^t A y^*.$$

(No better column strategy, no better row strategy.)

No row is better:
$\min_i A^{(i)} \cdot y = (x^*)^t A y^*$. [2]

No column is better:
$\max_j (A^t)^{(j)} \cdot x = (x^*)^t A y^*$.

[2] $A^{(i)}$ is $i$th row.

## Best Response

**Column goes first:**
Find $y$, where best row is not too low..

$$R = \max_y \min_x (x^t A y).$$

Note: $x$ can be $(0,0,\ldots,1,\ldots 0)$.

Example: Roshambo. Value of $R$?

**Row goes first:**
Find $x$, where best column is not high.

$$C = \min_x \max_y (x^t A y).$$

Agin: $y$ of form $(0,0,\ldots,1,\ldots 0)$.

Example: Roshambo. Value of $C$?

## Duality.

$$R = \max_y \min_x (x^t A y).$$
$$C = \min_x \max_y (x^t A y).$$

**Weak Duality:** $R \leq C$.
**Proof:** Better to go second. □

At Equilibrium $(x^*, y^*)$, payoff $v$:
row payoffs $(Ay^*)$ all $\geq v \implies R \geq v$.
column payoffs $((x^*)^t A)$ all $\leq v \implies v \geq C$.
$\implies R \geq C$

Equilibrium $\implies R = C$!

**Strong Duality:** There is an equilibrium point! and $R = C$!

Doesn't matter who plays first!

## Proof of Equilibrium.

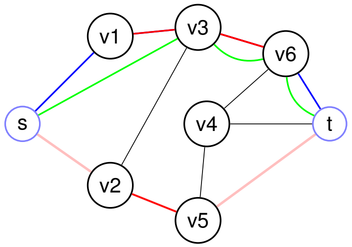
Later. Let's see some examples.

## An "asymptotic" game.

"Catch me."

Given: $G = (V, E)$.
Given $a, b \in V$.
Row ("Catch me"): choose path from $a$ to $b$.
Column("Catcher"): choose edge.
Row pays if column chooses edge on path.

Matrix:
row for each path: $p$
column for each edge: $e$
$A[p,e] = 1$ if $e \in p$.




**Catchme:**
Use Blue Path.
Blue with prob. 1/2.
Blue with prob. 1/3.
Green with prob. 1/2.
Green with prob. 1/6.
Pink with prob. 1/2.

**Catcher:**
Caught!
Caught sometimes.
With probability 1/2.

## Example.

Row solution: $Pr[p_1] = 1/2$, $Pr[p_2] = 1/3$, $Pr[p_3] = 1/6$.
Edge solution: $Pr[e_1] = 1/2$, $Pr[e_2] = 1/2$

**Offense (Best Response.):**

Catch me: route along shortest path.
(Knows catcher's distribution.)
Catcher: raise toll on most congested edge.
(Knows catch me's distribution.)

**Defense:**

Where should "catcher" play to catch any path? a cut.
**Minimum cut** allows the maximum toll on any edge!

What should "catch me" do to avoid catcher?
minimize maximum load on any edge!
**Max-Flow Problem.**

Note: exponentially many strategies for "catch me"!

## Toll/Congestion

Given: $G = (V, E)$.
Given $(s_1, t_1) \ldots (s_k, t_k)$.
Row: choose routing of all paths.
Column: choose edge.
Row pays if column chooses edge on any path.

Matrix:
row for each routing: $r$
column for each edge: $e$

$A[r, e]$ is congestion on edge $e$ by routing $r$

**Offense: (Best Response.)**
Router: route along shortest paths.
Toll: charge most loaded edge.

**Defense:** Toll: maximize shortest path under tolls.
Route: minimize max loaded on any edge.

Again: exponentially (squared) number of paths for route player.

## Summary...

You should now know about

Games
Nash Equilibrium
Pure Strategies
Zero Sum Two Person Games
Mixed Strategies.
Checking Equilibrium.
Best Response.
Statement of Duality Theorem.

## Finding Equilibrium.

...see you Tuesday.