Today

Johnson-Lindenstrass

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"Projecting and scaling by $\sqrt{\frac{d}{k}}$ preserves all pairwise distances w/in factor of $1 \pm \varepsilon$."

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Method 2:
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remove projection onto previous subspace.

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Project x into subspace spanned by $v_1, v_2, \cdots, v_k$.

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k is large enough $\rightarrow$
$\approx (1 \pm \varepsilon)\sqrt{\frac{k}{d}}$ with decent probability.

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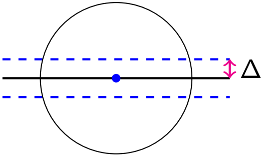

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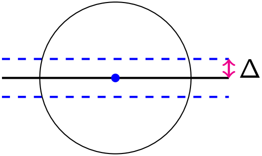

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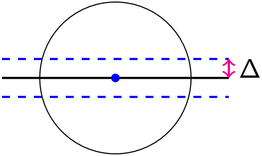

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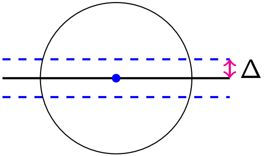
Point on "Δ-spherical cap".



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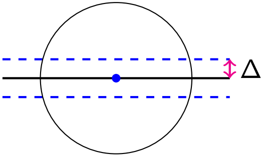

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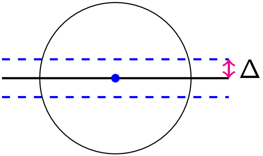

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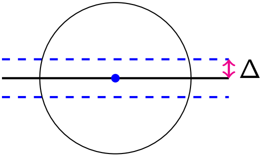

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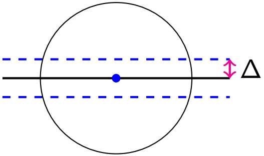

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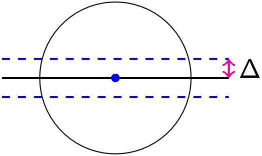

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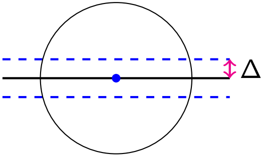

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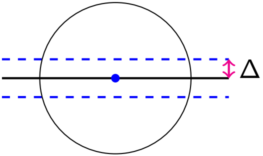

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Length? $z = z_1^2 + z_2^2 + \cdots z_k^2$.

$$\Pr[\left|\sqrt{z_1^2 + z_2^2 + \cdots + z_k^2} - \sqrt{\frac{k}{d}}\right| > t] \leq e^{-t^2 d}$$

Substituting $t = \varepsilon\sqrt{\frac{k}{d}}$, $k = \frac{c\log n}{\varepsilon^2}$.

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$\le n^2$ pairs

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$\le n^2$ pairs plus union bound
$\rightarrow$ prob any pair fails to be preserved with $\le \frac{1}{n^{c-2}}$.

Locality Preserving Hashing

Find nearby points in high dimensional space.

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Points could be images!

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Project high dimensional points into low dimensions.

Use grid hash function.

Implementing Johnson-Lindenstraus

Random vectors

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Random vectors have many bits

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Coordinate for bit vector b.

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$E[\sum_i C_i^2] = \frac{k}{d}$

Binary Johnson-Lindenstrass

Project onto $[-1, +1]$ vectors.

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$$\Pr\left[|C - \frac{k}{d}| \geq \varepsilon \frac{k}{d}\right] \leq e^{-\varepsilon^2 k}$$

Variance of C_i^2? $\left(\frac{k}{d^2}\right)(\sum_i {z_i}^4 + 4\sum_{i,j} z_i^2 z_j^2) \leq \left(\frac{k}{d^2}\right) 2(\sum_i z_i^2)^2 \leq \frac{2k}{d^2}$.

Roughly normal (gaussian):

Density $\propto e^{-t^2}/2$ for t std deviations away.

So, assuming normality

$\sigma = \frac{\sqrt{k}}{d}$, $t = \frac{\varepsilon \frac{k}{d}}{\frac{\sqrt{2k}}{d}} = \varepsilon\sqrt{k}/\sqrt{2}$.

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Sum up

Have a good break!