## Today

Load balancing.

Balls in Bins.

Power of two choices.

Cuckoo hashing.

$$\left(\frac{n}{k}\right)^k \le \binom{n}{k} \le \frac{n^k}{k!} \le \left(\frac{ne}{k}\right)^k$$

$\binom{n}{k} = \frac{n(n-1)\cdots(n-k+1)}{k(k-1)\cdot 1} = \frac{n}{k}\cdot\frac{n-1}{k-1}\cdots\frac{n-k+1}{1} \ge \frac{n}{k}\cdot\frac{n}{k}\cdots\frac{n}{k}$

$n(n-1)\cdots(n-k+1) \le n^k$

$k! \ge \left(\frac{k}{e}\right)^k$

## Simplest..

Load balance: $m$ balls in $n$ bins.

For simplicity: $n$ balls in $n$ bins.

Round robin: load 1 !

Centralized! Not so good.

Uniformly at random? Average load 1.

Max load?

$n$. Uh Oh!

Max load with probability $\ge 1-\delta$?

$\delta = \frac{1}{n^c}$ for today. $c$ is 1 or 2.

## Balls in bins.

For each of $n$ balls, choose random bin: $X_i$ balls in bin $i$.

$Pr[X_i \ge k] \le \sum_{S\subseteq[n],|S|=k} Pr[\text{balls in } S \text{ chooses bin } i]$

From Union Bound: $Pr[\cup_i A_i] \le \sum_i Pr[A_i]$

$Pr[\text{balls in } S \text{ chooses bin } i] = \left(\frac{1}{n}\right)^k$ and $\binom{n}{k}$ subsets $S$.

$$\begin{aligned} \Pr[X_i \ge k] &\le \binom{n}{k}\left(\frac{1}{n}\right)^k \\ &\le \frac{n^k}{k!}\left(\frac{1}{n}\right)^k = \frac{1}{k!} \end{aligned}$$

Choose $k$, so that $Pr[X_i \ge k] \le \frac{1}{n^2}$.

$Pr[\text{any } X_i \ge k] \le n \times \frac{1}{n^2} = \frac{1}{n} \to$ max load $\le k$ w.p. $\ge 1-\frac{1}{n}$

$k! \ge n^2$ for $k = 2e\log n$ (Recall $k! \ge (\frac{k}{e})^k$.)

**Lemma:** Max load is $\Theta(\log n)$ with probability $\ge 1-\frac{1}{n}$.

Much better than $n$.

Actually Max load is $\Theta(\log n/\log\log n)$ w.h.p.

(W.h.p. - means with probability at least $1 - O(1/n^c)$ for today.)

## Power of two..

$n$ balls in $n$ bins.

Choose two bins, pick least loaded.

still distributed, but a bit less than not looking.

Is max load lower? Yes? No? Yes.

How much lower?

$\log n/2$? $\sqrt{\log n}$? $O(\log\log n)$?

$O(\log\log n)$ ! ! ! !

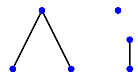
Exponentially better! Old bound is exponential of new bound.

## Analysis.

$n/8$ balls in $n$ bins.

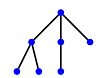
Each ball chooses two bins at random.
picks least loaded.

View as graph.
Bin is vertex.
Each ball is edge.



Analysis Intuition:
Add edge, add one to lower endpoint's "count."

Max load is max vertices count.
If max count is $k$.
neighbors with counts $\ge k-1, k-2, k-3, \ldots$.
and so on!



No cycles and max-load $k \to \ge 2^{k/2}$ nodes in tree.
No connected component of size $X$ and no cycles
$\implies$ max load $O(\log X)$.

Will show:
Max conn. comp is $O(\log n)$ w.h.p.
Average induced degree is small. (E.g.: cycle degree 2)

Extend tree intuition.

# Connected Component.

**Claim:** Component size in $n$ vertex, $\frac{n}{8}$ edge random graph is $O(\log n)$ w/ prob. $\geq 1-\frac{1}{n^c}$.
pause
**Proof:** Size $k$ component, $C$, contains $\geq k-1$ edges.

$$\Pr[|C| \geq k] \leq \binom{n}{k}\binom{n/8}{k-1}\left(\frac{k}{n}\right)^{2(k-1)} \quad (1)$$

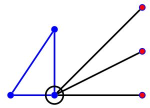
Possible $C$. Which edges. Prob. both endpoints inside $C$.

$$\begin{aligned}\Pr[|C| \geq k] &\leq \frac{n}{k}\binom{n}{k}\binom{n/8}{k}\left(\frac{k}{n}\right)^{2k} \\ &\leq \frac{n}{k}\left(\frac{ne}{k}\right)^k\left(\frac{ne}{8k}\right)^k\left(\frac{k}{n}\right)^{2k} = \frac{n}{k}\left(\frac{e^2}{8}\right)^k \leq \frac{n}{k}(0.93)^k \quad (2)\end{aligned}$$

Choose $k = -(c+1)\log_{.93} n$ make probability $\leq 1/n^c$.

# Not dense.

Induced degree of node on subset, $S$, is degree of internal edges.



Induced degree of nodes in blue subset is 2, not 5!

**Claim:** Average induced degree on any subset of nodes is $\leq 8$ with probability $\geq 1-O(\frac{1}{n^2})$.

**Proof:** Induced degree $\geq 8$
$\rightarrow$ $4k$ internal edges for subset of size $k$.

$$\Pr[\text{dense } S] \leq \binom{n}{k}\binom{n/8}{4k}\left(\frac{k}{n}\right)^{8k} \leq \left(\frac{e^{1.25}}{32}\right)^{4k}\left(\frac{k}{n}\right)^{3k} \leq \left(\frac{k}{n}\right)^{3k}$$

Starts at $1/n^3$, decreasing till $k \leq n/8$ (at least)
$\rightarrow$ Total $O(1/n^2)$.

# Removal Process!

**Random Graph:** Component size is $c\log n$ and max-induced degree is 8 w.h.p.

**Process:** Remove degree $\leq 16$ nodes
and incident edges. Repeat.
Claim: $O(\log X)$ iterations where $X$ is max component size.

For any connected component:
Average induced degree 8 $\rightarrow$ half nodes w/degree $\leq 16$.
$\rightarrow$ half nodes removed in each iteration.
$\rightarrow$ $\log X$ iterations to remove all nodes.

**Claim:** Max load is $O(\log\log n)$ w.h.p.

Recall edge corresponds to ball.
Height of ball, $h_i$, is load of bin when it is placed in bin.
Corresponding edge removed in iteration $r_i$.
**Property:** $h_i \leq 16r_i$.
Case $r_i = 1$ - only 16 balls incident to bin $\rightarrow$ $h_i \leq 16$.
Induction: Previous removed edges(ball) induce load $\leq 16(r_i-1)$.
$+16$ edges/balls this iteration.
$\rightarrow$ $h_i \leq 16r_i$.

# Power of two choices.

Max load: $\log X$ where $X$ is max component size.

$X$ is $O(\log n)$ with high probability.

Max load is $O(\log\log n)$.

# Cuckoo hashing.

Hashing with two choices: max load $O(\log\log n)$.

Cuckoo hashing:
Array. Two hash functions $h_1$, $h_2$.

Insert $x$: place in $h_1(x)$ or $h_2(x)$ if space.
Else bump elt $y$ in $h_i(x)$ u.a.r.
Bump $y,x$: place $y$ in $h_i(y) \neq h_i(x)$ if space.
Else bump $y'$ in $h_i(y)$.

If go too long. Fail. Rehash entire hash table.

Fails if cycle.

$C_l$ - event of cycle of length $l$.

$$\Pr[C_l] \leq \binom{m}{l+1}\binom{n}{l}\left(\frac{l}{n}\right)^{2(l+1)} \leq \left(\frac{e^2}{8}\right)^l \quad (3)$$

Probability that an insert makes a cycle of length $l \leq \frac{l}{n}\left(\frac{e^2}{8}\right)^l$

Rehash every $\Omega(n)$ inserts (if $\leq n/8$ items in table.)

$O(1)$ time on average.

# Sum up

Balls in bins: $\Theta(\log n/\log\log n)$ load.

Power of two: $\Theta(\log\log n)$.

Cuckoo hashing.

See you on Thursday...