Today

Lagrangian Dual.

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Lagrangian Dual. Already saw example!

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Convex Separator.

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Farkas Lemma.

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Find x, subjet to

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$f_i(x) \leq 0, i = 1, \ldots m.$

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(A) there is no feasible x.

(B) there is no x,λ with $L(x,\lambda) < 0$.

Lagrangian:constrained optimization.

$$\begin{aligned} \min \quad & f(x) \\ \text{subject to } & f_i(x) \leq 0, \qquad i = 1, ..., m \end{aligned}$$

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Primal problem:
x, that minimizes $L(x,\lambda)$ over all $\lambda > 0$.

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$\min cx, Ax \geq b$

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Duals!

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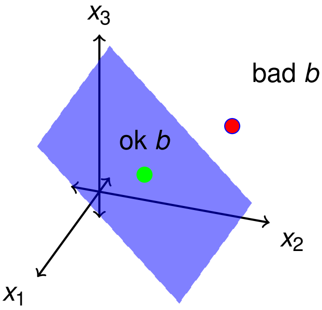

Strong Duality.

Strong Duality.

Later.

Strong Duality.

Later. Actually. No.

Strong Duality.

Later. Actually. No. Now

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Later. Actually. No. Now ...ish.

Special Cases:

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min-max 2 person games and experts.

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- Special Cases:
  - min-max 2 person games and experts.
  - Max weight matching and algorithm.

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    Approximate: facility location primal dual.

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Today: Geometry!

Convex Body and point.

For a convex body P and a point b, $b \in P$ or hyperplane separates P from b.

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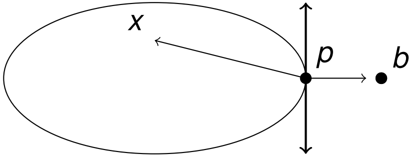
point p where $(x-p)^T(b-p) < 0$



Proof.

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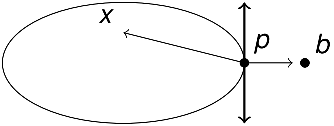

Proof: Choose p to be closest point to b in P.

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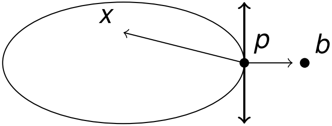

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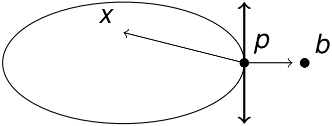

Proof: Choose p to be closest point to b in P.

Done or $\exists\, x \in P$ with $(x-p)^T(b-p) \geq 0$

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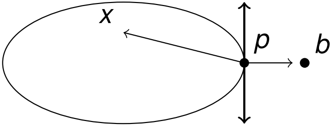

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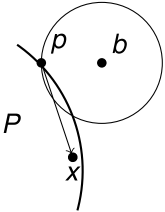

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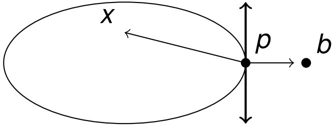

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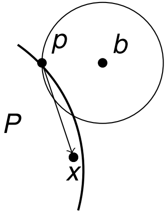

$(x-p)^T(b-p) \geq 0$

$\rightarrow \leq 90^\circ$ angle between $\overrightarrow{x-p}$ and $\overrightarrow{b-p}$.

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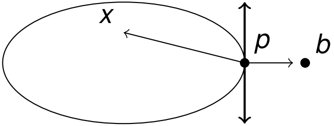

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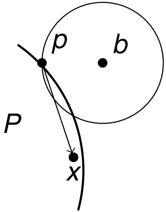
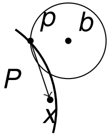


$(x-p)^T(b-p) \geq 0$

$\rightarrow \leq 90^\circ$ angle between $\overrightarrow{x-p}$ and $\overrightarrow{b-p}$.

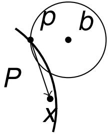
Must be closer point on line to from p to x.

More formally.



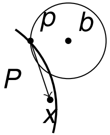
Squared distance to b from $p + (x - p)\mu$

More formally.



Squared distance to b from $p+(x-p)\mu$
point between p and x

More formally.



Squared distance to b from $p+(x-p)\mu$

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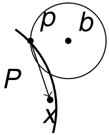
$$(|p-b|-\mu|x-p|cos\theta)^2+(\mu|x-p|sin\theta)^2$$

More formally.



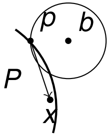
Squared distance to b from $p + (x-p)\mu$

point between p and x

$(|p-b| - \mu|x-p|cos\theta)^2 + (\mu|x-p|sin\theta)^2$

θ is the angle between $x-p$ and $b-p$.

More formally.



Squared distance to b from $p+(x-p)\mu$

point between p and x

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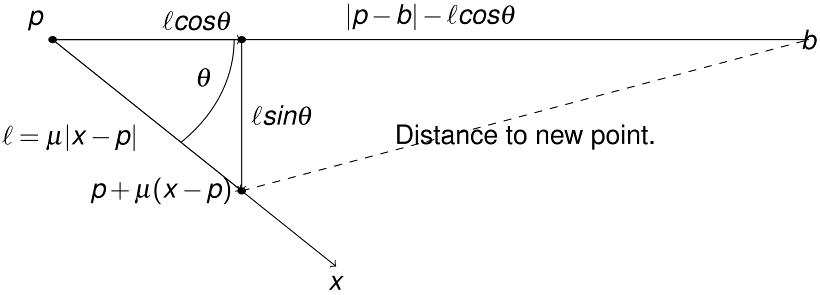

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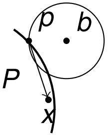

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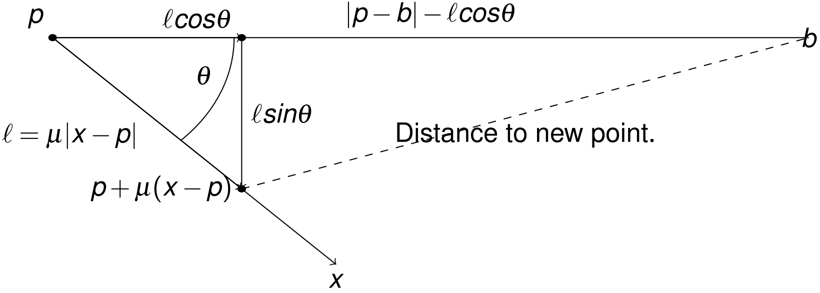

Simplify:

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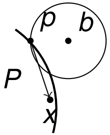

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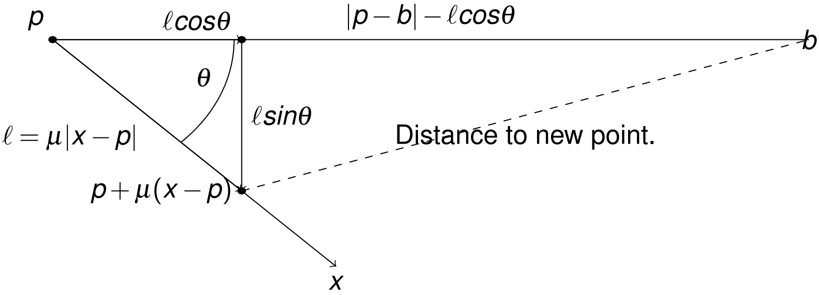

Simplify:
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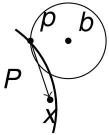
More formally.



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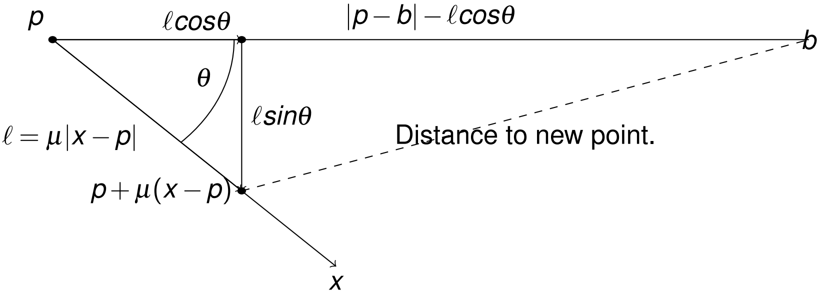

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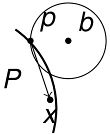
Derivative with respect to μ ...

More formally.



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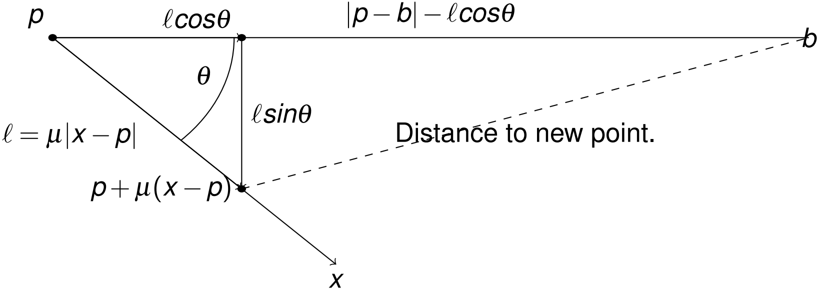

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Derivative with respect to μ ...

$-2|p-b||x-p|cos\theta+2(\mu|x-p|)$.

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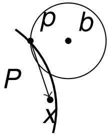

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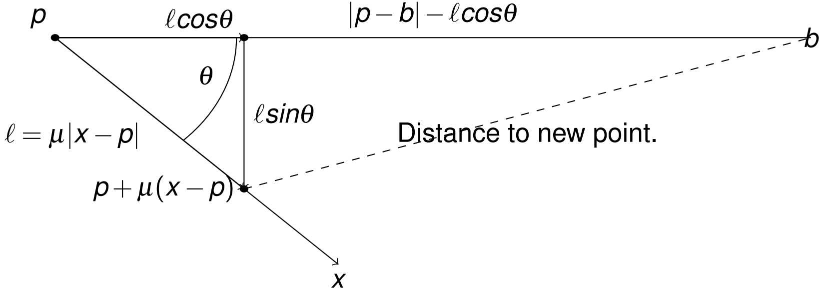

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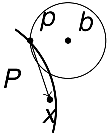

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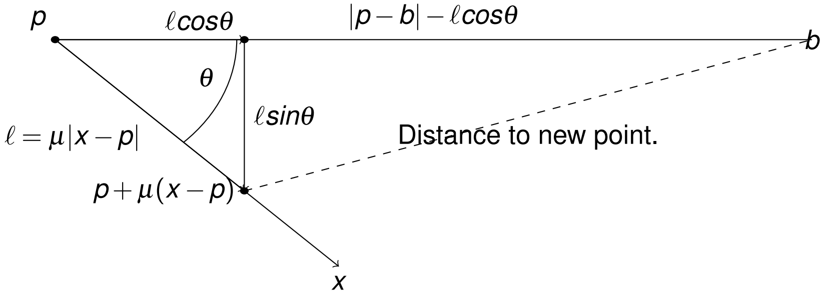

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Generalization: exercise.

There is a separating hyperplane between any two convex bodies.

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Let closest pair of points in two bodies define direction.

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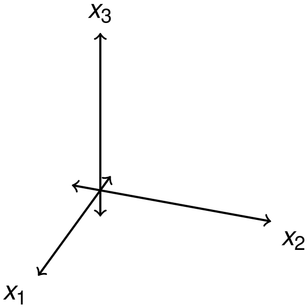
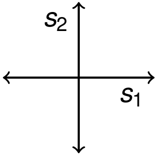


Coordinates $s = b - Ax$.
$x \geq 0$ where $s = 0$?



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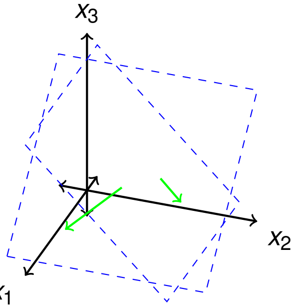

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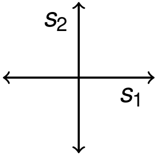

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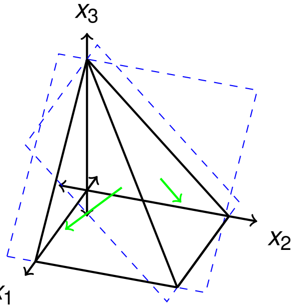

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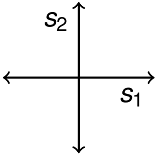

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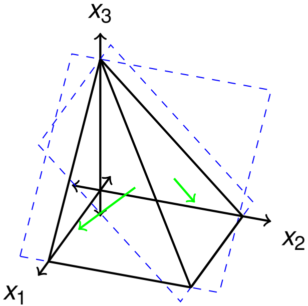

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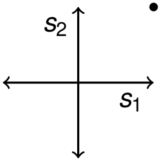

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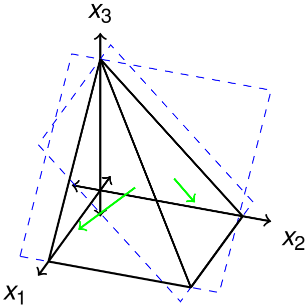

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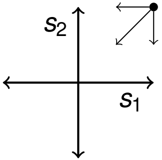

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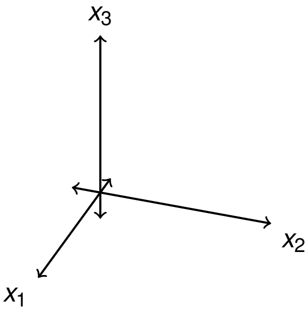

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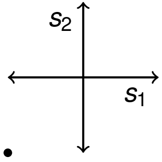

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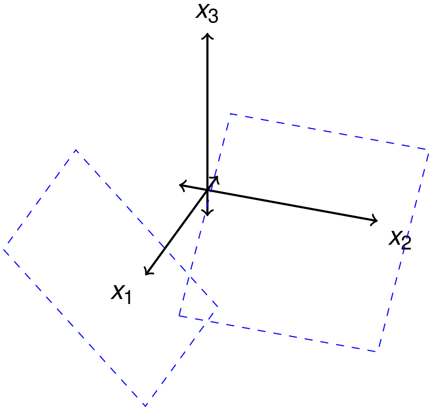

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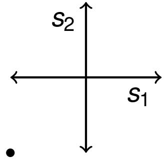

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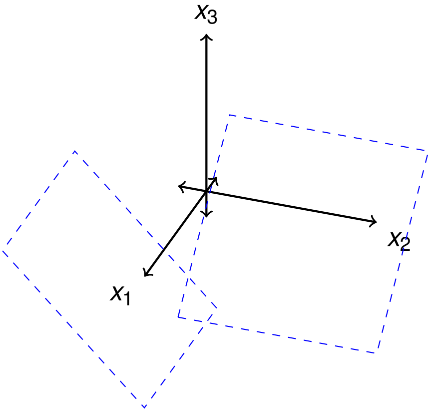

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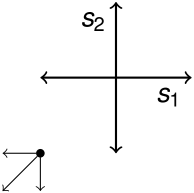

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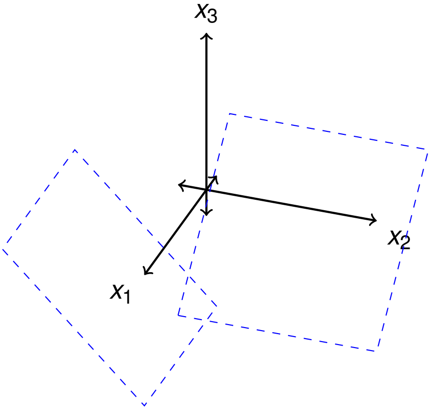

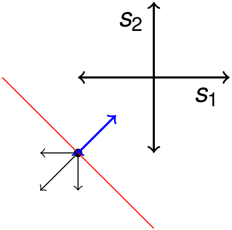

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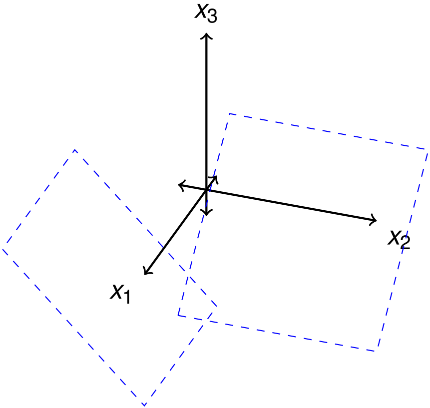

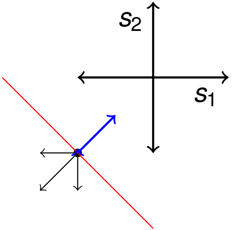

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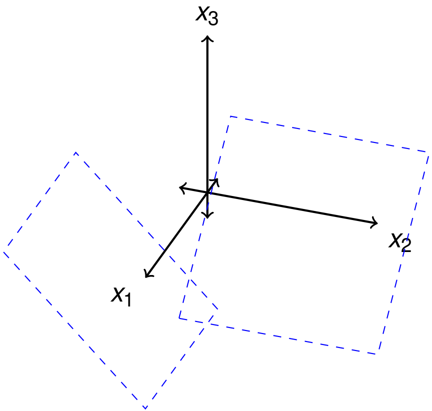

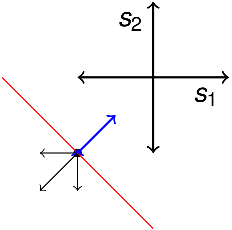

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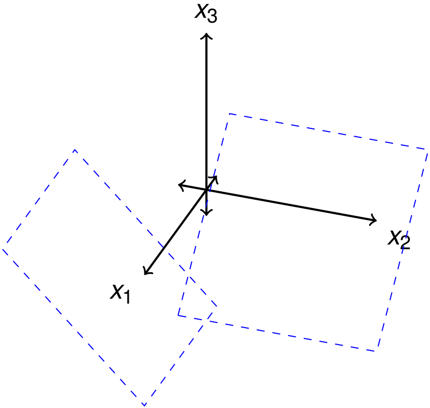

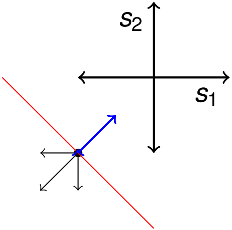

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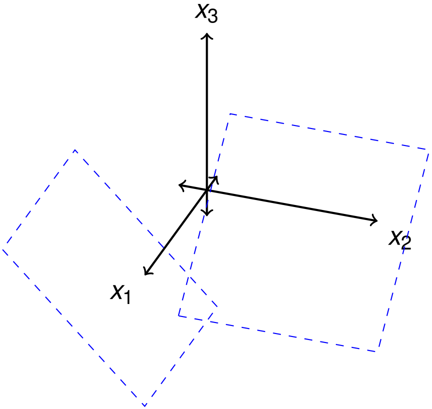

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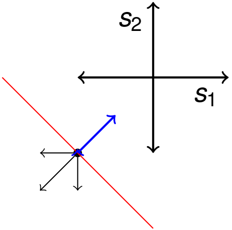

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Farkas 2

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Strong Duality

(From Goemans notes.)

Primal P $\quad z^* = \min c^T x$

$$Ax = b$$

$$x \geq 0$$

Dual D $: w^* = \max b^T y$

$$A^T y \leq c$$

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$$\begin{aligned} x^T c - b^T y &= x^T c - x^T A^T y \\ &= x^T (c - A^T y) \\ &\geq 0 \end{aligned}$$

Strong duality If P or D is feasible and bounded then $z^* = w^*$.

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If none, then Farkas B says
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$$\begin{pmatrix} A & -b \end{pmatrix} \begin{pmatrix} x \\ \lambda \end{pmatrix} = 0$$

$$\begin{pmatrix} c^T & -z^* \end{pmatrix} \begin{pmatrix} x \\ \lambda \end{pmatrix} < 0$$

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$\exists x, \lambda$ with $Ax - b\lambda = 0$ and $c^t x - z^* \lambda < 0$

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$\exists x, \lambda$ with $Ax - b\lambda = 0$ and $c^t x - z^* \lambda < 0$

Case 1: $\lambda > 0$.

Strong duality If P or D is feasible and bounded then $z^* = w^*$.

Primal feasible, bounded, value z^*.

Claim: Exists a solution to dual of value at least z^*.

$\exists y, y^T A \leq c, b^T y \geq z^*$.

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See you on Thursday.