Today

Lagrangian Dual. Already saw example!

Convex Separator.

Farkas Lemma.

Lagrangian Dual.

Find x, subjet to

$f_i(x) \le 0, i = 1, \ldots m.$

Remember calculus (constrained optimization.)

Lagrangian: $L(x,\lambda) = \sum_{i=1}^{m} \lambda_i f_i(x)$

λ_i - Lagrangian multiplier for inequality i.

For feasible solution x, $L(x,\lambda)$ is

(A) non-negative in expectation

(B) positive for any λ.

(C) non-positive for any valid λ.

If λ, where $L(x,\lambda)$ is positive for all x

(A) there is no feasible x.

(B) there is no x,λ with $L(x,\lambda) < 0$.

Lagrangian:constrained optimization.

$$\begin{aligned} \min \quad & f(x) \\ \text{subject to } & f_i(x) \le 0, \quad i = 1,\ldots,m \end{aligned}$$

Lagragian function:

$L(x,\lambda) = f(x) + \sum_{i=1}^{m} \lambda_i f_i(x)$

If (primal) x value v

For all $\lambda \ge 0$ with $L(x,\lambda) \le v$

Maximizing λ only positive when $f_i(x) = 0$.

If there is λ with $L(x,\lambda) \ge \alpha$ for all x

For optimum value of program is at least α

Primal problem:

x, that minimizes $L(x,\lambda)$ over all $\lambda > 0$.

Dual problem:

λ, that maximizes $L(x,\lambda)$ over all x.

Linear Program.

$\min cx, Ax \ge b$

$$\begin{aligned} \min \quad & c \cdot x \\ \text{subject to } & b_i - a_i \cdot x \le 0, \quad i = 1,\ldots,m \end{aligned}$$

Lagrangian (Dual):

$L(\lambda,x) = cx + \sum_i \lambda_i (b_i - a_i x_i).$

or

$L(\lambda,x) = -(\sum_j x_j(a_j\lambda - c_j)) + b\lambda.$

Best λ?

$\max b \cdot \lambda$ where $a_j \lambda = c_j$.

$\max b\lambda, \lambda^T A = c, \lambda \ge 0$

Duals!

Linear Equations.

$Ax = b$

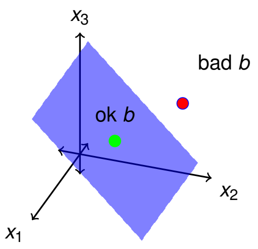
A is $n \times n$ matrix...

..has a solution.

If rows of A are linearly independent.

$y^T A \ne 0$ for any y

..or if b in subspace of A.



Strong Duality.

Later. Actually. No. Now ...ish.

Special Cases:

min-max 2 person games and experts.

Max weight matching and algorithm.

Approximate: facility location primal dual.

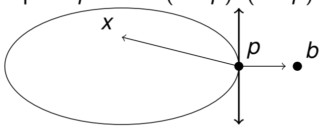
Today: Geometry!

Convex Body and point.

For a convex body P and a point b, $b \in P$ or hyperplane separates P from b.

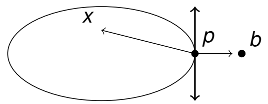
v, α, where $v \cdot x \leq \alpha$ and $v \cdot b > \alpha$.

point p where $(x-p)^T(b-p) < 0$



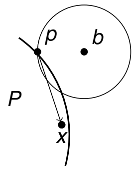
Proof.

For a convex body P and a point b, $b \in A$ or hyperplane point p where $(x-p)^T(b-p) < 0$



Proof: Choose p to be closest point to b in P.

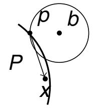
Done or $\exists\, x \in P$ with $(x-p)^T(b-p) \geq 0$



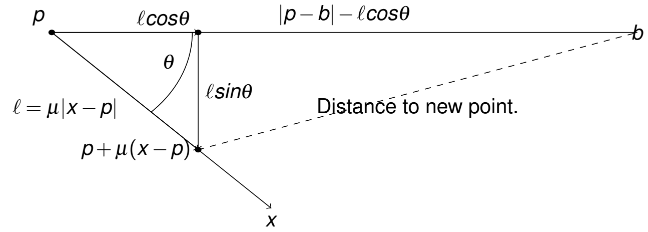
$(x-p)^T(b-p) \geq 0$
$\rightarrow \leq 90^\circ$ angle between $\overrightarrow{x-p}$ and $\overrightarrow{b-p}$.

Must be closer point on line to from p to x.

More formally.



Squared distance to b from $p + (x-p)\mu$
point between p and x
$(|p-b| - \mu|x-p|cos\theta)^2 + (\mu|x-p|sin\theta)^2$
θ is the angle between $x-p$ and $b-p$.



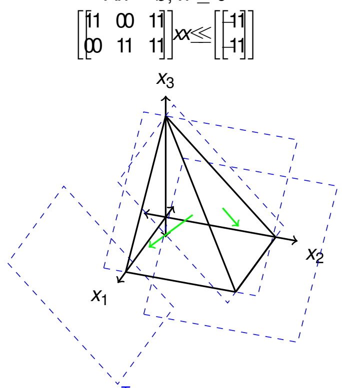
Simplify:
$|p-b|^2 - 2\mu|p-b||x-p|cos\theta + (\mu|x-p|)^2$.
Derivative with respect to μ ...
$-2|p-b||x-p|cos\theta + 2(\mu|x-p|)$.
which is negative for a small enough value of μ (for positive $cos\theta$.)

Generalization: exercise.

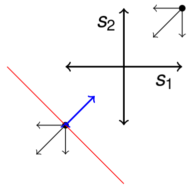
There is a separating hyperplane between any two convex bodies.

Let closest pair of points in two bodies define direction.

$Ax = b$, $x \geq 0$



Coordinates $s = b - Ax$.
$x \geq 0$ where $s = 0$?



y where $y^T(b - Ax) < 0$ for all x $\rightarrow y^Tb < 0$ and $y^TA \geq 0$.
Farkas A: Solution for exactly one of:
(1) $Ax = b, x \geq 0$
(2) $y^TA \geq 0, y^Tb < 0$.

Farkas 2

Farkas A: Solution for exactly one of:
(1) $Ax = b, x \geq 0$
(2) $y^TA \geq 0, y^Tb < 0$.

Farkas B: Solution for exactly one of:
(1) $Ax \leq b$
(2) $y^TA = 0, y^Tb < 0, y \geq 0$.

Strong Duality

(From Goemans notes.)

Primal P $z^* = \min c^T x$
$Ax = b$
$x \geq 0$

Dual D : $w^* = \max b^T y$
$A^T y \leq c$

Weak Duality: x, y- feasible P, D: $x^T c \geq b^T y$.

$$\begin{aligned} x^T c - b^T y &= x^T c - x^T A^T y \\ &= x^T (c - A^T y) \\ &\geq 0 \end{aligned}$$

Strong duality If P or D is feasible and bounded then $z^* = w^*$.

Primal feasible, bounded, value z^*.

Claim: Exists a solution to dual of value at least z^*.

$\exists y, y^T A \leq c, b^T y \geq z^*$.

Want y.
$$\begin{pmatrix} A^T \\ -b^T \end{pmatrix} y \leq \begin{pmatrix} c \\ -z^* \end{pmatrix}.$$

If none, then Farkas B says
$\exists x, \lambda \geq 0$.

$$\begin{pmatrix} A & -b \end{pmatrix} \begin{pmatrix} x \\ \lambda \end{pmatrix} = 0 \qquad \begin{pmatrix} c^T & -z^* \end{pmatrix} \begin{pmatrix} x \\ \lambda \end{pmatrix} < 0$$

$\exists x, \lambda$ with $Ax - b\lambda = 0$ and $c^t x - z^* \lambda < 0$

Case 1: $\lambda > 0$. $A(\frac{x}{\lambda}) = b$, $c^T(\frac{x}{\lambda}) < z^*$. Better Primal!!

Case 2: $\lambda = 0$. $Ax = 0, c^T x < 0$.
Feasible $\tilde{x}$ for Primal.
(a) $\tilde{x} + \mu x \geq 0$ since $\tilde{x}, x, \mu \geq 0$.
(b) $A(\tilde{x} + \mu x) = A\tilde{x} + \mu Ax = b + \mu \cdot 0 = b$. Feasible
$c^T(\tilde{x} + \mu x) = x^T \tilde{x} + \mu c^T x \to -\infty$ as $\mu \to \infty$
Primal unbounded!

See you on Thursday.