CS270: Lecture 1.

1. Overview
2. Administration
3. Dueling Subroutines: Congestion/Tolls.

Algorithms.

Undergraduate.
This class.

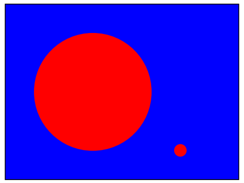
1. Classical.
   Flavor of the week?
   Modern.
2. Cleanly Stated Problems. Shortest Paths, max-flow, MST.
   Vaguely stated problems!
   Address problems, messy or not.
3. Solutions: effective precise bounds!
   Ineffective...imprecise!
   Analysis sometimes based on modelling world.
4. Techniques: Greedy Dyn. Programming Linear Programming.
   Heuristic, in practice.
5. Techniques tend to be Combinatorial.
   Probabilistic, linear algebra methods, continuous.

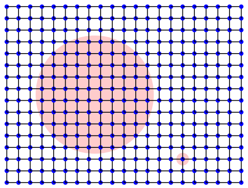
Example Problem: clustering.

- Points: documents, dna, preferences.
- Graphs: applications to VLSI, parallel processing, image segmentation.

Image example.

Image Segmentation





Which region? Normalized Cut: Find S, which minimizes

$$\frac{w(S,\overline{S})}{w(S) \times w(\overline{S})}.$$

Ratio Cut: minimize

$$\frac{w(S,\overline{S})}{w(S)},$$

$w(S)$ no more than half the weight. (Minimize cost per unit weight that is removed.)
Either is generally useful!

Example: recommendations.

Sarah Palin likes True Grit (the old one.)
Sarah Palin doesn't like The Social Network.
Sarah Palin doesn't like Black Swan.
Sarah Palin likes Sarah Palin on Discovery channel.

Hillary Clinton doesn't like True Grit (the old one.)
Hillary Clinton likes The Social Network.
Hillary Clinton likes Black Swan.

Should you recommend the discovery channel to Hillary?

What about you?

Are you Hillary? Are you Sarah? A bit of both?

High dimensional data: dimension for each movie.
More than three dimensions!
Nearest neighbors. Principal Components methods.
Topic Models.
Reasoning about these methods.

Linear Systems.

Revolution!

Physical Simulation. Airflow.

Solve $Ax = b$.

How long?

$n \times n$ matrix A.

Middle School: substitution, adding equations ...
Time: $O(n^3)$.

Now: $\tilde{O}(m)$. Hmmm. What's that tilde?

Techniques:
Relate graph theory to matrix properties.
Dense matrix (graph) to sparse matrix (graph).
Approximating distances by trees.
Electrical networks analysis.

Combinatorial Applications: Better Max Flow!

Other Algorithmic Techiniques

Sketching:
Large stream of data: $a_1, a_2, \ldots$

Find digest.
Graphs: Sparse graph.
Data: average, statistics.
Points: center point, k-medians, .

High Dimensional optimization.
Gradient Descent. Convexity.

Linear Algebra.
Eigenvalues.
Semidefinite Programming.

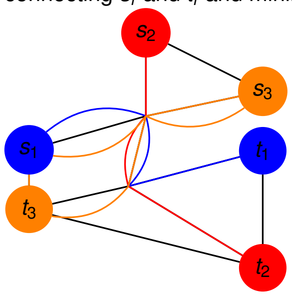
Dueling Subroutines. Duality.
Lower bounding, upper bounder.
Upper uses lower's evidence to find better solutions.
Lower uses upper's evidence to prove better lower bounds.

CS270: Administration.

1. Staff:
   Satish Rao
   Di Wang
2. Piazza. Log in! Pay attention to "spam everyone" especially.
3. Assessment.
   3.1 Homeworks (40%).
       Homework 1 out tonight/tomorrow.
   3.2 1 Homework/Midterm (25 %)
   3.3 Project (35%)
       Groups of 2 or 3.
       Connect research to class.
       Or explore/digest a topic from class.
   3.4 No Discussion this week.

Path Routing.

Given $G = (V, E)$, $(s_1, t_1), \ldots, (s_k, t_k)$, find a set of k paths connecting s_i and t_i and minimize max load on any edge.



~~Value: 3~~

Value: 2

Terminology

Routing: Paths $p_1, p_2, \ldots, p_k$, p_i connects s_i and t_i.

Congestion of edge, e: $c(e)$
  number of paths in routing that contain e.

Congestion of routing: maximum congestion of any edge.

Find routing that minimizes congestion (or maximum congestion.)

Algorithms?

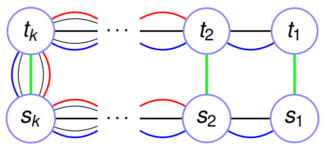
Route along any path.
Feasible...but good?
How far from optimal could it be?

(A) It is optimal!

(B) A factor of two.

(C) A factor of k, in general.

(C) and (A).



Value: k.

Opt: 1.

Stupid..also depth first search lexicographically!
Route along shortest path! Duh.
  Optimal use of "resources" ..or edges.

Shortest Path Routing.
minimizes $\sum_i \ell(p_i)$.
Total congestion: $\sum_e c(e)$
where $c(e)$ congestion of edge.

Why?

Let $\ell(p_i)$ be the length of path p_i.

(A) $\sum_i \ell(p_i) = \sum_e c(e)$?
(B) $\sum_i \ell(p_i) > \sum_e c(e)$?
(C) $\sum_i \ell(p_i) < \sum_e c(e)$?

(A). Proof?
Path i uses $\ell(p_i)$ edges.
Edge used by $c(e)$ paths.
Totals be the same.
$\sum_i \ell(p_i) = \sum_i \sum_{e \in p_i} 1 = \sum_e \sum_{p_i \ni e} 1 = \sum_e c(e)$
Shortest path routing minimizes total congestion.

Shortest Path Routing and Congestion.

Minimize each path length minimizes total congestion.

Also minimizes average: $\frac{1}{m}\sum_e c(e)$. Just a scaling!

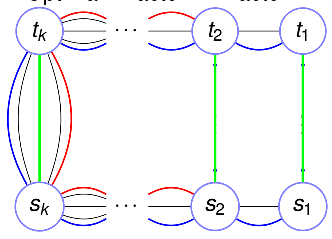
Average load is lower bound on the lowest max congestion!

Shortest path routing minimizes average load.

Does it minimize maximum load?

One problem...

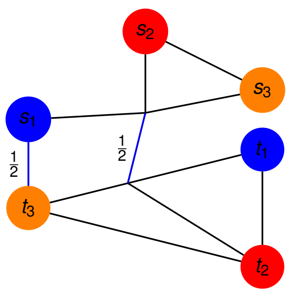
How far from optimal?
Optimal? Factor 2? Factor k?



Value: k.
$> 2k$.
Opt: 1.

Another problem.

Given $G = (V, E)$, $(s_1, t_1), \ldots, (s_k, t_k)$, find a set of k paths assign one unit of "toll" to edges to maximize total toll for connecting pairs.



Assign $\frac{1}{11}$ on each of 11 edges.
Total toll: $\frac{3}{11} + \frac{3}{11} + \frac{3}{11} = \frac{9}{11}$

Can we do better?

Assign $1/2$ on these two edges.
Total toll: $\frac{1}{2} + \frac{1}{2} + \frac{1}{2} = \frac{3}{2}$

Toll problem and Routing problem.

Given $G = (V, E)$, $(s_1, t_1), \ldots, (s_k, t_k)$, find a set of k paths assign one unit of "toll" to edges to maximize total toll for connecting pairs.

Possible solution: $\frac{1}{m}$ on each edge.

Toll collected: $\geq \frac{\sum_i \ell(p_i)}{m}$.

Familiar?

Find $d : e \to R$ with $\sum_e d(e) = 1$ which maximizes

$$\sum_i d(s_i, t_i).$$

$d(s_i, t_i)$- shortest path between s_i and t_i under $d(\cdot)$.

Digression?

$d(e)$ suggests a weighted average.

Remember uniform average congestion is lower bound on congestion of routing!

Optimal toll solution (weighted average congestion) is lower bound on congestion.

Proving lower bound: notation.

$d(e)$ - toll assigned to edge e.
$d(p)$ - total toll assigned to path p.
$d(u, v)$ - total assigned to shortest path between u and v. $d(\cdot)$ - polymorpic: edges, paths, pairs.

Proving lower bound.

Routing solution: p_i connects (s_i, t_i) and has length $d(p_i)$.

$c(e)$ - congestion on edge e under routing.
Max $c(e)$?
$\max_e c(e) \geq \sum_e c(e)d(e)$ since $\sum_e d(e) = 1$.

$$\sum_e c(e)d(e) = \sum_i d(p_i)$$

$$\begin{aligned}
\sum_i d(p_i) &= \sum_i \sum_{e \in p_i} d(e) \\
&= \sum_e \sum_{i: e \ni p_i} d(e) \\
&= \sum_e d(e) \sum_{i: e \ni p_i} 1 \\
&= \sum_e d(e)c(e)
\end{aligned}$$

A path uses "volume" $d(p_i)$.

Volume on edge is $d(e)c(e)$.

$\sum_i d(p_i) = \sum_e d(e)c(e)$.

$\max_e c(e) \geq \sum_e d(e)c(e) = \sum_i d(p_i) \geq \sum_i d(s_i, t_i)$.
Routing solution cost $\geq$ Any toll solution cost.

Toll is lower bound.

From before:
Max bigger than minimum weighted average:
$\max_e c(e) \geq \sum_e c(e)d(e)$
Total length is total congestion: $\sum_e c(e)d(e) = \sum_i d(p_i)$
Each path, p_i, in routing has length $d(p_i) \geq d(s_i, t_i)$.

$$\max_e c(e) \geq \sum_e c(e)d(e) = \sum_i d(p_i) \geq \sum_i d(s_i, t_i).$$

A toll solution is lower bound on any routing solution.
Any routing solution is an upper bound on a toll solution.

Algorithm.

Assign tolls.
How to route? Shortest paths!
Assign routing.
How to assign tolls? Higher tolls on congested edges.
Toll: $d(e) \propto 2^{c(e)}$.

Equilibrium:
The shortest path routing has has $d(e) \propto 2^{c(e)}$.

The routing does not change, the tolls do not change.

How good is equilibrium?

Path is routed along shortest path and $d(e) \propto 2^{c(e)}$.
For e with $c(e) \leq c_{max} - 2\log m$; $2^{c(e)} \leq 2^{c_{max} - 2\log m} = \frac{2^{c_{max}}}{m^2}$.

$$\begin{aligned}
c_{opt} &\geq \sum_i d(s_i, t_i) = \sum_e d(e)c(e) \\
&= \sum_e \frac{2^{c(e)}}{\sum_{e'} 2^{c(e')}} c(e) = \frac{\sum_e 2^{c(e)} c(e)}{\sum_e 2^{c(e)}} \quad \text{Let } c_t = c_{max} - 2\log m. \\
&\geq \frac{\sum_{e: c(e) > c_t} 2^{c(e)} c(e)}{\sum_{e: c(e) > c_t} 2^{c(e)} + \sum_{e: c(e) \leq c_t} 2^{c(e)}} \\
&\geq \frac{(c_t) \sum_{e: c(e) > c_t} 2^{c(e)}}{(1 + \frac{1}{m}) \sum_{e: c(e) > c_t} 2^{c(e)}} \\
&\geq \frac{(c_t)}{1 + \frac{1}{m}} = \frac{c_{\max} - 2\log m}{(1 + \frac{1}{m})}
\end{aligned}$$

Or $c_{max} \leq (1 + \frac{1}{m})c_{opt} + 2\log m$.
(Almost) within $2\log m$ of optimal!

The end: sort of.

Got to here in class. Feel free to continue reading.

Getting to equilibrium.

Maybe no equilibrium!

Approximate equilibrium:

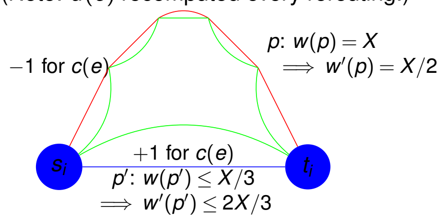
Each path is routed along a path with length
within a factor of 3 of the shortest path and $d(e) \propto 2^{c(e)}$.

Lose a factor of three at the beginning.
$c_{opt} \geq \sum_i d(s_i, t_i) \geq \frac{1}{3} \sum_e d(p_i)$.
We obtain $c_{max} = 3(1 + \frac{1}{m})c_{opt} + 2\log m$.

This is worse!
What do we gain?

An algorithm!

Algorithm: reroute paths that are off by a factor of three.
(Note: $d(e)$ recomputed every rerouting.)



Potential function: $\sum_e w(e)$, $w(e) = 2^{c(e)}$
Moving path:
Divides $w(e)$ along long path (with $w(p)$ of X) by two.
Multiplies $w(e)$ along shorter ($w(p) \leq X/3$) path by two.

$-\frac{X}{2} + \frac{X}{3} = -\frac{X}{6}$.

Potential function decreases. $\implies$ termination and existence.

Done for the day.....

...see you on Thursday.

Tuning...

Replace $d(e) = (1+\varepsilon)^{c(e)}$.

Replace factor of 3 by $(1+2\varepsilon)$

$c_{max} \leq (1+2\varepsilon)c_{opt} + 2\log m/\varepsilon$.. (Roughly)

Fractional paths?

Wrap up.

Dueling players:
Toll player raises tolls on congested edges.
Congestion player avoids tolls.

Converges to near optimal solution!

A lower bound is “necessary” (natural),
and helpful (mysterious?)!