

## Lecture 2 (portion)

These are slightly modified versions of the previous year's notes.

### 1 Two Player Games

The framework of two player games is useful for designing algorithms using the dueling subroutines approach. In this note, we introduce two player games and discuss the notion of equilibrium.

As an example of a two player game, let us consider rock, paper, scissors game (roshambo). The game can be represented by the payoff matrix  $A$ , where entry  $a_{ij}$  is the amount paid by the row player to the column player if they play strategies  $(i, j)$ .

$$A = \begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix}$$

This is an example of a zero-sum game where the sum of the payoffs of the two players is zero for all strategies.

The players are assumed to be rational, the row player attempts to minimize the payoff while the column player attempts to maximize it. For example a run of Roshambo might proceed as follows: The row player plays rock the column player plays paper. In the next round the row player plays scissors and wins over paper played by the column player. The column player plays rock to win over scissors played by the row players. This process continues, each player changing strategy on losing.

In fact, the row player wishes to win or not lose, and thus tries to play strategies to keep the column from winning and vice versa. We know from experience that for roshambo the game converges to a situation where both the row and column player choose their strategies uniformly at random. For the uniformly random strategy, the players have no incentive to change their strategy even after knowing the other player's strategy.

**DEFINITION 1** *A pair of strategies  $(x^*, y^*)$  is a Nash equilibrium if there is no incentive for a player to deviate from the equilibrium strategy if the other player does not deviate,*

$$(x^*)^t A y^* = \max_y (x^*)^t A y = \min_x x^t A y^*$$

A pair of equilibrium strategies are best responses to each other. For an equilibrium strategy, even if a player knows the other player's strategy in advance, there is no incentive to deviate from the equilibrium strategy. All two player games are guaranteed to have Nash equilibria.

Equilibrium strategies are usually not uniform or mixed. For example, let us find the equilibrium for the game obtained by adding another a preemptive strategy to the rock paper

scissors game which beats rock, paper, scissors, and ties if the other player plays preemptive. The payoff matrix as follows.

$$A = \begin{bmatrix} 0 & 1 & -1 & 1 \\ -1 & 0 & 1 & 1 \\ 1 & -1 & 0 & 1 \\ -1 & -1 & -1 & 0 \end{bmatrix}$$

The only equilibrium for this game is for both players to play the preemptive strategy with probability 1. If we instead added a row strategy which beats and ties with rock and paper, and wins against scissors then the payoff matrix would be,

$$A = \begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

For this game, the equilibrium strategy for the row player is never to play rock, play paper with 1/3, play scissors with probability 1/6 and finally play the last strategy with probability 1/2. The column player plays rock, paper, scissors, with probabilities 1/3, 1/2 and 1/6 respectively.

Exercise: Verify that this is a Nash equilibrium.

### 1.1 Routing/toll game

For the routing/toll game, the row player each row corresponds to the routing of a set of paths which connect the  $k$  pairs. The column player chooses an edge. The entry in the payoff matrix for the pair  $(r, e)$  is the congestion that the routing  $r$  places on the edge  $e$ .

The best response for the toll player given a routing is to place the entire toll on the most congested edge. The best response for the routing player given a mixed strategy over tolls is to route along shortest paths under the toll function. The algorithm from lecture 1 proceeds with the routing player moving toward the optimal best response for the toll player's strategy while the toll player places tolls more heavily on congested edges.

We remark that the iterative process converges to the optimal routing even though knowledge of the the optimal toll function may not help much. For example, in the 'single commodity flow' game the other player's strategy is to play the minimum cut. However, knowledge of the minimum cut by itself does not appear to help in finding the maximum flow.

### 1.2 The minimax theorem

The minimax theorem is central to the theory of two player zero sum games. In later lectures we will prove the theorem and see algorithms that converge to approximate equilibria. The minimax theorem is motivated by the question: Do the players have an advantage if they know the opponents strategy in advance?

**Weak duality:** If the column player chooses a strategy first and the row player responds after seeing the chosen strategy, the value of the game is  $R = \max_y \min_x x^t A y$ .

If the row player chooses a strategy first and the column player responds after seeing the chosen strategy, the value of the game is  $C = \min_x \max_y x^t A y$ .

The name  $R$  indicates that the row player has the advantage as the column player's strategy was declared first, and similarly for  $C$ .

The optimal strategy  $x^*$  when the row player plays first guarantees that the payoff is at most  $C$  against any strategy  $y$  played by the column player. Playing second the row player can ensure that the payoff is at most  $C$  by playing  $x^*$  against the strategy  $y^*$  used by the column player. The argument shows that  $C \geq R$ , the minimax theorem asserts that  $R$  and  $C$  are actually equal.

#### THEOREM 1

For all  $m \times n$  matrices  $A$  with real entries,

$$\max_{y \in \Delta_n} \min_{x \in \Delta_m} x^t A y = \min_{x \in \Delta_m} \max_{y \in \Delta_n} x^t A y = v \quad (1)$$

Here  $\Delta_m$  and  $\Delta_n$  denote the unit simplices from which mixed strategies for the two players are drawn. The  $v$  in equation (1) is called the value of the game.