Static Arbitrage Bounds on Basket Option Prices*

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Abstract

We consider the problem of computing upper and lower bounds on the price of a European basket call option, given prices on other similar baskets. Although this problem is very hard to solve exactly in the general case, we show that in some instances the upper and lower bounds can be computed via simple closed-form expressions, or linear programs. We also introduce an efficient linear programming relaxation of the general problem based on an integral transform interpretation of the call price function. We show that this relaxation is tight in some of the special cases examined before.

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Notation

For two n-vectors x, y, $x \geq y$ (resp. $x < y$) means $x_i \geq y_i$ (resp. $x_i < y_i$), $i = 1, \ldots, n$; x_+ denotes the positive part of x, which is the vector with components $\max(x_i, 0)$. e is the n-vector with all components equal to one, and e_i is the i-th unit vector of $\mathbf{R}^n$. The set $\mathbf{R}^n_+$ denotes the set of n-vectors with non-negative components, and $\mathbf{R}^n_{++}$ its interior. The cone of nonnegative measures with support included in $\mathbf{R}^n_+$ is denoted by $\mathcal{K}$. For $w \in \mathbf{R}^s$, $K \in \mathbf{R}$ and $g \in \mathbf{R}^{s+1}$, the notation $\langle g, (w, K)\rangle$ denotes the scalar product $\tilde{g}^T w + g_{m+1} K$, where $\tilde{g}$ contains the first s elements of g.

1 Introduction

1.1 Problem setup

Let $p \in \mathbf{R}^m_+$, $K \in \mathbf{R}^m_+$, $w \in \mathbf{R}^n$, $w_i \in \mathbf{R}^n$, $i = 1, \ldots, m$ and $K_0 \geq 0$. We consider the problem of computing upper and lower bounds on the price of an European basket call option with strike K_0 and weight vector w_0:

$$\mathbf{E}_\pi (w_0^T x - K_0)_+, \tag{1}$$

with respect to all probability distributions $\pi \in \mathcal{K}$ on the asset price vector x, consistent with a given set of observed prices p_i of options on other baskets, that is, given

$$\mathbf{E}_\pi (w_i^T x - K_i)_+ = p_i, \quad i = 1, \ldots, m. \tag{2}$$

Note that we implicitly assume that all the options have the same maturity, and that, without loss of generality, the risk-free interest rate is zero (we compare prices in the forward market).

We seek non-parametric bounds, that is, we do not assume any specific model for the underlying asset prices; our sole assumption is the absence of a static arbitrage today (i.e. the absence of an arbitrage that only requires trading today and at maturity). The primary objective of these bounds is not to detect and exploit arbitrage opportunities in the basket vs. vanilla market near the money, the amplitude of the Bid-Ask spreads being likely to make those opportunities very rare. However, the data on basket prices (index options in equity markets or swaptions in fixed income) is usually *very sparse* and traders often rely on intuitive guesses to extrapolate the remaining points. Our results provide a simple method to check the validity of these extrapolated prices where they are the most likely to create static arbitrage opportunities, i.e. very far in or out of the money.

From a financial point of view, our approach can be seen as a one-period, non-parametric computation of the upper and lower hedging prices defined in El Karoui and Quenez ([6] and [7], see also [11]). The necessary conditions we detail in section 2.3 in a multidimensional setup have been extensively used in the unidimensional case to infer information on the state-price density given option prices (see [4] or [12] among others).

From an optimization point of view, problems such as the one above have received a significant amount of attention in various forms. First, we can think of (2) as a *linear semi-infinite program*, *i.e.* a linear program with a finite number of linear constraints on an infinite dimensional variable. We use this interpretation and the related duality results to compute closed-form solutions, for a particular subclass of problems. Secondly, we can see

(2) as *generalized moment constraints*. This approach was successfully used in dimension one by Bertsimas and Popescu [1]. In higher dimensions however, their relaxation algorithm requires the solution of a number of linear programs that is potentially exponential in n, the number of assets. This makes the method prohibitive for large-scale problems. Finally, as in Henkin and Shananin [9], one can think of (2) as an *integral transform inversion problem.* This is the approach we adopt to design an efficient relaxation in the general case, based on shape constraints on the call price as a function of the weight vector w and strike price K.

We examine in detail a special case of the problem, in which prices on options of individual assets, as well as forward prices, are given, and the option to be priced involves a non-negative weight vector w. Our contribution there is to provide a solution that is polynomial-time in the number of assets, involving a linear program with $O(n)$ variables and constraints, where n is the number of assets. We consider two cases for each of the upper and lower bounds: one where the forward price constraints are included and the simpler case when these price constraints are ignored. Except for the lower bound with forward price constraints included, we prove that our bounds are exact, that is, they are attained (possibly in the limit) by some distribution π consistent with observed option prices. We obtain expressions for these optimal measures, and use them to prove tightness of the linear programming upper bounds applied to the special case of individual option prices, including forward price information.

Our paper is organized as follows. Our results are based on a dual formulation of the general problem that is described in 1.2. We obtain in section 2 upper and lower bounds, in the special cases referred to above, and then for the general problem. We discuss the tightness of the bounds in section 3. Our results are summarized in 4. Finally, section 5 provides some numerical examples.

1.2 Dual of the semi-infinite program

In the general case, we can write the upper bound problem as a semi-infinite program:

$$p^{\sup} := \sup_{\pi \in \mathcal{K}} \int_{\mathbf{R}_+^n} \psi(x)\pi(x)dx \text{ subject to } \int_{\mathbf{R}_+^n} \phi(x)\pi(x)dx = p, \quad \int_{\mathbf{R}_+^n} \pi(x)dx = 1, \tag{3}$$

where

$$\psi(x) := (w^T x - K_0)_+, \quad \phi_i(x) := (w_i^T x - K_i)_+, \quad i = 1, \ldots, m.$$

We define the Lagrangian (on $\mathcal{K} \times \mathbf{R}^{m+1}$):

$$L(\pi, \lambda, \lambda_0) = \int_{\mathbf{R}_+^n} \psi(x)\pi(x)dx + \lambda^T \left(p - \int_{\mathbf{R}_+^n} \phi(x)\pi(x)dx \right) + \lambda_0 (1 - \int_{\mathbf{R}_+^n} \pi(x)dx),$$

and, as in [10], we can explicit the dual of (3):

$$\begin{array}{rcl} d^{\sup} & := & \inf\limits_{\lambda_0, \lambda} : \ \lambda^T p + \lambda_0 \ : \ \lambda^T \phi(x) + \lambda_0 \geq \psi(x) \text{ for every } x \in \mathbf{R}_+^n \\ & = & \inf\limits_{\lambda} : \ \sup\limits_{x \geq 0} : \lambda^T p + \psi(x) - \lambda^T \phi(x). \end{array} \tag{4}$$

Both primal and dual problems have very intuitive financial interpretations. The primal problem looks for a state price density (see for example [5]) that maximizes the target option while satisfying the pricing constraints imposed by the current market conditions. The dual problem looks for the least expensive portfolio of options plus cash, $\lambda^T\phi(x)+\lambda_0$, that dominates the option payoff $\psi(x)$. Of course, the dual problem above yields an upper bound on the upper bound.

Similarly, the computation of the lower bound involves

$$p^{\inf} := \inf_{\pi\in\mathcal{K}} \int_{\mathbf{R}_+^n} \psi(x)\pi(x)dx \text{ subject to } \int_{\mathbf{R}_+^n} \phi(x)\pi(x)dx = p, \quad \int_{\mathbf{R}_+^n} \pi(x)dx = 1, \tag{5}$$

whose dual is

$$\begin{array}{rcl} d^{\inf} & := & \sup\limits_{\lambda_0,\lambda} : \lambda^T p + \lambda_0 \ : \ \lambda^T\phi(x) + \lambda_0 \le \psi(x) \text{ for every } x \in \mathbf{R}_+^n \\ & = & \sup\limits_{\lambda} : \inf\limits_{x\ge 0} : \lambda^T p + \psi(x) - \lambda^T\phi(x). \end{array} \tag{6}$$

Here, the dual problem provides a lower bound on the lower bound.

General results on semi-infinite linear programs establish the equivalence between the primal and dual formulations. We cite here a sufficient constraint qualification condition for perfect duality from [10], which makes an assumption about the support of optimal distributions. (We focus now on the lower bound; a similar result holds for the upper bound problem.)

Proposition 1 *Assume that for problem (6), without loss of generality, the support of the asset price distribution can be restricted to a given compact set $B \subset \mathbf{R}_+^n$. Assume further that there exist a pair $(\lambda_0, \lambda) \in \mathbf{R}^{n+1}$ such that:*

$$\lambda^T\phi(x) + \lambda_0 < \psi(x) \text{ for all } x \in B.$$

Then if $d^{\inf}$ is finite, perfect duality holds, namely $d^{\inf} = p^{\inf}$.

This constraint qualification condition trivially holds when $\phi(x)$ and $\psi(x)$ are Call option payoffs hence we have $d^{\inf} = p^{\inf}$, provided that the support of distributions feasible for our problem can be restricted to some compact $B \subset \mathbf{R}_+^n$. However, this may not be the case for the bounds detailed below and we will prove perfect duality directly whenever possible.

2 Upper and Lower Bounds

In this section, we address the problem of computing the bounds. We first consider the case when the observed prices correspond to options on each individual assets. In practice, these observations always include the forward contract prices $\mathbf{E}_\pi x_i = q_i$, $i = 1, \ldots, n$, which are quoted by the market, and we seek to exploit the forward price information. Then we specialize in 2.2 these results to the case when the forward prices are ignored; we examine this case because it is useful in the proofs of perfect duality in section 3. Finally we address the general case in 2.3.

2.1 Option and forward price constraints

We examine the problem of computing upper and lower bounds on

$$\mathbf{E}_\pi (w^T x - K_0)_+,$$

given the $2n$ constraints

$$\mathbf{E}_\pi (x_i - K_i)_+ = p_i, \quad \mathbf{E}_\pi x_i = q_i, \quad i = 1, \ldots, n, \tag{7}$$

where $K_0 > 0$ and w, K, p, q are given vectors of $\mathbf{R}^n_{++}$.

We will assume that $0 \leq p \leq q \leq p + K$, which is a necessary and sufficient condition for the problem above to be feasible. Sufficiency is obtained with the discrete distribution defined by

$$x = \begin{cases} 2p + K & \text{with probability } 1/2, \\ 2(q-p) - K & \text{with probability } 1/2. \end{cases} \tag{8}$$

From the form of the constraints, we also observe that the constraints $0 \leq p \leq q \leq p + K$ are necessary.

2.1.1 Upper bound

In this section, we apply the duality formalism to the upper bound problem with constraints described in (7).

In view of the general result (4), the dual problem can be expressed as

$$d^{\sup} \;=\; \inf_{\lambda + \mu \geq w} \sup_{x \geq 0} \; \lambda^T p + \mu^T q + (w^T x - K_0)_+ - \lambda^T (x - K)_+ - \mu^T x, \tag{9}$$

where, without loss of generality, we have included the constraint $\lambda + \mu \geq w$, in order to ensure that the inner supremum is finite. We introduce a partition of $\mathbf{R}^n_+$ as follows. To a given subset I of $\{1, \ldots, n\}$, we associate a subset D_I of $\mathbf{R}^n_+$, defined by

$$D_I = \{x \;:\; x_i > K_i, \; i \in I, \;\; 0 \leq x_i \leq K_i, \; i \in J\},$$

where J denotes the complement of I in $\{1, \ldots, n\}$. For $z \in \mathbf{R}^n$, let z_I be the vector formed with the elements $(z_i)_{i \in I}$, in the ascending order of indices in I.

We have

$$\begin{aligned} d^{\sup} &= \inf_{\lambda + \mu \geq w} \max_{t \in \{0,1\}} \max_{I \subseteq \{1,\ldots,n\}} \sup_{x \in D_I} \lambda^T p + \mu^T q + t(w^T x - K_0) - \lambda_I^T (x_I - K_I) - \mu^T x \\ &= \inf_{\lambda + \mu \geq w} \max_{t \in \{0,1\}} \max_{I \subseteq \{1,\ldots,n\}} \lambda^T p + \mu^T q + h(\lambda, \mu, I, t), \end{aligned}$$

where $h(\lambda, \mu, I, t)$ is given by

$$\begin{aligned} h(\lambda, \mu, I, t) &:= \sup_{x \in D_I} t(w^T x - K_0) - \lambda_I^T (x_I - K_I) - \mu^T x \\ &= \sup_{0 \leq x_J \leq K_J} (t w_J - \mu_J)^T x_J - t K_0 + \lambda_I^T K_I + \sup_{x_I > K_I} (t w_I - \mu_I - \lambda_I)^T x_I \\ &= \begin{cases} (t w_J - \mu_J)_+^T K_J - t K_0 + (t w_I - \mu_I)^T K_I & \text{if } \lambda_I + \mu_I \geq t w_I, \\ +\infty & \text{otherwise.} \end{cases} \end{aligned}$$

We note that finiteness of $h(\lambda, \mu, I, t)$ is guaranteed by $\lambda + \mu \geq w$ and $t \geq 0$. When these conditions hold, the maximum value of $h(\lambda, \mu, I, t)$ over $I \subseteq \{1, \ldots, n\}$ is obtained when the complement J is the full set, that is, when I is empty. We obtain

$$\max_{I \subseteq \{1,\ldots,n\}} h(\lambda, \mu, I, t) \quad = \quad (tw - \mu)_+^T K - tK_0.$$

Optimizing over t, we obtain

$$\max_{t \in \{0,1\}} \max_{I \subseteq \{1,\ldots,n\}} h(\lambda, \mu, I, t) = \max\left((-\mu)_+^T K, (w - \mu)_+^T K - K_0\right).$$

This results in the following expression for d^{sup}:

$$\begin{aligned} d^{\text{sup}} &= \inf_{\lambda+\mu \geq w} \lambda^T p + \mu^T q + \max\left((-\mu)_+^T K, (w-\mu)_+^T K - K_0\right) \\ &= \inf_{\mu} w^T p + \mu^T (q - p) + \max\left((-\mu)_+^T K, (w-\mu)_+^T K - K_0\right), \end{aligned} \tag{10}$$

which admits the following linear programming representation:

$$d^{\text{sup}} = \inf_{\mu,t,v,z} : w^T p + \mu^T (q - p) + t \qquad \begin{array}{l} t \geq v^T K, \;\; v \geq 0, \;\; v + \mu \geq 0 \\ t \geq z^T K - K_0, \;\; z \geq 0, \;\; z + \mu \geq w. \end{array}$$

The problem is feasible, and is thus equivalent to its dual. After some elimination of dual variables, the dual writes

$$d^{\text{sup}} = \max_{y,\beta} w^T p + w^T y - \beta K_0 \;\; : \;\; \begin{array}{l} (1-\beta)K \geq q - p - y \geq 0 \\ \beta K \geq y \geq 0. \end{array}$$

Note that the above problem is feasible if and only if $p \leq q \leq p + K$. We thus recover the primal feasibility condition mentioned before. This condition ensures that the dual bound d^{sup} is finite. The above further reduces to the one-dimensional problem:

$$d^{\text{sup}} = \max_{0 \leq \beta \leq 1} : w^T p + \sum_i w_i \min(q_i - p_i, \beta K_i) - \beta K_0. \tag{11}$$

The above problem is the maximization of a piecewise linear concave function of one variable, thus the maximum is attained at one of the break points $\beta_j := (q_j - p_j)/K_j \in [0 : 1]$, $j = 1, \ldots, n$, or for $\beta = 0, 1$. This way, we can obtain a closed-form expression for the upper bound, namely

$$d^{\text{sup}} = \max_{0 \leq j \leq n+1} w^T p + \sum_i w_i \min(q_i - p_i, \beta_j K_i) - \beta_j K_0,$$

with the convention $\beta_0 = 0$, $\beta_{n+1} = 1$.

We can check that the above bound satisfies some basic properties: it is convex in w and concave in p, q. Also, when $w = e_i$ (the i-th unit vector), and $K_0 = K_i$, we obtain $d^{\text{sup}} = p_i$, while for $K_i = 0$, we obtain $d^{\text{sup}} = q_i$.

2.1.2 Lower bound

In the lower bound case, the dual problem is

$$d^{\inf} = \sup_{\lambda+\mu \le w} \inf_{x \ge 0} \lambda^T p + \mu^T q + (w^T x - K_0)_+ - \lambda^T (x-K)_+ - \mu^T x,$$

where we exploited the fact that the inner infimum is $-\infty$ unless $\lambda + \mu \le w$.

Let us use the same notation as before. We have

$$\begin{aligned} d^{\inf} &= \sup_{\lambda+\mu \le w} \min_{I \subseteq \{1,\dots,n\}} \inf_{x \in D_I} \lambda^T p + \mu^T q + (w^T x - K_0)_+ - \lambda_I^T (x_I - K_I) - \mu^T x \\ &= \sup_{\lambda+\mu \le w} \min_{I \subseteq \{1,\dots,n\}} \lambda^T p + \mu^T q + h(\lambda, \mu, I), \end{aligned}$$

where

$$h(\lambda, \mu, I) \ = \ \inf_{x, y_0} y_0 - \lambda_I^T (x_I - K_I) - \mu^T x \ : \ x \in D_I, \ y_0 \ge w^T x - K_0, \ y_0 \ge 0.$$

We have by linear programming duality

$$h(\lambda, \mu, I) = \sup \, (\alpha w - \mu)^T K - \alpha K_0 - (\alpha w_J - \mu_J)_+^T K_J \ : \ \begin{array}{l} \alpha w_I - \lambda_I - \mu_I \ge 0 \\ 0 \le \alpha \le 1 \end{array}$$

Thus

$$d^{\inf} \ = \ \sup_{\lambda+\mu \le w} \lambda^T p + \mu^T (q - K) + \min_{I \subseteq \{1,\dots,n\}} f(\lambda, \mu, I),$$

where

$$f(\lambda, \mu, I) := \sup_{\underline{\alpha}(\lambda,\mu,I) \le \alpha \le 1} \alpha(w^T K - K_0) - (\alpha w_J - \mu_J)_+^T K_J,$$

and

$$\underline{\alpha}(\lambda, \mu, I) := \max_{i \in I} \frac{(\lambda_i + \mu_i)_+}{w_i},$$

with the convention that $\underline{\alpha}(\lambda, \mu, I) = 0$ when I is empty.

Let I be a non-empty subset of $\{1, \dots, n\}$. Let $i \in \arg\max_{i \in I} (\lambda_i + \mu_i)_+ / w_i$. We observe that

$$\underline{\alpha}(\lambda, \mu, I) = \underline{\alpha}(\lambda, \mu, \{i\}),$$

and

$$f(\lambda, \mu, I) \ge f(\lambda, \mu, \{i\}),$$

which dramatically reduces the complexity of the minimization subproblem: instead of computing the minimum over all 2^n sets $I \subseteq \{1, \dots, n\}$ it is sufficient to pick I in the set of *singletons* of $\{1, \dots, n\}$, or $I = \emptyset$. Therefore, the problem reads as a linear program

$$d^{\inf} = \sup_{\lambda, \mu, \alpha_0, \dots, \alpha_n} \lambda^T p + \mu^T (q - K) + h \ : \ \begin{array}{l} \lambda + \mu \le w \\ h \le \alpha_0 (w^T K - K_0) - (\alpha_0 w - \mu)_+^T K, \ 0 \le \alpha_0 \le 1 \\ \forall i \ : \ h \le \alpha_i (w^T K - K_0) - \sum_{j \ne i} (\alpha_i w_j - \mu_j)_+ K_j \\ (\lambda_i + \mu_i)_+ / w_i \le \alpha_i \le 1, \end{array} \tag{12}$$

and can be solved efficiently, since it has $O(n)$ constraints and variables.

2.2 Ignoring forward price constraints

In this section, we examine the problem in the case when the forward price constraints $\mathbf{E}_\pi x = q$ are ignored. The simple bounds we obtain in this setting will prove useful for obtaining perfect duality results later.

2.2.1 Upper bound

The new upper bound is readily obtained by setting the variable μ, which is the variable dual to the constraint $\mathbf{E}_\pi x = q$, to zero in the expression (10). We get the simple closed-form expression

$$d^{\sup} = w^T p + (w^T K - K_0)_+, \tag{13}$$

which can be obtained as a direct consequence of Jensen's inequality applied to the function $x \to x_+$.

2.2.2 Lower bound

A closed-form expression. For the lower bound, we again set the dual variable μ to zero in the expression (12). We obtain

$$d^{\inf} = \sup_{0 \le \xi \le e} p(w)^T \xi + h \ : \ h \le 0, \ h \le \xi_i(w_i K_i - K_0), \ 1 \le i \le n. \tag{14}$$

We note that $d^{\inf}$ can be expressed as the solution of a non-linear, convex optimization problem:

$$d^{\inf} = \sup_{\xi} \ p(w)^T \xi - \max_{1 \le i \le n} \ \xi_i (K_0 - w_i K_i)_+ \ : \ 0 \le \xi \le e, \tag{15}$$

or its dual:

$$d^{\inf} = \inf_{\nu} \sum_{i=1}^{n} (p_i w_i - \nu_i (K_0 - w_i K_i)_+)_+ \ : \ \nu^T e = 1, \ \nu \ge 0. \tag{16}$$

We can reduce the optimization problem to a line search over a scalar parameter, by elimination of the variable ξ. We obtain

$$d^{\inf} = \sum_{i \ : \ K_i w(i) \ge K_0} p_i w(i) + \sup_{v \ge 0} \sum_{i \ : \ K_i w(i) < K_0} p_i w(i) \min(1, \frac{v}{K_0 - K_i w(i)}) - v.$$

The minimization above can be further reduced to a closed-form expression by noting that the piecewise-linear function (of v) involved has break points at $\gamma_i = K_0 - K_i w_i$ (for i such that $\gamma_i > 0$) and 0. Thus:

$$d^{\inf} = \sum_{i \ : \ K_i w_i \ge K_0} p_i w_i + \max_{j \ : \ K_j w_j < K_0} \left(\sum_{i \ : \ K_i w_i < K_0} p_i w_i \min(1, \frac{K_0 - K_j w_j}{K_0 - K_i w_i}) - K_0 + w_j K_j \right)_+ . \tag{17}$$

Interpretation in terms of portfolios. Although the development above has the definite advantage of being completely constructive, we can get a more direct and perhaps more intuitive proof of (17) by interpreting its equivalent form (15) in terms of portfolio inequalities. Without loss of generality, we can assume that $w_0 = e$, where e is the n-vector of ones. To show

$$d^{\inf} = \sup_{0 \leq \xi \leq e} p^T \xi - \max_{1 \leq i \leq n} \xi_i (K_0 - K_i)_+,$$

it suffices to show that

$$\xi^T (x - K)_+ - \max_{1 \leq i \leq n} \xi_i (K_0 - K_i)_+ \leq \left(e^T x - K_0\right)_+ \text{ for all } x \in \mathbf{R}^n_+ \tag{18}$$

holds for every ξ such that $0 \leq \xi \leq e$. The above can be interpreted as a portfolio inequality: the price of the options portfolio $\xi^T (x - K)_+$, together with a certain amount of cash (negative values meaning borrowing), is dominated by the payoff.

Let us prove the portfolio inequality above. Let ξ be such that $0 \leq \xi \leq e$. Condition (18) trivially holds when $x = 0$. Let us now consider $x \in \mathbf{R}^n_+$, $x \neq 0$. Then, $e^T x > 0$. First, assume $(0 <) e^T x \leq K_0$, then:

$$\xi^T (x - K)_+ \leq \xi^T \left(\frac{x}{e^T x} K_0 - K\right)_+,$$

and by convexity of the function $x \to x_+$, we have:

$$\begin{aligned} \xi^T \left(\frac{x}{e^T x} K_0 - K\right)_+ &\leq \sum_{i=1}^n \xi_i \frac{x_i}{e^T x} (K_0 - K_i)_+ \\ &\leq \max_{i=1,\dots,n} \xi_i (K_0 - K_i)_+ . \end{aligned} \tag{19}$$

Assume now that $e^T x \geq K_0$, and let $i_0 = \arg\max_{i=1,\dots,n} \xi_i (K_0 - K_i)_+$, we can write

$$\sum_{i=1, i \neq i_0}^n \xi_i x_i + \xi_{i_0} (x_{i_0} - K_i)_+ - \max_{i=1,\dots,n} \xi_i (K_0 - K_i)_+ \leq \left(e^T x - K_0\right)_+$$

as

$$\sum_{i=1, i \neq i_0}^n \xi_i x_i + \xi_{i_0} (x_{i_0} - K_i)_+ - \max_{i=1,\dots,n} \xi_i (K_0 - K_i)_+ \leq e^T x - K_0,$$

which holds since $0 \leq \xi \leq e$ and

$$\xi_{i_0} \left((x_{i_0} - K_{i_0})_+ - (K_0 - K_{i_0})_+\right) \leq x_{i_0} - K_0.$$

The above, together with (19), proves the inequality (18). This shows (15) directly.

2.3 Relaxation for the general case

2.3.1 An integral transform

Let us come back to the original problem, for $p \in \mathbf{R}^m_+$, $K \in \mathbf{R}^m_+$, $w_0 \in \mathbf{R}^n$, $w_i \in \mathbf{R}^n$, $i = 1, ..., m$ and $K_0 \geq 0$. We consider the problem of computing upper and lower bounds on the price of an European call basket option with strike K_0 and weight vector w_0:

$$\mathbf{E}_\pi (w_0^T x - K_0)_+,$$

with respect to all probability distributions $\pi \in \mathcal{K}$ on the asset price vector x, consistent with a given set of m observed prices p_i of options on other baskets and forward prices q_i, that is, given

$$\mathbf{E}_\pi (w_i^T x - K_i)_+ = p_i, \quad i = 1, \ldots, m \quad \text{and} \quad \mathbf{E}_\pi x_j = q_j, \quad j = 1, \ldots, n.$$

If we write, for some $\pi \in \mathcal{K}$:

$$\begin{aligned} C(w, K) &= \mathbf{E}_\pi (w^T x - K)_+ \\ &= \int_{\mathbf{R}_+^n} (w^T x - K)_+ d\pi(x), \end{aligned}$$

we can think of $C_\pi(w, K)$ as a particular integral transform of the measure π. We can compute the inverse of this integral transform. If we assume that the measure π is absolutely continuous with respect to the Lebesgue measure with density $\pi(x)$, then for almost all K we have:

$$\hat{f}(w, K) := \frac{\partial^2 C(w, K)}{\partial K^2} = \int_{\mathbf{R}_+^n} \delta(w^T x - K) \pi(x) dx,$$

where $\delta(x)$ is the Dirac Delta function. This means that $\hat{f}(w, K)$ is the Radon transform (see [8] or [13]) of the measure π. The general pricing problem above can then be rewritten as the following infinite dimensional problem:

$$\begin{array}{ll} \text{minimize/maximize} & f(w_0, K_0) \\ \text{subject to} & f(w_i, K_i) = p_i, \quad i = 1, \ldots, m \\ & f(w, K) \in \mathcal{R}_C, \end{array}$$

where $\mathcal{R}_C$ is the range of the (linear) integral transform

$$\begin{array}{rl} C: & \mathcal{K} \to \mathcal{R}_C \\ & \pi \to C(w, K) = \int_{\mathbf{R}_+^n} (w^T x - K)_+ d\pi(x). \end{array}$$

Thus, the problem of finding all possible arbitrage-free option prices becomes equivalent to that of characterizing the range of the Radon transform on the set of nonnegative measures $\mathcal{K}$. This has been done by [9] in the context of production functions (which can be thought of as Put options). Using Call-Put parity, we can directly derive from the theorem 3.2 in [9] the following result:

Proposition 2 *A function $C(w, K)$, with $w \in \mathbf{R}_+^n$ and $K > 0$ belongs to $\mathcal{R}_C$, i.e. it can be represented in the form*

$$C(w, K) = \int_{\mathbf{R}_+^n} (w^T x - K)_+ d\pi(x),$$

where π is a nonnegative measure on a compact of $\mathbf{R}_+^n$, if and only if the following conditions hold.

- *$C(w, K)$ is convex and homogenous of degree one;*

- *for every $w \in \mathbf{R}^n_{++}$, we have*

$$\lim_{K\to\infty} C(w,K) = 0 \text{ and } \lim_{K\to 0^+} \frac{\partial C(w,K)}{\partial K} = -1;$$

- *the function*

$$F(w) = \int_0^\infty e^{-K} d\left(\frac{\partial C(w,K)}{\partial K}\right)$$

belongs to $C_0^\infty(\mathbf{R}^n_+)$ and for some $\tilde{w} \in \mathbf{R}^n_+$ the inequalities:

$$(-1)^{k+1} D_{\xi_1}...D_{\xi_k} F(\lambda\tilde{w}) \geq 0$$

hold for all positive integers k and $\lambda \in \mathbf{R}_{++}$ and all $\xi_1, ..., \xi_k$ in $\mathbf{R}^n_+$.

2.3.2 Linear programming relaxation

The conditions above are not tractable in the general case but we can formulate a relaxation of the original program by simply dropping the last condition, and replacing it with a (necessary) linearity condition on $C(w,0)$ with respect to w. We get an upper bound on the upper bound (resp. a lower bound on the lower bound) solution by computing:

$$\begin{array}{ll} \sup / \inf & C(w_0, K_0) \\ \text{subject to} & C(w,K) \text{ convex in } (K,w) \\ & C(w,K) \text{ homogeneous of degree 1} \\ & -1 \leq \partial C(w,K)/\partial K \leq 0 \text{ and } C(w,K) \text{ nondecreasing in } w \\ & C(w_i, 0) = w_i^T q,\ i = 1, ..., m \\ & C(w_i, K_i) = p_i,\ i = 1, ..., m. \end{array} \tag{20}$$

This is an infinite dimensional linear program in the variable $C(w,K) \in C(\mathbf{R}^{n+1} \to \mathbf{R}_+)$. As we show below, this infinite program can be reduced to a finite LP if we define $p_i = w_i^T q$ and $K_i = 0$ for $i = m+1, ..., m+n$ and $p_{m+n+1} = w^T q$ with $K_{m+n+1} = 0$.

Proposition 3 *If the following finite LP in the variables $p_0 \in \mathbf{R}_+$ and $g_i \in \mathbf{R}^{n+1}$ for $i = 0, ..., m+n+1$:*

$$\begin{array}{ll} \textit{maximize/minimize} & p_0 \\ \textit{subject to} & \langle g_i, (w_j, K_j) - (w_i, K_i)\rangle \leq p_j - p_i, \quad i,j = 0, ..., m+n+1 \\ & g_{i,j} \geq 0, -1 \leq g_{i,n+1} \leq 0, \quad i = 0, ..., m+n+1, \quad j = 1, ..., n \\ & \langle g_i, (w_i, K_i)\rangle = p_i, \quad i = 0, ..., m+n+1, \end{array} \tag{21}$$

is strictly feasible and its optimal value is finite (hence it is attained), the infinite program (20) and its discretization (21) have the same optimal value. Furthermore, an optimal point of (20) can be constructed from the optimal solution to (21).

As in [3], we first notice that as a discretization of the infinite program (20), the finite LP will compute a lower (or upper) bound on its optimal value. Let us now show that this bound is in fact equal to the optimal value of (20). If we note $z^* = \left[p_0^*, g_0^{*T}, \ldots, g_k^{*T}\right]^T$ the optimal solution to the LP problem above and if we define:

$$s(w,K) = \max_{i=0,\ldots,m+n+1} \left\{p_i^* + \langle g_i^*, (w,K) - (w_i, K_i)\rangle\right\},$$

$s(x)$ satisfies

$$s(x_i) = p_i, \quad i = 1, \ldots, m+n+1,$$

and, by construction, $s(x_0)$ attains the lower bound p_0 computed in the finite LP. Also, $s(x)$ is convex as the pointwise maximum of affine functions and is piecewise affine with gradient g_i, which implies that it also verifies the convexity and monotonicity conditions in (20), hence it is a feasible point of the infinite dimensional problem. This means that both problems have the same optimal value and $s(x)$ is an optimal solution to the Infinite Linear Program in (20).

3 Some Cases of Perfect Duality

In this section, we prove that the bounds we obtained before are tight in some special cases.

3.1 Upper bound without forwards

We first compute the optimal probability measures corresponding to the upper and lower price bounds, when forward prices are ignored. We thus consider the problem examined in 2.2.1. Based on (13), we can recover an optimal distribution, or a sequence of distributions which achieve the bound in the limit. This provides a direct proof of the fact that $p^{\sup} = d^{\sup}$ in the case when we ignore the forward price information.

If $w^T K \geq K_0$, we choose a distribution π of asset prices such that $x = p + K$ with probability one. Then, constraints (2) are trivially satisfied, and the objective (1) becomes

$$\mathbf{E}_\pi (w^T x - K_0)_+ = (w^T (p + K) - K_0)_+ = w^T p + w^T K - K_0 = d^{\sup}.$$

If $w^T K < K_0$, we have $d^{\sup} = w^T p$, and the upper bound is only attained in the limit. For a given $\epsilon > 0$, we define a probability distribution $\pi(\epsilon)$ on the asset prices as follows:

$$x = \begin{cases} \epsilon^{-1} p + K & \text{with probability } \epsilon, \\ 0 & \text{with probability } 1 - \epsilon. \end{cases} \tag{22}$$

Then, we have

$$\mathbf{E}_{\pi(\epsilon)} (x - K)_+ = \epsilon(\epsilon^{-1} p + K - K))_+ + (1-\epsilon)(-K)_+ = p,$$

while the objective becomes

$$\begin{aligned} \mathbf{E}_{\pi(\epsilon)} (w^T x - K_0)_+ &= \epsilon(w^T(\epsilon^{-1} p + K) - K_0)_+ + (1-\epsilon)(-K_0)_+ \\ &= (w^T p + \epsilon(w^T K - K_0))_+. \end{aligned}$$

When $\epsilon \to 0$, the above quantity goes to $w^T p = d^{\sup}$, as claimed.

3.2 Upper bound with forwards

We now consider the upper bound result with option and forward price constraints, obtained in 2.1.1. Without loss of generality, we assume $e^T w = 1$. In (11) we obtained:

$$d^{\sup} = \sup_{0 \le \beta \le 1} : w^T p + \sum_i w_i \min(q_i - p_i, \beta K_i) - \beta K_0,$$

which can be rewritten (the min is taken elementwise):

$$\sup_{0 \le \beta \le 1} w^T \min(q - \beta K_0 e, p + \beta (K - K_0)),$$

or again:

$$\sup_{0 \le \beta \le 1} \inf_{t \in [0,1]^m} w^T((1-t)(q - \beta K_0 e) + t(p + \beta (K_i - K_0))).$$

Using LP duality we know that this is also equal to (with $e^T w = 1$):

$$\inf_{t \in [0,1]^m} \sup_{0 \le \beta \le 1} \beta \left(w^T t K - K_0 \right) + w^T (1-t) q + w^T t p.$$

We express the above as

$$\inf_{t \in [0,1]^m} w^T (1-t) q + w^T t p + \left(w^T t K - K_0 \right)_+ .$$

This problem can be solved exactly as a finite linear program, and we obtain t^* such that:

$$d^{\sup} = w^T \left((1-t^*) q + t^* p \right) + \left(w^T t^* K - K_0 \right)_+ . \tag{23}$$

We recognize here the expression of the upper bound on the price of a basket, where we are only given the following option price constraints (see 2.2.1):

$$\mathbf{E}_\pi (x_i - \hat{K}_i)_+ = \hat{p}_i, \quad i = 1, \ldots, n,$$

where $\hat{K} := t^* K$ and $\hat{p} := (1-t^*) q + t^* p$. This means that we can directly recover the upper bound probability as in (22), substituting $(\hat{p}, \hat{K})$ with (p, K), setting $\pi(\epsilon)$ such that:

$$x = \begin{cases} \epsilon^{-1} \hat{p} + \hat{K} & \text{with probability } \epsilon, \\ 0 & \text{with probability } 1 - \epsilon, \end{cases}$$

and taking the limit when $\epsilon \to 0$.

3.3 Lower bound without forwards

We consider the problem examined in 2.2.2. The linear programming expression (16) allows us to recover a sequence of distributions that are optimal in the limit, as follows.

Let ν be an optimal vector for problem (16). We remark that ν can be interpreted as a probability distribution. Let $\mathcal{I}$ be the set of indices i such that $K_0 > w_i K_i$. We note that

$i \notin \mathcal{I}$ implies $\nu_i = 0$. For simplicity we assume that $\mathcal{I} = \{1, \ldots, m\}$, where $0 \le m \le n$ (the choice $m = 0$ corresponding to empty $\mathcal{I}$).

First we examine the case when $m = 0$, that is, $\mathcal{I}$ is empty. In other words, $\min_i w(i)K_i \ge K_0$, and therefore $d^{\inf} = p^T w$. For a given $\epsilon > 0$, we choose the probability distribution on the asset prices given by (22), and follow the same steps taken before, for the upper bound. We obtain that $d^{\inf}$ is attained as $\epsilon \to 0$.

Next, we assume $m \ge 1$. Let $\alpha = (n-m)/m$. For ϵ such that $0 < \alpha^{-1} \min_{1 \le i \le m} \nu_i (\ne 0)$, we define vector $\nu(\epsilon)$ by

$$\nu_i(\epsilon) = \begin{cases} \nu_i - \alpha\epsilon & \text{if } 1 \le i \le m, \\ \epsilon & \text{otherwise.} \end{cases}$$

Since ϵ is small enough, vector $\nu(\epsilon)$ satisfies the constraints of problem (16).

We now define a distribution $\pi(\epsilon)$ on the asset price vector x as follows.

$$x = x^\epsilon(i) \text{ with probability } \nu_i(\epsilon),$$

where

$$x_j^\epsilon(i) = \begin{cases} \dfrac{p_j}{\nu_j(\epsilon)} + K_j & \text{if } j = i, \\ 0 & \text{otherwise.} \end{cases}$$

Note that $x_j^\epsilon(i)$ is always well-defined, since $\nu_j(\epsilon) > 0$ for every j.

Let us check that the distribution $\pi(\epsilon)$ of asset prices satisfies the constraints (2). For every j, $1 \le j \le n$, we have

$$\begin{aligned} \mathbf{E}(x_j - K_j)_+ &= \textstyle\sum_{i=1}^n \nu_i(\epsilon)(x_j^\epsilon(i) - K_j)_+ \\ &= \nu_j(\epsilon)(x_j^\epsilon(j) - K_j)_+ \\ &= p_j. \end{aligned}$$

Let us now check that with this choice of asset price distribution, the objective (1) attains the lower bound $d^{\inf}$, when we let $\epsilon \to 0$. We have

$$\begin{aligned} \mathbf{E}_{\pi(\epsilon)}(w^T x - K_0)_+ &= \textstyle\sum_{i=1}^n \nu_i (w^T x^\epsilon(i) - K_0)_+ \\ &= \textstyle\sum_{i=1}^n \nu_i(\epsilon)(\sum_{j=1}^n w_j x_j^\epsilon(i) - K_0)_+ \\ &= \textstyle\sum_{i=1}^n \nu_i(\epsilon)(w_i x_i^\epsilon(i) - K_0)_+ \\ &= \textstyle\sum_{i=1}^n \nu_i(\epsilon)(w_i(\frac{p_i}{\nu_i(\epsilon)} + K_i) - K_0)_+ \\ &= \textstyle\sum_{i=1}^n (w_i p_i - \nu_i(\epsilon)(K_0 - w_i K_i))_+. \end{aligned}$$

Letting $\epsilon \to 0$, we obtain

$$\begin{aligned} \lim_{\epsilon \to 0} \mathbf{E}_{\pi(\epsilon)}(w^T x - K_0)_+ &= \textstyle\sum_{i=1}^m (w_i p_i - \nu_i(K_0 - w_i K_i))_+ + \sum_{i=m+1}^n w_i p_i \\ &= \textstyle\sum_{i=1}^n (w_i p_i - \nu_i(K_0 - w_i K_i)_+)_+ \\ &= d^{\inf}, \end{aligned}$$

as claimed. This concludes our proof that $d^{\inf} = p^{\inf}$ in the absence of constraints on forward prices.

3.4 Tightness of the linear programming upper bound relaxation

We now show that for the special case considered in section 2.1.1, namely when we have option and forward price constraints on individual assets, and we seek to compute the upper bound, the linear programming relaxation devised in (21) yields a tight result. In order for our problem to be feasible, we have assumed $0 \leq p \leq q \leq p + K$.

In this case, the LP (21) is feasible and its feasible set is compact, which ensures that there exist an optimal solution. Indeed, we can form a piecewise affine function that is feasible for (20) by taking $C(w_0, k_0) = E_\pi(w_0^T x - K_0)_+$, where π is the probability measure defined in (8), precisely

$$E_\pi(w_0^T x - K_0)_+ = \max\left\{w_0^T q - K_0, w_0^T p - K_0/2, w_0^T(q-p) - K_0/2, 0\right\}.$$

This function also turns out to correspond to a feasible point of (21); the variables g_i in (21) are simply the subgradients of $C(w_0, k_0)$ at the data points. Finally, the LP in (21) is finite, since we always have $0 \leq E_\pi(w_0^T x - K_0)_+ \leq w_0^T q$ and the feasible set of (21) is compact. This means that the optimum in (21) is attained.

First, we prove tightness of the LP relaxation in the case when forward price information is ignored. The setting of section 2.2.1 assumes that $m = n$, and $w_0 \in \mathbf{R}_+^n$. We note e_i, the i-th unit vector. Without loss of generality, we set $w_0^T e = 1$. Since the function $C(w_0, K_0) = w_0^T p + \left(w_0^T K - K_0\right)_+$ is a feasible point of the infinite LP (20), if we call V^{LP} the upper bound computed by the linear program (21), we must have:

$$V^{\mathrm{LP}} \geq w_0^T p + \left(w_0^T K - K_0\right)_+.$$

Now, using the necessary conditions in (20) and the convexity of $\mathbf{E}_{\pi_\varepsilon}\left(w^T x - K\right)_+$ in (w, K) we can write

$$\begin{aligned}
\mathbf{E}_{\pi_\varepsilon}\left(w_0^T x - K_0\right)_+ &= \mathbf{E}_{\pi_\varepsilon}\left(w_0^T x - \left(w_0^T K + \left(K_0 - w_0^T K\right)\right)\right)_+ \\
&\leq \sum_{i=1}^n w_{0,i} \mathbf{E}_{\pi_\varepsilon}\left(x_i - \left(K_i + \left(K_0 - w_0^T K\right)\right)\right)_+ \\
&= \sum_{i=1}^n w_{0,i} C\left(e_i, K_i + \left(K_0 - w_0^T K\right)\right).
\end{aligned}$$

The conditions on the slope of the function $C(w, K)$ imply

$$\sum_{i=1}^n w_{0,i} C\left(e_i, K_i + \left(K_0 - w_0^T K\right)\right) \leq w_0^T p + \left(w_0^T K - K_0\right)_+.$$

Hence, $V^{\mathrm{LP}} \leq w_0^T p + \left(w_0^T K - K_0\right)_+$ and finally

$$V^{\mathrm{LP}} = w_0^T p + \left(w_0^T K - K_0\right)_+, \tag{24}$$

where we recover the expression found in (13). This means that the upper bound computed by the LP relaxation is tight in the particular case considered above.

Now we turn to the case when forward price constraints $\mathbf{E}_\pi x_i = q_i$ for $i = 1, \ldots, n$, are included. As already observed in 2.1.1, the function

$$d^{\text{sup}}(w_0, K_0) = \max_{0 \le j \le n+1} : w_0^T p + \sum_i w_{0,i} \min(q_i - p_i, \beta_j K_i) - \beta_j K_0,$$

is convex in (w_0, K_0). Also, when $w_0 = e_i$, and $K_0 = K_i$, we obtain $d^{\text{sup}} = p_i$, while for $K_i = 0$, we obtain $d^{\text{sup}} = q_i$. This means that $d^{\text{sup}}(w, K)$ is a feasible point of the infinite program (20) and hence $V^{\text{LP}} \ge d^{\text{sup}}(w_0, K_0)$.

Since the finite LP (21) is attained, at a point denoted by $z^* = \left[p_0^*, g_0^{*T}, \ldots, g_k^{*T}\right]^T$, we can define the call price function

$$d^{\text{LP}}(w, K) = \max_{i=0,\ldots,m+n+1} \left\{p_i^* + \langle g_i^*, (w, K) - (w_i, K_i)\rangle\right\},$$

corresponding to the strike prices $\hat{K} = t^* K$ and option prices $\hat{p} = (1 - t^*)q + t^* p$, as in 3.2. By convexity of $d^{\text{LP}}(w, K)$, we have $d^{\text{LP}}(e_i, \hat{K}) \le \hat{p}_i$ for $i = 1, \ldots, n$. We know then from (24) that $d^{\text{LP}}(w_0, K_0) = V^{\text{LP}} \le d^{\text{sup}}(w_0, K_0)$, hence finally $d^{\text{LP}}(w_0, K_0) = d^{\text{sup}}(w_0, K_0)$. This shows that the LP relaxation of the upper bound is tight when the input is composed of options and forward prices as in (7).

4 Summary of Results

We are ready to summarize our results.

Theorem 4 *Tight upper and lower bounds on the price p_{basket} of an European basket call option involving n assets, with weight vector $w > 0$ and strike K_0, given the n prices p_i of individual European call options with strike $K_i > 0$, are given by*

$$p^{\text{inf}} \le p_{\text{basket}} = \mathbf{E}_\pi (w^T x - K_0)_+ \le p^{\text{sup}} = \sum_{i=1}^n p_i w_i + (\sum_{i=1}^n w_i K_i - K_0)_+,$$

where

$$p^{\text{inf}} = \sum_{i\ :\ K_i w_i \ge K_0} p_i w_i + \max_{j\ :\ K_j w_j < K_0} \left(\sum_{i\ :\ K_i w_i < K_0} p_i w_i \min(1, \frac{K_0 - K_j w_j}{K_0 - K_i w_i}) - K_0 + w_j K_j \right)_+ .$$

When one includes the forward contract prices information $\mathbf{E}_\pi x = q$, then the problem is feasible if and only if $p \le q \le p + K$. The tight upper bound then becomes

$$p^{\text{sup}} = \max_{0 \le j \le n+1} w^T p + \sum_i w_i \min(q_i - p_i, \beta_j K_i) - \beta_j K_0,$$

with the convention $\beta_0 = 0$, $\beta_{n+1} = 1$, and $\beta_j := (q_j - p_j)/K_j$, $j = 1, \ldots, n$.

The lower bound p^{inf} is given by the solution of the linear program defined in (12). This bound is tight when forward prices are ignored.

In the general version of the problem, the linear programming relaxation (21) provides bounds in polynomial-time. The upper bound is tight in the special cases considered above.

Note that we have not proven the tightness of the lower bound in the case when individual option prices are given, and forward price constraints are included. We conjecture that the lower bound computed by (12) is tight, and we leave this topic for further research.

We observe that the results pertaining to our special cases (those involving individual option and forward prices only) are readily extended to a situation where we only have upper and lower bounds on these prices: simply replace the prices p_i by their upper bound in the expression for the upper bound of the basket price, and by their lower bound to compute the lower bound on the basket price.

5 Numerical results

We test here the various bounds obtained above on a simulated arbitrage-free dataset. We first evaluate by Monte-Carlo simulation the following option prices:

$$C(w_0, K_0) = \mathbf{E}\left(w_0^T x - K_0\right)_+,$$

where $x_{i,T} = S_i \exp\left(g_i\sqrt{T} - \frac{1}{2}V_{i,i}T\right)$ for $i = 1, \ldots, 5$, with g a centered multivariate Gaussian variable with given covariance matrix V. The x_i are the simulated Black Scholes [2] lognormal

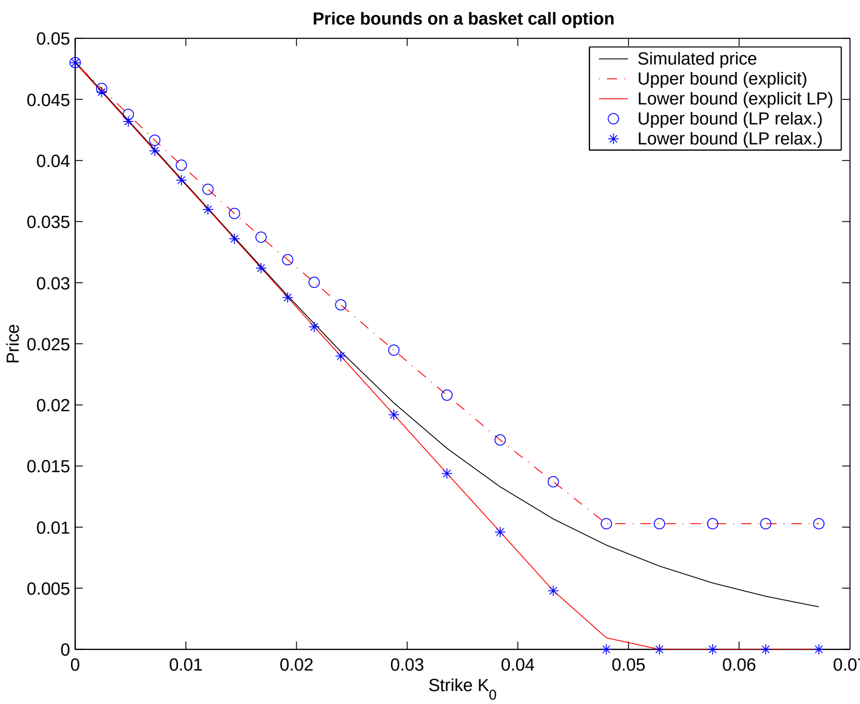

Figure 1: Upper and lower price bounds obtained for various strikes using both the explicit bounds and the LP relaxation method.

asset prices at maturity, with S the initial stock values. The numerical values used here are $S = \{0.7, 0.5, 0.4, 0.4, 0.4\}$, $w_0 = \{0.2, 0.2, 0.2, 0.2, 0.2\}$, $T = 5$ years and the covariance matrix is given by:

$$V = \frac{11}{100}\begin{pmatrix} 0.64 & 0.59 & 0.32 & 0.12 & 0.06 \\ 0.59 & 1 & 0.67 & 0.28 & 0.13 \\ 0.32 & 0.67 & 0.64 & 0.29 & 0.14 \\ 0.12 & 0.28 & 0.29 & 0.36 & 0.11 \\ 0.06 & 0.13 & 0.14 & 0.11 & 0.16 \end{pmatrix}.$$

All individual options are at-the-money, hence $K = \{0.7, 0.5, 0.4, 0.4, 0.4\}$. We get $p = \{0.0161, 0.0143, 0.0093, 0.0070, 0.0047\}$. In figure (1), we plot the upper and lower bounds obtained for various strikes using both the explicit bounds and the LP relaxation methods. We can notice that the lower bound computed using (12) is tighter than that provided by the LP relaxation in (21). We also observe that, as showed in 3.4, the two upper bounds coincide.

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