Worst-Case Simulation of Uncertain Systems

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Abstract

In this paper we consider the problem of worst-case simulation for a discrete-time system with structured uncertainty. The approach is based on the recursive computation of ellipsoids of confidence for the system state, based on semidefinite programming.

1 Introduction

This paper is concerned with the problem of estimating the state of an uncertain discrete-time system of the form

$$x_{k+1} = \left[\begin{array}{cc} \mathbf{A}_k & \mathbf{b}_k \end{array} \right] \left[\begin{array}{c} x_k \\ 1 \end{array} \right], \quad k = 0, 1, 2, \ldots$$

where the initial state $x_0 \in \mathbf{R}^n$ is only known within a given ellipsoid, and the system matrix $[\mathbf{A}_k \ \mathbf{b}_k]$ is only known to belong to a given set $\mathcal{U}$ (we will be more specific about $\mathcal{U}$ shortly).

The basic problem we consider is to compute an *ellipsoid of confidence* for the state x_{k+1}, based on a deterministic uncertainty model for the system matrices,previous confidence ellipsoid for the state x_k. This setting corresponds to the classical problem of state prediction, but in the deterministic (or *set-membership*) framework. The problems of measurement-based prediction (smoothing and filtering) may also be treated by the method presented in this paper and are object of current research.

The idea of propagating ellipsoids of confidence for systems with deterministic uncertainty was first proposed by Schweppe [10] and Bertzekas [1], and later developed by several authors, including Kurzhanski [6] and Chernousko [3]. These authors consider the case with additive uncertainty, meaning that the system matrix $[\mathbf{A} \ \mathbf{b}]$ is assumed to be exactly known. Of course, the assumption that the dynamic and measurement matrices are exactly known is very strong. The benefit of this simplification is that it yields recursive equations for the predictive filter that are simple to implement and have a structure similar to that of the Kalman filter equations. Recently, Savkin and Petersen [9] have considered a problem with a special kind of structured uncertainty, with the assumption that the uncertainty is bounded in an energy sense. These assumptions lead to recursive Riccati equations for the confidence ellipsoid, similar in spirit to the above-mentioned approach.

The main result of this paper is that ellipsoids of confidence, of size minimal in a certain geometrical sense, can be recursively computed in polynomial time via *semidefinite programming* (SDP). SDPs are convex optimization problems that generalize linear programming, and which can be solved with great theoretical and practical efficiency, using interior-point methods [7, 12].

The considered uncertainty model encompasses a very wide variety of perturbation structures, for example it can be used for uncertain systems described by matrices depending rationally on unknown-but-bounded parameters. It can also be used with more classical uncertainty models, e.g. for systems with independent additive perturbations on the state and measurement equations (these are deterministic equivalents of systems with independent process and measurement noise, as used in Kalman filtering).

Our approach basically extends the existing results to the case with structured uncertainty on *every* system matrix. To understand why the problem is much more difficult when $\mathbf{A}_k$ is allowed to be uncertain, note that if $\mathbf{A}_k$ is exactly known, and if both x_k and $\mathbf{b}_k$ belong to a convex set, then x_{k+1} also belongs to a convex set; this is not true if $\mathbf{A}_k$ is uncertain. We pay a price for being able to handle more general perturbation models, of course: we do not end up with recursive equations, but recursive optimization problems. However, the price is not to too high, since computations can still be done in polynomial-time.

The methods developed here belong to the class of methods now known as *robust programming* in the field of optimization, and developed by Oustry, El Ghaoui and Lebret [?, ?] and Ben Tal and Nemirovskii [?]. Robust optimization is concerned with decision (optimization) problems with uncertain (unknown-but-bounded) data, and tries to compute (via semidefinite programming) robust solutions, that is, solutions that are guaranteed to satisfy the (uncertain) constraints of the optimization problem, despite the perturbations.

2 Preliminaries

2.1 Notation

For a square matrix X, $X \succ 0$ (resp. $X \succeq 0$) means X is symmetric, and positive-definite (resp. semidefinite). For a square matrix U, $U^\dagger$ denotes the (Moore-Penrose) pseudo-inverse of U.

For $P \in \mathbf{R}^{n\times n}$, with $P \succ 0$, and $x \in \mathbf{R}^n$, the notation $E(P,x)$ denotes the ellipsoid

$$E(P,x) = \left\{ \xi \;\middle|\; (\xi - x)^T P^{-1} (\xi - x) \leq 1 \right\},$$

where x is the center, and P determines the "shape" of the ellipsoid.

2.2 Measures of size of an ellipsoid

The size of an ellipsoid is a function of the shape matrix P; we will denote it by $\phi(P)$ in the sequel. There are many alternative measures of size for an ellipsoid: volume, largest

measure function ϕ	measure
$\log\det P$	volume (convex in P^{-1})
$\mathbf{Tr}P$	sum of squared semi-axis lengths (convex in both P and P^{-1})
$\lambda_{\max}(P)$	largest semi-axis length (convex in P, quasi-convex in P^{-1})

Table 1: Examples of functions related to the size of ellipsoid $E(P,x)$, that are (quasi-) convex functions of P or of P^{-1} over the cone of positive-definite matrices.

semi-axis length, etc. Our method will work on any such size function ϕ, provided it is a (quasi-) convex function of the "shape" matrix P, or of its inverse, over the set of positive-definite matrices. Examples of such functions, and their geometrical interpretation, is given in Table 1.

In the sequel, we concentrate on the measure of size given by the sum of squares of semi-axis lengths; the extension to the other measures given in Table 1 is left to the reader.

2.3 Semidefinite programming

We will seek to formulate our estimation problems in terms of semidefinite programming problems, which are convex optimization problems involving linear matrix inequalities (LMIs). An LMI is a constraint on a vector $x \in \mathbf{R}^m$ of the form

$$\mathcal{F}(x) = \mathcal{F}_0 + \sum_{i=1}^{m} x_i \mathcal{F}_i \succeq 0 \qquad (1)$$

where the symmetric matrices $\mathcal{F}_i = \mathcal{F}_i^T \in \mathbf{R}^{N\times N}$, $i = 0, \ldots, m$ are given. The minimization problem

$$\text{minimize } c^T x \text{ subject to } \mathcal{F}(x) \succeq 0 \qquad (2)$$

where $c \in \mathbf{R}^m$, is called a semidefinite program (SDP). SDPs are convex optimization problems and can be solved in polynomial-time with *e.g.* primal-dual interior-point methods [7, **?**].

3 Models of Uncertain Systems

3.1 LFR models

In this paper, we will consider uncertain systems modelled as

$$x_{k+1} = \left[\begin{array}{c|c} \mathbf{A}(\Delta_k) & \mathbf{b}(\Delta_k) \end{array}\right] \left[\begin{array}{c} x_k \\ 1 \end{array}\right], \quad k = 0, 1, 2, \ldots \qquad (3)$$

where Δ_k is a (possibly time-varying) uncertainty matrix. We assume that the matrix-valued functions $\mathbf{A}(\Delta)$, $\mathbf{b}(\Delta)$, etc, are given by a *linear-fractional representation* (LFR):

$$\left[\begin{array}{c|c} \mathbf{A}(\Delta) & \mathbf{b}(\Delta) \end{array}\right] = \left[\begin{array}{c|c} A & b \end{array}\right] + L\Delta\,(I - H\Delta)^{-1} \left[\begin{array}{c|c} R_A & R_b \end{array}\right], \qquad (4)$$

where A, b, L, $R = [R_A \ R_b]$, and H are constant matrices, while $\Delta \in \mathbf{\Delta}_1$, where

$$\mathbf{\Delta}_1 = \{\Delta \in \mathbf{\Delta} \ \mid \|\Delta\| \leq 1\} \ ,$$

and $\mathbf{\Delta}$ is a linear subspace. We denote by $\mathcal{U}$ the set

$$\mathcal{U} = \left\{ \left[\ \mathbf{A}(\Delta) \,\middle|\, \mathbf{b}(\Delta) \ \right] \ \mid \Delta \in \mathbf{\Delta}_1 \right\}$$

The subspace $\mathbf{\Delta}$, referred to as the *structure set* in the sequel, defines the structure of the perturbation, which is otherwise only bounded in norm. Together, the matrices A, b, C, d, L, R, H, and the subspace $\mathbf{\Delta}$, constitute a *linear-fractional representation* (LFR) of our uncertain system.

The above LFR models are not necessarily well-posed over $\mathbf{\Delta}_1$, meaning that if might happen that $\det(I - H\Delta) = 0$ for some $\Delta \in \mathbf{\Delta}_1$; we return to this issue in §3.1.1.

In the sequel, we denote by $\mathcal{B}(\mathbf{\Delta})$ a linear subspace constructed from the subspace $\mathbf{\Delta}$, referred to as the *scaling subspace*, as follows:

$$\mathcal{B}(\mathbf{\Delta}) = \left\{(S, T, G) \mid \ S\Delta = \Delta T, \ G\Delta = -\Delta^T G^T \text{ for every } \Delta \in \mathbf{\Delta}\right\}. \tag{5}$$

We will give examples of LFR models, and explicit representations of associated sets $\mathbf{\Delta}$ and $\mathcal{B}(\mathbf{\Delta})$ shortly. This uncertainty framework includes the case when parameters perturb each coefficient of the data matrices in a (polynomial or) rational manner. This is thanks to the representation lemma given below.

Lemma 1 *For any rational matrix function $\mathbf{M} : \mathbf{R}^k \to \mathbf{R}^{n \times c}$, with no singularities at the origin, there exist nonnegative integers $r_1, \ldots, r_L$, and matrices $M \in \mathbf{R}^{n \times c}$, $L \in \mathbf{R}^{n \times N}$, $R \in \mathbf{R}^{N \times c}$, $H \in \mathbf{R}^{N \times N}$, with $N = r_1 + \ldots + r_L$, such that $\mathbf{M}$ has the following Linear-Fractional Representation (LFR): For all δ where $\mathbf{M}$ is defined,*

$$\mathbf{M}(\delta) = M + L\Delta \left(I - H\Delta\right)^{-1} R, \ \text{where } \Delta = \mathbf{diag}\left(\delta_1 I_{r_1}, \ldots, \delta_L I_{r_L}\right). \tag{6}$$

A Linear-Fractional Representation (LFR) is thus a matrix-based way to describe a multivariable rational matrix-valued function. It is a generalization, to the multivariable case, of the well-known state-space representation of transfer functions. A constructive proof of the above result is given in [?]. The proof is based on a simple idea: first devise LFRs for simple (*e.g.*, linear) functions, then use combination rules (such as multiplication, addition, etc), to devise LFRs for arbitrary rational functions.

3.1.1 Well-posedness

The LFRs introduced earlier are not necessarily well-posed over $\mathbf{\Delta}_1$, meaning that if might happen that $\det(I - H\Delta) = 0$ for some $\Delta \in \mathbf{\Delta}_1$. Checking well-posedness is a NP-hard problem, known in robust control theory as the μ analysis problem, that is addressed *e.g.*, in [5]. In [5], the authors have proved that if there exist a triple $(S, T, G) \in \mathcal{B}(\mathbf{\Delta})$ such that $S \succ 0$, $T \succ 0$, and

$$H^T T H + H^T G + G^T H \prec S, \tag{7}$$

then the LFR is well-posed over $\mathbf{\Delta}_1$.

In this paper we will make the assumption that the LFR is well-posed. It turns out that this is not a strong assumption in our context, since the conditions we will obtain always imply the above condition, which in turn guarantees well-posedness.

3.1.2 Robustness lemma

We will need the following results.

Lemma 2 (unstructured perturbations) *Let $\mathcal{F} = \mathcal{F}^T$, $\mathcal{L}$, $\mathcal{R}$ and $\mathcal{H}$ be given matrices of appropriate size. We have*

$$\mathcal{F} + \mathcal{L}\Delta(I - \mathcal{H}\Delta)^{-1}\mathcal{R} + (\mathcal{L}\Delta(I - \mathcal{H}\Delta)^{-1}\mathcal{R})^T \succeq 0 \text{ for every } \Delta,\ \|\Delta\| \leq 1$$

if and only if there exists a scalar τ such that

$$\begin{bmatrix} \mathcal{F} & \mathcal{L} \\ \mathcal{L}^T & 0 \end{bmatrix} \succeq \tau \begin{bmatrix} \mathcal{R} & \mathcal{H} \\ 0 & I \end{bmatrix}^T \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix} \begin{bmatrix} \mathcal{R} & \mathcal{H} \\ 0 & I \end{bmatrix}, \quad \tau \geq 0. \tag{8}$$

Lemma 3 (structured perturbations) *Let $\mathcal{F} = \mathcal{F}^T$, $\mathcal{L}$, $\mathcal{R}$ and $\mathcal{H}$ be given matrices of appropriate size. We have*

$$\mathcal{F} + \mathcal{L}\Delta(I - \mathcal{H}\Delta)^{-1}\mathcal{R} + (\mathcal{L}\Delta(I - \mathcal{H}\Delta)^{-1}\mathcal{R})^T \succeq 0 \text{ for every } \Delta \in \mathbf{\Delta}_1$$

if there exist block-diagonal matrices

$$\begin{array}{c} S = \mathbf{diag}\,(S_1, \ldots, S_l),\quad S_i = S_i^T \in \mathbf{R}^{r_i \times r_i}, \\ G = \mathbf{diag}\,(G_1, \ldots, G_l),\quad G_i = -G_i^T \in \mathbf{R}^{r_i \times r_i}, \end{array}$$

such that

$$\begin{bmatrix} \mathcal{F} & \mathcal{L} \\ \mathcal{L}^T & 0 \end{bmatrix} \succeq \begin{bmatrix} \mathcal{R} & \mathcal{H} \\ 0 & I \end{bmatrix}^T \begin{bmatrix} S & G \\ G^T & -S \end{bmatrix} \begin{bmatrix} \mathcal{R} & \mathcal{H} \\ 0 & I \end{bmatrix}, \quad S \succeq 0. \tag{9}$$

4 LMI Conditions for Ellipsoid Update

In this section, we give the main results on the simulation problem in the general context of systems with structured uncertainty. For ease of notation, we will drop the time index k on quantities at time k, and the quantities at time $k+1$ will be denoted with a subscript $+$. With this convention, x stands for x_k, and x_+ stands for x_{k+1}.

Our aim is to determine a confidence ellipsoid $\mathcal{E}_+ = E(P_+, \hat{x}_+)$ for the state at the next time instant x_+, given the measurement information y at the current time instant, and given that x belongs to a given ellipsoid $\mathcal{E} = E(P, \hat{x})$. To avoid inverting the matrix P, we introduce the following equivalent representation for $\mathcal{E}$:

$$\mathcal{E} = \{\hat{x} + Ez \ :\ \|z\|_2 \leq 1\},$$

where $P = EE^T$ and E^T is the Cholesky factor of P.

4.1 The case with no uncertainty

Consider first the case when $\mathbf{A} = A$, $\mathbf{b} = b$ are perfectly known. We want

$$P_+ \succeq (x_+ - \hat{x}_+)(x_+ - \hat{x}_+)^T$$

to hold whenever

$$x_+ = Ax + b, \;\; x = \hat{x} + Ez, \;\; \|z\| \leq 1.$$

We get the following robust LMI condition:

$$\begin{bmatrix} P_+ & A\hat{x} + b - \hat{x}_+ + AEz \\ (A\hat{x} + b - \hat{x}_+ + AEz)^T & 1 \end{bmatrix} \succeq 0 \text{ whenever } \|z\|_2 \leq 1.$$

Using the robustness lemma (lemma 2), we obtain an equivalent condition: there exists τ such that

$$\begin{bmatrix} P_+ & A\hat{x} + b - \hat{x}_+ & AE \\ (A\hat{x} + b - \hat{x}_+)^T & 1 - \tau & 0 \\ E^T A^T & 0 & \tau I \end{bmatrix} \succeq 0.$$

4.2 Robust version

We now seek $P_+, \hat{x}_+, \tau$ such that

$$\begin{bmatrix} P_+ & \mathbf{A}\hat{x} + \mathbf{b} - \hat{x}_+ & \mathbf{A}E \\ (\mathbf{A}\hat{x} + \mathbf{b} - \hat{x}_+)^T & 1 - \tau & 0 \\ E^T \mathbf{A}^T & 0 & \tau I \end{bmatrix} \succeq 0 \text{ for every } [\mathbf{A} \; \mathbf{b}] \in \mathcal{U}.$$

To obtain the robust counterpart, we just apply the robustness lemma. We have the LFR

$$\begin{bmatrix} P_+ & \mathbf{A}\hat{x} + \mathbf{b} - \hat{x}_+ & \mathbf{A}E \\ (\mathbf{A}\hat{x} + \mathbf{b} - \hat{x}_+)^T & 1 - \tau & 0 \\ E^T \mathbf{A}^T & 0 & \tau I \end{bmatrix} = \mathcal{F} + \mathcal{L}\Delta(I - \mathcal{H}\Delta)^{-1}\mathcal{R} + (\mathcal{L}\Delta(I - \mathcal{H}\Delta)^{-1}\mathcal{R})^T,$$

where

$$\mathcal{F} = \begin{bmatrix} P_+ & A\hat{x} + b - \hat{x}_+ & AE \\ (A\hat{x} + b - \hat{x}_+)^T & 1 - \tau & 0 \\ E^T A^T & 0 & \tau I \end{bmatrix},$$

$$\mathcal{L} = \begin{bmatrix} L \\ 0 \\ 0 \end{bmatrix}, \;\; \mathcal{R} = \begin{bmatrix} 0 \\ R_A\hat{x} + R_b \\ R_A E \end{bmatrix}^T, \;\; \mathcal{H} = H.$$

We now apply the robustness lemma.

Theorem 1 *The ellipsoid*

$$\mathcal{E}_+ = \{\hat{x}_+ + E_+ z \;\mid\; \|z\|_2 \leq 1\}$$

is an ellipsoid of confidence for the new state if $\hat{x}_+$ *and* $P_+ := E_+E_+^T$ *satisfy the LMI*

$$\begin{array}{l}
\left[\begin{array}{cccc} P_+ & A\hat{x}+b-\hat{x}_+ & AE & L \\ (A\hat{x}+b-\hat{x}_+)^T & 1-\tau & 0 & 0 \\ E^TA^T & 0 & \tau I & 0 \\ L^T & 0 & 0 & 0 \end{array}\right] \succeq \\
\left[\begin{array}{cccc} 0 & R_A\hat{x}+R_b & R_AE & H \\ 0 & 0 & 0 & I \end{array}\right]^T \left[\begin{array}{cc} S & G \\ G^T & -S \end{array}\right] \left[\begin{array}{cccc} 0 & R_A\hat{x}+R_b & R_AE & H \\ 0 & 0 & 0 & I \end{array}\right], \\
S \succeq 0,
\end{array}$$

for some block-diagonal matrices

$$\begin{array}{c} S = \mathbf{diag}\,(S_1,\ldots,S_l), \;\; S_i = S_i^T \in \mathbf{R}^{r_i\times r_i}, \\ G = \mathbf{diag}\,(G_1,\ldots,G_l), \;\; G_i = -G_i^T \in \mathbf{R}^{r_i\times r_i}. \end{array}$$

Using the elimination lemma [2], we may eliminate the variable $\hat{x}$, as follows.

Theorem 2 *The ellipsoid*

$$\mathcal{E}_+ = \{\hat{x}_+ + E_+z \;\mid\; \|z\|_2 \le 1\}$$

is an ellipsoid of confidence for the new state if τ *and* $P_+ := E_+E_+^T$ *satisfy the LMI*

$$\begin{array}{l}
\left[\begin{array}{ccc} P_+ & AE & L \\ E^TA^T & \tau I & 0 \\ L^T & 0 & 0 \end{array}\right] \succeq \left[\begin{array}{ccc} 0 & R_AE & H \\ 0 & 0 & I \end{array}\right]^T \left[\begin{array}{cc} S & G \\ G^T & -S \end{array}\right] \left[\begin{array}{ccc} 0 & R_AE & H \\ 0 & 0 & I \end{array}\right], \;\; S \succeq 0, \\
\left[\begin{array}{ccc} 1-\tau & 0 & 0 \\ 0 & \tau I & 0 \\ 0 & 0 & 0 \end{array}\right] \succeq \left[\begin{array}{ccc} R_A\hat{x}+R_b & R_AE & H \\ 0 & 0 & I \end{array}\right]^T \left[\begin{array}{cc} S & G \\ G^T & -S \end{array}\right] \left[\begin{array}{ccc} R_A\hat{x}+R_b & R_AE & H \\ 0 & 0 & I \end{array}\right],
\end{array} \tag{10}$$

for some block-diagonal matrices

$$\begin{array}{c} S = \mathbf{diag}\,(S_1,\ldots,S_l), \;\; S_i = S_i^T \in \mathbf{R}^{r_i\times r_i}, \\ G = \mathbf{diag}\,(G_1,\ldots,G_l), \;\; G_i = -G_i^T \in \mathbf{R}^{r_i\times r_i}. \end{array} \tag{11}$$

The minimum-trace ellipsoid is obtained by solving the semidefinite programming problem

$$\text{minimize } \mathbf{Tr} P_+ \text{ subject to } (10), \;\; (11). \tag{12}$$

At the optimum, we have

$$P_+ = \left[\begin{array}{cc} AE & L \end{array}\right] X^\dagger \left[\begin{array}{cc} AE & L \end{array}\right]^T,$$

and a corresponding center is given by

$$\hat{x}_+ = A\hat{x} + b + \left[\begin{array}{cc} AE & L \end{array}\right] X^\dagger \left[\begin{array}{c} E^TR_A^TS(R_A\hat{x}+R_b) \\ (SH+G)^T(R_A\hat{x}+R_b) \end{array}\right],$$

where

$$X = \left[\begin{array}{cc} \tau I & 0 \\ 0 & 0 \end{array}\right] - \left[\begin{array}{cc} R_AE & H \\ 0 & I \end{array}\right]^T \left[\begin{array}{cc} S & G \\ G^T & -S \end{array}\right] \left[\begin{array}{cc} R_AE & H \\ 0 & I \end{array}\right].$$

5 Numerical Implementation

In this section, we discuss the numerical implementation of the solution to problem (12) using interior-point methods for semidefinite programming (SDP). Numerous algorithms are available today for SDP; in our experiments we have used the SDPpack package.

In order to work, these methods require that the problem be strictly feasible, and that its dual (in the SDP sense, see [12]) be also strictly feasible. Primal strict feasibility means that the feasible set is not "flat" (contained in a hyperplane in the space of decision variables). Dual strict feasibility means, roughly speaking, that the objective of the primal problem is bounded below on the (primal) feasible set.

For simplicity, we will reduce our discussion to the case when the matrix H is zero; this means that the perturbations enter *affinely* in the state-space matrices. We will also assume that the affine term R_b is zero. Finally, we constrain the matrix variables G in problem (12) to be zero. The extension of our results to the general case is straightforward.

The reduced problem we examine now takes the form

$$\begin{array}{c} \text{minimize } \mathbf{Tr} P_+ \text{ subject to} \\ \left[\begin{array}{ccc} P_+ & AE & L \\ E^T A^T & \tau I - E^T R_A^T S R_A E & 0 \\ L^T & 0 & S \end{array}\right] \succeq 0, \quad \left[\begin{array}{cc} 1-\tau-\hat{x}^T R_A^T S R_A \hat{x} & -\hat{x}^T R_A^T S R_A E \\ E^T R_A^T S R_A \hat{x} & \tau I - E^T R_A^T S R_A E \end{array}\right] \succeq 0, \\ S = \mathbf{diag}(S_1, \ldots, S_l), \quad S_i = S_i^T \in \mathbf{R}^{r_i \times r_i}. \end{array} \tag{13}$$

In the above, we have used the fact that the constraint $S \succeq 0$ is implied by the above LMIs.

5.1 Strict feasibility of primal problem

We have the following result.

Theorem 3 *If the system is well-posed, and if the matrix* $\left[\begin{array}{cc} R_A\hat{x} & R_A E \end{array}\right]$ *is not zero, then the primal problem (13) is strictly feasible; a strictly feasible primal point is given by*

$$\begin{aligned} \tau &= 0.5 \\ S &= \frac{1}{4\left\|\left[\begin{array}{cc} R_A\hat{x} & R_A E \end{array}\right]\right\|^2} \cdot I, \\ P_+ &= I + \left[\begin{array}{cc} AE & L \end{array}\right] X^\dagger \left[\begin{array}{cc} AE & L \end{array}\right]^T, \end{aligned}$$

where

$$X = \left[\begin{array}{cc} \tau I & 0 \\ 0 & 0 \end{array}\right] - \left[\begin{array}{cc} R_A E & H \\ 0 & I \end{array}\right]^T \left[\begin{array}{cc} S & 0 \\ 0 & -S \end{array}\right] \left[\begin{array}{cc} R_A E & H \\ 0 & I \end{array}\right].$$

Remark 1 *If the matrix* $\left[\begin{array}{cc} R_A\hat{x} & R_A E \end{array}\right]$ *is zero, then the new iterate* x_+ *is independent of perturbation, and the new ellipsoid of confidence reduces to the singleton* $\{A\hat{x}+b\}$.

Remark 2 *If each* $n \times r_i$ *block* L_i *of* L *is full rank, and if* E *is also full rank, then the optimal* P_+ *(and corresponding* E_+*) is full rank.*

5.2 Strict feasibility of dual problem

The problem dual to the SDP (13) is

$$\text{maximize } -2\left(\mathbf{Tr} AEZ_{12}^T + \mathbf{Tr} LZ_{13}^T\right) \text{ subject to}$$

$$Z = \begin{bmatrix} I & Z_{12} & Z_{13} \\ Z_{12}^T & Z_{22} & Z_{23} \\ Z_{13}^T & Z_{23}^T & Z_{33} \end{bmatrix} \succeq 0, \quad X = \begin{bmatrix} x_{11} & x_{12}^T \\ x_{12} & X_{22} \end{bmatrix} \succeq 0, \quad x_{11} = \mathbf{Tr}(X_{22} + Z_{22}),$$
$$Z_{33}^{(i)} = R_i\left(E(X_{22} + Z_{22})E^T + x_{11}\hat{x}\hat{x}^T + \hat{x}x_{12}^T E^T + Ex_{12}\hat{x}^T\right) R_i^T, \quad i = 1, \ldots, l.$$

In the above, the notation $Z^{(i)}$ refers to the i-th $r_i \times r_i$ block in matrix X, and R_i is the i-th $r_i \times n$ block in R_A.

Theorem 4 *If, for each i, $i = 1, \ldots, l$, the matrix $R_i E$ is full rank, then the dual, problem is strictly feasible. A strictly feasible dual point is obtained by setting the dual variables X, Z to be zero, except for the block-diagonal terms:*

$$\begin{aligned} X_{22} &= I \\ Z_{22} &= I \\ x_{11} &= 2n \\ Z_{33}^{(i)} &= R_i(EE^T + n\hat{x}\hat{x}^T)R_i^T, \quad i = 1, \ldots, l. \end{aligned}$$

The next theorem summarizes the sufficient conditions we have obtained to guarantee that our algorithm behaves numerically well.

Theorem 5 *If the initial ellipsoid is not "flat" (that is, the initial E is full rank), and if for each i, the blocks L_i, R_i are also full rank, then at each step of the simulation algorithm, the SDP to solve is both primal and dual stricly feasible.*

6 Example

Consider a second-order, continuous-time uncertain system

$$\ddot{y} + \mathbf{a}_1(t)\dot{y} + \mathbf{a}_2(t)y = \mathbf{a}_2(t),$$

where the uncertain, time-varying parameters $\mathbf{a}_i$, $i = 1, 2$ are subject to bounded variation of given relative amplitude ρ, precisely:

$$\mathbf{a}_i(t) = a_i^{\text{nom}}(1 + \rho\delta_i(t)), \quad i = 1, 2, \quad t \geq 0,$$

where $-1 \leq \delta_i(t) \leq 1$ for every t, and a_i^{nom}, $i = 1, 2$, is the nominal value of the parameters.

By discretizing this system using a forward-Euler scheme with discretization period h, we obtain a system of the form (**??**), with the following LFR

$$\left[\begin{array}{c|c} \mathbf{A} & \mathbf{b} \end{array}\right] = \left[\begin{array}{cc|c} 1 & h & 0 \\ -h\mathbf{a}_2 & -h\mathbf{a}_1 & h\mathbf{a}_2 \end{array}\right] = \left[\begin{array}{c|c} A & b \end{array}\right] + L\Delta R,$$

where

$$\left[\, A \,|\, b \,\right] = \left[\begin{array}{cc|c} 1 & h & 0 \\ -ha_2^{\text{nom}} & -ha_1^{\text{nom}} & ha_2^{\text{nom}} \end{array}\right],$$

$$L = -h\rho \left[\begin{array}{cc} 0 & 0 \\ a_1^{\text{nom}} & a_2^{\text{nom}} \end{array}\right], R = \left[\begin{array}{ccc} 0 & 1 & 0 \\ 1 & 0 & -1 \end{array}\right], \Delta = \mathbf{diag}\,(\delta_1, \delta_2).$$

Figures 6 and 6 compare the responses in output y for $h = 0.1$, $a_1^{\text{nom}} = 3$, $a_2^{\text{nom}} = 9$, for a time horizon of $T = 50$ steps, with 100 Monte-Carlo simulations, for two values of ρ: $\rho = 0.2$ and $\rho = 0.4$, respectively.

The first plot ($\rho = 0.2$) shows a case when the Monte-Carlo and worst-case analyses agree on the qualitative behavior of the uncertain system. In the second plot ($\rho = 0.4$), the worst-case analysis appear to predict instability of the system, while the random trials predict stability. The worst-case analysis seems to be conservative, but the reader should be aware that the actual worrst-case behavior cannot be accurately predicted, in general, by taking random samples.

7 Conclusions

In this paper we presented a recursive scheme for computing a minimal size ellipsoid (ellipsoid of confidence) that is guaranteed to contain the state at time $k+1$ of a linear discrete-time system affected by deterministic uncertainty in all the system matrices, given a previous ellipsoid of confidence at time k. The ellipsoid of confidence can be recursively computed in polynomial time via semidefinite programming. We remark that the presented results are valid on a finite time horizon, while steady-state and convergence issues are currently under investigation.

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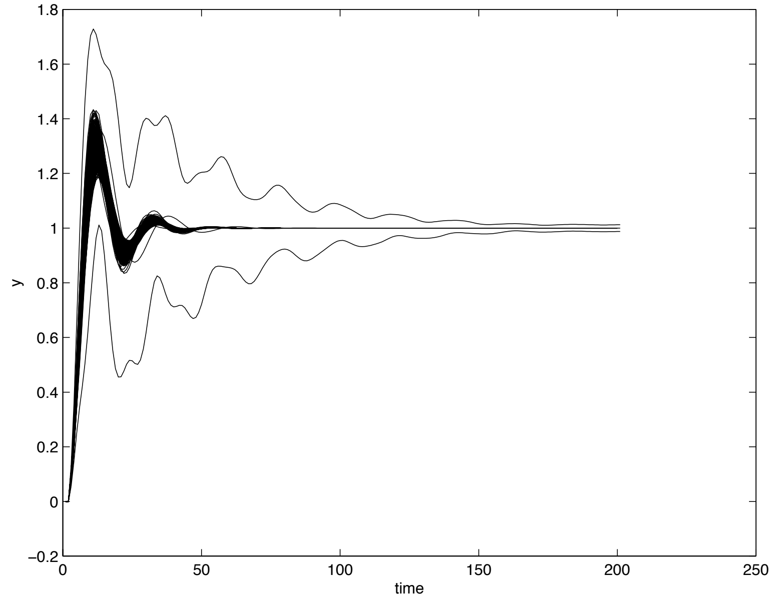

Figure 1: Worst-case and random simulation for a second-order uncertain system ($\rho = 0.2$).

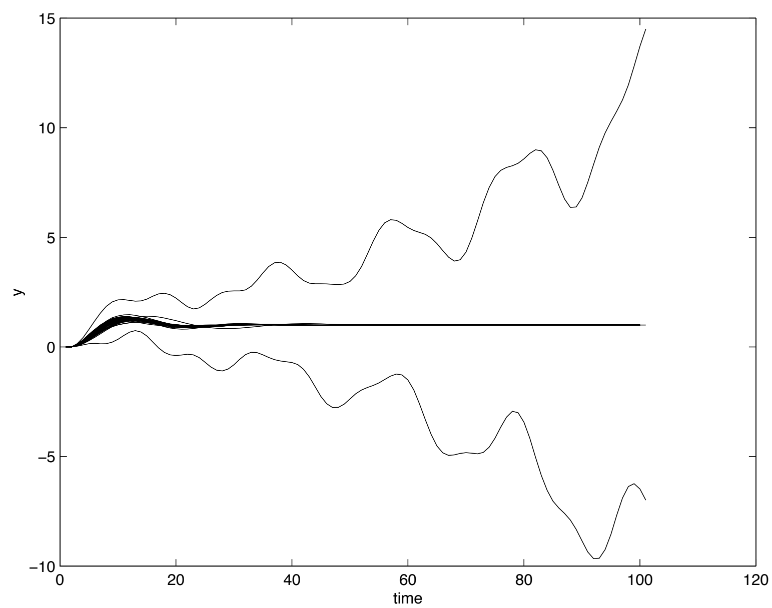

Figure 2: Worst-case and random simulation for a second-order uncertain system ($\rho = 0.4$).

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